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RESEARCH ARTICLE

Relevant Functions to (*p*,*q*)-Gamma Function and (*p*,*q*)-Beta Function

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Abstract

In this article, we study some new functions (namely $\tilde{\Gamma}_{p,q}(z) = \Gamma_{q/p}(z)$ and $\tilde{\mathrm{B}}_{p,q}(s,t) = \Gamma_{q/p}(s,t)$) and we will show how these functions are relevant to (*p*,*q*)-Gamma function and (*p*,*q*)-Beta function.

Keywords: (*p*,*q*)-Gamma function; (*p*,*q*)-Beta function.

1. Introduction

The *q*-deformed algebras [14, 15] and their generalizations ((*p*,*q*)-deformed algebras [11, 12] attract much attention these last years. The main reason is that these topics stand for a meeting point of today's fast developping areas in methematics and physics like the theory of quantum orthogonal polynomials and special functions, quantum groups, conformal field theories and statistics. From these works, many generalizations of special functions arise. There is a considerable list of references.

In this work, we give a new generalization of the Gamma and the Beta functions, namely, the (*p*,*q*)-Gamma and the (*p*,*q*)-Beta functions and we study the relation between them. In the last section we study a new functions (namely $\tilde{\Gamma}_{p,q}(z) = \Gamma_{q/p}(z)$ and $\tilde{\mathrm{B}}_{p,q}(s,t) = \Gamma_{q/p}(s,t)$) and we will show how these functions are relevant to (*p*,*q*)-Gamma function and (*p*,*q*)-Beta function.

2. Notations and miscellaneous relations

Before we study the new generalization of Gamma function and Beta Function, which is (*p*,*q*)-analogue for each one. As of now some notations and definitions of (*p*,*q*)-calculus will be introduced in this section.

Definition1: for any positive integer number, the (*p*,*q*)-numbers define as:

$$[t]_{p,q} = p^{t-1} + p^{t-2}q + p^{t-3}q^2 + \ldots + pq^{t-2} + q^{t-1} = \frac{p^t - q^t}{p-q}. \tag{2.1}$$

Since $[t]_{p,q} = \frac{p^t - q^t}{p-q}$.

$$= p^{t-1}\frac{\left(p-p^{1-t}q^{t}\right)}{\left(p-q\right)} = p^{t-1}\frac{\left(p^{1-t}\frac{q^{t}}{p}-1\right)}{\left(\frac{q}{p}-1\right)},$$

$$= p^{t-1}\frac{\left(\left(q/p\right)^{t}-1\right)}{\left(\left(q/p\right)-1\right)} = p^{t-1}\left[t\right]_{q/p},$$

Clearly, we can also write $[t]_{p,q}$as: $[t]_{p,q} = p^{t-1}[t]_{q/p}$

Definition2: For $t \in \mathbb{N}$the (p,q)-factorial is given by:

$$\left[t\right]_{p,q}! = \prod_{n=1}^{t}\left[n\right]_{p,q}!, \qquad t \ge 1,\ \left[0\right]_{p,q}! = 1. \tag{2.2}$$

Definition3: the (p,q)-binomial coefficient is defined as:

$$\begin{bmatrix} a \\ b \end{bmatrix}_{p,q} = \frac{\left[a\right]_{p,q}!}{\left[a-b\right]_{p,q}!\left[b\right]_{p,q}!}, \qquad 0 \le b \le a\ . \tag{2.3}$$

Clearly, we can see by definition that:

$$\begin{bmatrix} a \\ b \end{bmatrix}_{p,q} = \begin{bmatrix} a \\ a-b \end{bmatrix}_{p,q}, \qquad 0 \le b \le a\ .$$

Definition4: The (p,q)-powers is given by:

$$(z \oplus b)_{p,q}^{k} = (z+b)(pz+bq)\ldots(zp^{k-1}+bq^{k-1}) = \prod_{j=0}^{k-1}(zp^{j}+q^{j}b) \tag{2.4}$$

$$(z \ominus b)_{p,q}^{k} = (z-b)(pz-bq)\ldots(zp^{k-1}-bq^{k-1}) = \prod_{j=0}^{k-1}(zp^{j}-q^{j}b) \tag{2.5}$$

and these definitions are extended to the following expressions:

$$(z \oplus b)_{p,q}^{\infty} = (z+b)(pz+bq)(zp^2+bq^2)\ldots = \prod_{j=0}^{\infty}(zp^j + q^j b) \tag{2.6}$$

$$(z \ominus b)_{p,q}^{\infty} = (z-b)(pz-bq)(zp^2-bq^2)\ldots = \prod_{j=0}^{\infty}(zp^j - q^j b) \tag{2.7}$$

Note that, the convergence is required in these equations.

Definition5: the (p,q)-derivative of the function g is given by:

$$D_{p,q}g(z) = \frac{g(pz)-g(qz)}{(p-q)z}, \qquad z \neq 0 \ . \tag{2.8}$$

and if g is differentiable at 0, then $D_{p,q}g(0) = g'(0)$. Also notice that for p=1, the (p,q)-derivative minimize to the q-derivative.

The following is the product rules for the (p,q)-derivative:

$$D_{p,q}\left(g(z)f(z)\right) = g(pz)D_{p,q}f(z) + f(qz)D_{p,q}g(z), \tag{2.9}$$

$$D_{p,q}\left(g(z)f(z)\right) = f(pz)D_{p,q}g(z) + g(qz)D_{p,q}f(z), \tag{2.10}$$

Proposition 2.1: For any integer$t \geq 1$, we have:

$$(1) D_{p,q}(z \ominus b)_{p,q}^{t} = [t]_{p,q}(pz \ominus b)_{p,q}^{t-1}, \tag{2.11}$$

$$(2)(1)D_{p,q}(b \ominus z)_{p,q}^{t} = -[t]_{p,q}(b \ominus qz)_{p,q}^{t-1}, \tag{2.12}$$

Note that, $D_{p,q}(z \ominus b)_{p,q}^{0} = 0$,

Proof(1):

By using the (p,q) -derivative definition (2.8) we have:

$$D_{p,q}(z \ominus b)_{p,q}^{t} = \frac{(pz \ominus b)_{p,q}^{t} - (qz \ominus b)_{p,q}^{t}}{(p-q)z},$$

$$= \frac{(pz-b)(p^2z-qb)\dots(p^tz-q^{t-1}b) - (qz-b)(qpz-qb)\dots(qp^{t-1}z-q^{t-1}b)}{(p-q)z},$$

$$= \frac{(pz-b)(p^2z-qb)\dots(p^{t-1}z-q^{t-2}b)(p^tz-q^{t-1}b) - (qz-b)q(pz-b)q(p^2z-qb)\dots q(p^{t-1}z-q^{t-2}b)}{(p-q)z}$$

$$= \frac{(pz-b)(p^2z-qb)\dots(p^{t-1}z-q^{t-2}b)\big(p^tz-q^{t-1}b-q^{t-1}(qz-b)\big)}{(p-q)z},$$

$$= \frac{(pz-b)(p^2z-qb)\dots(p^{t-1}z-q^{t-2}b)(p^tz-q^{t-1}b-q^tz+q^{t-1}b)}{(p-q)z},$$

$$= (pz-b)(p^2z-qb)\dots(p^{t-1}z-q^{t-2}b)\frac{(p^tz-q^tz)}{(p-q)z},$$

$$= (pz \ominus b)_{p,q}^{n-1}\frac{(p^t-q^t)}{(p-q)} = [t](pz \ominus b)_{p,q}^{n-1},$$

Proof(2):

$$D_{p,q}(b \ominus z)_{p,q}^{t} = \frac{(b \ominus pz)_{p,q}^{t} - (b \ominus qz)_{p,q}^{t}}{(p-q)z},$$

$$=\frac{(b-pz)(bp-pqz)(bp^2-pq^2z)\dots(bp^{t-1}-pq^{t-1}z)-(b-qz)(bp-q^2z)(bp^2-q^3z)\dots(bp^{t-1}-q^tz)}{(p-q)z},$$

$$=\frac{(qz-b)(q^2z-bp)\dots(q^{t-1}z-bp^{t-2})(q^tz-bp^{t-1})-(pz-b)p(qz-b)p(q^2z-bp)\dots p(q^{t-1}z-bp^{t-2})}{(p-q)z},$$

$$=\frac{(qz-b)(q^2z-bp)\dots(q^{t-1}z-bp^{t-2})(q^tz-bp^{t-1}-(pz-b)p^{t-1})}{(p-q)z},$$

$$=\frac{(qz-b)(q^2z-bp)\dots(q^{t-1}z-bp^{t-2})(q^t-p^t)}{(p-q)};$$

$$=-\frac{(p^t-q^t)}{(p-q)}(b-qz)(bp-q^2z)\dots(bp^{t-2}-q^{t-1}z),$$

$$=-[t]_{p,q}(b\ominus qz)_{p,q}^{t-1}$$

Proposition 2.2: It's easily to verify the following identities:

(1) $(s\ominus t)_{p,q}^{k}=\frac{(s\ominus t)_{p,q}^{\infty}}{(sp^k\ominus tq^k)_{p,q}^{\infty}}$,

(2) $(s\ominus t)_{p,q}^{k+j}=(s\ominus t)_{p,q}^{k}(sp^k\ominus tq^k)_{p,q}^{j}$,

(3) $\left(sp^k\ominus q^kt\right)_{p,q}^{j}=\frac{(s\ominus t)_{p,q}^{j}(sp^j\ominus tq^j)_{p,q}^{k}}{(s\ominus t)_{p,q}^{k}}$,

(4) $(sp^j\ominus q^jt)_{p,q}^{k-j}=\frac{(s\ominus t)_{p,q}^{k}}{(s\ominus t)_{p,q}^{j}}$,

Proof (1):

$$\frac{(s\ominus t)_{p,q}^{\infty}}{(sp^k\ominus tq^k)_{p,q}^{\infty}}=\frac{(s-t)(sp-qt)\dots(sp^{k-1}-q^{k-1}t)(sp^k\ominus tq^k)_{p,q}^{\infty}}{(sp^k\ominus tq^k)_{p,q}^{\infty}}=(s\ominus t)_{p,q}^{k},$$

Proof (2):

$$(s \ominus t)_{p,q}^{k+j} = (s-t)(sp-qt)\dots(sp^{k-1}-q^{k-1}t)(sp^k-tq^k)\dots(sp^{k+j-1}-tq^{k+j-1}),$$

$$= (s \ominus t)_{p,q}^{k}(sp^k-tq^k)\dots(sp^{k+j-1}-tq^{k+j-1}),$$

$$= (s \ominus t)_{p,q}^{k}(sp^k \ominus tq^k)_{p,q}^{j},$$

Proof (3):

$$(s \ominus t)_{p,q}^{j+k} = (s \ominus t)_{p,q}^{j}(sp^j \ominus tq^j)_{p,q}^{k},$$

$$(s \ominus t)_{p,q}^{k}(sp^k \ominus tq^k)_{p,q}^{j} = (s \ominus t)_{p,q}^{j}(sp^j \ominus tq^j)_{p,q}^{k},$$

$$(sp^k \ominus tq^k)_{p,q}^{j} = \frac{(s \ominus t)_{p,q}^{j}(sp^j \ominus tq^j)_{p,q}^{k}}{(s \ominus t)_{p,q}^{k}},$$

Proof (4):

$$(sp^j \ominus q^j t)_{p,q}^{k-j} = (sp^j \ominus q^j t)_{p,q}^{k}(sp^{j+k} \ominus q^{j+k} t)_{p,q}^{-j},$$

$$= \frac{(sp^j \ominus q^j t)_{p,q}^{k}}{(sp^k \ominus q^k t)_{p,q}^{j}} = \frac{(s \ominus t)_{p,q}^{j+k}(s \ominus t)_{p,q}^{k}}{(s \ominus t)_{p,q}^{j}(s \ominus t)_{p,q}^{j+k}} = \frac{(s \ominus t)_{p,q}^{k}}{(s \ominus t)_{p,q}^{j}},$$

Definition 6: The (p,q)-integral of $g(x)$ on $[0,b]$ is given by:

$$\int_0^b g(x) d_{p,q}x = (p-q)b\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}g\left(\frac{q^j}{p^{j+1}}b\right), \quad 0<q<p<1$$

Definition 7: The (p,q)-IBP is given by:

$$\int_b^c f(pz)D_{p,q}g(z)\,d_{p,q}z = f(c)g(c) - f(b)g(b) - \int_b^c g(qz)D_{p,q}f(z)\,d_{p,q}z, \qquad (2.13)$$

3. The Definitions of (p,q)-Gamma function and (p,q)-Beta Function

Definition 1:The (p,q)-GF for a nonnegative integer z is given by :

$$\Gamma_{p,q}(z) = \frac{(p\ominus q)_{p,q}^{\infty}}{(p^z\ominus q^z)_{p,q}^{\infty}}(p-q)^{1-z} = \frac{(p\ominus q)_{p,q}^{z-1}}{(p-q)}, \quad 0 < q < p.$$

(3.1)

Remark 1: Notice that if $p = 1$, then $\Gamma_{p,q}(z)$ is reduces to $\Gamma_q(z)$.

Lemma 3.1: For all $z \in \mathbb{N}$, the (p,q)- Gamma function obtain the fundamental relation:

$$\Gamma_{p,q}(z+1) = [z]_{p,q}\Gamma_{p,q}(z), \qquad (3.2)$$

proof :

$$\Gamma_{p,q}(z+1) = \frac{(p\ominus q)_{p,q}^{\infty}}{(p^{z+1}\ominus q^{z+1})_{p,q}^{\infty}}(p-q)^{-z} \quad .$$

$$= (p-q)^{1-z}\frac{(p\ominus q)_{p,q}^{\infty}}{(p^z\ominus q^z)_{p,q}^{\infty}}\frac{(p^z\ominus q^z)_{p,q}^{\infty}}{(p^{z+1}\ominus q^{z+1})_{p,q}^{\infty}(p-q)} \quad .$$

$$= \Gamma_{p,q}(z)\frac{(p^z\ominus q^z)_{p,q}^{\infty}}{(p^{z+1}\ominus q^{z+1})_{p,q}^{\infty}(p-q)} = \Gamma_{p,q}(z)\frac{(p^z - q^z)(p^{z+1}\ominus q^{z+1})_{p,q}^{\infty}}{(p^{z+1}\ominus q^{z+1})_{p,q}^{\infty}(p-q)} \quad .$$

$$= \Gamma_{p,q}(z)\frac{(p^z - q^z)}{p-q} = [z]_{p,q}\Gamma_{p,q}(z).$$

we obtain also:

$\Gamma_{p,q}(z+1)=\frac{(p\ominus q)_{p,q}^{z}}{(p-q)^{z}}=[z][z-1]\Gamma_{\mathrm{p,q}}(z)=[z]_{p,q}!,\quad 0<q<p,$(4.2.3)

Definition 2: The (p,q) −BF for $s,t\in\mathbb{N}$ is given by:

$\mathrm{B}_{p,q}(s,t)=\int_0^1(pz)^{s-1}(p\ominus pqz)_{p,q}^{t-1}\,d_{p,q}z.$

(3.4)

Theorem 3.1: The relation between (p,q)-Gamma function and (p,q)-Beta Function for $s,t\in\mathbb{N}$ is defined by:

$$\mathrm{B}_{p,q}(s,t)=p^{[t(2s+t-2)+t-2]/2}\,\frac{\Gamma_{\mathrm{p,q}}(\mathrm{s})\Gamma_{\mathrm{p,q}}(\mathrm{t})}{\Gamma_{\mathrm{p,q}}(\mathrm{s+t})} \tag{3.5}$$

Proof:

First, for $s,t\in\mathbb{N}$ we have:

$$\mathrm{B}_{p,q}(s,t)=\int_0^1(pz)^{s-1}(p\ominus pqz)_{p,q}^{t-1}\,d_{p,q}z,$$

Now, apply (p,q)-integral by parts for $f(x)=z^{s-1}$and $(x)=-\frac{(p\ominus pz)_{p,q}^{t}}{p[t]_{p,q}}$,

where$D_{p,q}(p\ominus pz)_{p,q}^{t}=-[t]_{p,q}p(p\ominus pqz)_{p,q}^{t-1}$,

Then we get :

$$\mathrm{B}_{p,q}(s,t)=\frac{[s-1]_{p,q}}{p^{s-1}[t]_{p,q}}\int_0^1(pz)^{s-2}(p\ominus pqz)_{p,q}^{t}\,d_{p,q}z,$$

$$=\frac{[s-1]_{p,q}}{p^{s-1}[t]_{p,q}}\mathrm{B}_{p,q}(s-1,t+1), \tag{3.6}$$

We can write, for positive integer :

$$\mathrm{B}_{p,q}(s,t+1)=\int_0^1 (pz)^{s-1}(p\ominus pqz)_{p,q}^{t-1+1}\,d_{p,q}z=\int_0^1 (pz)^{s-1}(p\ominus pqz)_{p,q}^{t-1}\,(p^t\ominus pq^t z)d_{p,q}z,$$

$$=p^t\int_0^1 (pz)^{s-1}(p\ominus pqz)_{p,q}^{t-1}d_{p,q}z-q^t\int_0^1 (pz)^{s}(p\ominus pqz)_{p,q}^{t-1}d_{p,q}z\,,$$

$$=p^t\mathrm{B}_{p,q}(s,t)-q^t\mathrm{B}_{p,q}(s+1,t) \tag{3.7}$$

After that, using (3.6) to get:

$$\mathrm{B}_{p,q}(s,t+1)=p^t\mathrm{B}_{p,q}(s,t)-q^t\,\frac{[s]_{p,q}}{p^s[t]_{p,q}}\,\mathrm{B}_{p,q}(s,t+1), \tag{3.8}$$

Which mean :

$$\mathrm{B}_{p,q}(s,t+1)\left(1+q^t\,\frac{[s]_{p,q}}{p^s[t]_{p,q}}\right)=p^t\mathrm{B}_{p,q}(s,t),$$

$$\mathrm{B}_{p,q}(s,t+1)\left(\frac{p^s(p^t-q^t)+q^t(p^s-q^s)}{p^s(p^t-q^t)}\right)=p^t\mathrm{B}_{p,q}(s,t),$$

$$\mathrm{B}_{p,q}(s,t+1)=p^{s+t}\,\frac{(p^t-q^t)}{(p^{t+s}-q^{t+s})}\,\mathrm{B}_{p,q}(s,t), \tag{3.9}$$

We know :

$$\mathrm{B}_{p,q}(s,1)=\int_0^1 (pz)^{s-1}\,d_{p,q}z=\frac{p^{s-1}}{[s]_{p,q}}, \tag{3.10}$$

For t positive integer we get:

$$\mathrm{B}_{p,q}(s,t) = p^{s+t-1}\frac{(p^{t-1}-q^{t-1})}{(p^{t+s-1}-q^{t+s-1})}\mathrm{B}_{p,q}(s,t-1),$$

$$= p^{s+t-1}\frac{(p^{t-1}-q^{t-1})}{(p^{t+s-1}-q^{t+s-1})}p^{s+t-2}\frac{(p^{t-2}-q^{t-2})}{(p^{t+s-2}-q^{t+s-2})}\mathrm{B}_{p,q}(s,t-2),$$

$$= p^{s+t-1}\frac{(p^{t-1}-q^{t-1})}{(p^{t+s-1}-q^{t+s-1})}p^{s+t-2}\frac{(p^{t-2}-q^{t-2})}{(p^{t+s-2}-q^{t+s-2})}\dots p^{s+1}\frac{(p-q)}{(p^{s+1}-q^{s+1})}\mathrm{B}_{p,q}(s,1),$$

$$= p^{s+t-1}\frac{(p^{t-1}-q^{t-1})}{(p^{t+s-1}-q^{t+s-1})}p^{s+t-2}\frac{(p^{t-2}-q^{t-2})}{(p^{t+s-2}-q^{t+s-2})}\dots p^{s+1}\frac{(p-q)}{(p^{s+1}-q^{s+1})}\frac{p^{s-1}}{[s]_{p,q}} \ ,$$

$$= \frac{p^{(s-1)+(s)+(s+1)+\cdots+(s+t-1)}}{p^{s}}\frac{(p\ominus q)_{p,q}^{t-1}}{(p^{s}\ominus q^{s})_{p,q}^{t}}(p-q),$$

It mean that:

$$\mathrm{B}_{p,q}(s,t) = p^{m}\frac{(p\ominus q)_{p,q}^{t-1}}{(p^{s}\ominus q^{s})_{p,q}^{t}}(p-q), \tag{3.11}$$

Letting $m = [t(2s+t-2)+t-2]/2$.

By using proposition (2.2) part (2) and (3.11) we have :

$$\mathrm{B}_{p,q}(s,t) = p^{m}\frac{(p\ominus q)_{p,q}^{t-1}}{(p^{s}\ominus q^{s})_{p,q}^{t}}(p-q),$$

$$= p^{m}\frac{(p\ominus q)_{p,q}^{t-1}}{(p-q)^{t-1}}\frac{(p\ominus q)_{p,q}^{s-1}}{(p-q)^{s-1}}\frac{(p-q)^{s-1}}{(p\ominus q)_{p,q}^{s-1}}\frac{(p-q)^{t-1}}{(p^{s}\ominus q^{s})_{p,q}^{t}}(p-q),$$

$$= p^{m}\frac{(p\ominus q)_{p,q}^{t-1}}{(p-q)^{t-1}}\frac{(p\ominus q)_{p,q}^{s-1}}{(p-q)^{s-1}}\frac{(p-q)^{s+t-1}}{(p\ominus q)_{p,q}^{s+t-1}} = p^{m}\frac{\Gamma_{p,q}(t)\Gamma_{p,q}(s)}{\Gamma_{p,q}(s+t)},$$

4. The analogous definition of (p,q)-Gamma Function and (p,q)-Beta Function

We defined the (p,q)-analogue for Gamma function and Beta Function and we found the relation between them in the previous section. In this section we want to study a new functions (namely $\tilde{\Gamma}_{p,q}(z)=\Gamma_{q/p}(z)$ and $\tilde{\mathrm{B}}_{p,q}(s,t)=\mathrm{B}_{q/p}(s,t)$) and we will show how these functions are relevant to (p,q)-Gamma function and (p,q)-Beta function.

Definition 1: for $(0<q<p<1)$ we get:

$$D_{p,q}g(z)=D_{q/p}g(z/p), \tag{4.1}$$

Proof : By definition of the (p,q)-derivative we know:

$$D_{p,q}g(z)=\frac{g(pz)-g(qz)}{(p-q)z}, \qquad z\neq 0 \ .$$

$$=\frac{g\left(\frac{q}{p}pz\right)-g(pz)}{\left(\frac{q}{p}-1\right)pz}=D_{q/p}g(pz),$$

Definition 2: for $(0<q<p<1)$ we get :

$$\int_0^b g(z)d_{p,q}z=\int_0^b g(z/p)d_{q/p}z. \text{ (where } 0<q/p<1.)$$

Proof:

$$\int_0^b g(z)d_{p,q}z = (p-q)b\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}g\left(b\frac{q^j}{p^{j+1}}\right),$$

$$= (1-q/p)b\sum_{j=0}^{\infty}\left(\frac{q}{p}\right)^j g\left(b\left(\frac{q}{p}\right)^j\frac{1}{p}\right),$$

$$= \int_0^b g(z/p)d_{q/p}z.$$

Definition 3: The (q/p) -Integration by parts is given by :

$$\int_0^b g(z)D_{q/p}f(z)d_{q/p}z = g(z)f(z)\Big|_0^b - \int_0^b f(\frac{q}{p}z)D_{q/p}g(z)d_{q/p}z, \tag{4.2}$$

Definition 4: For ($0 < q/p < 1$) we get :

$$\tilde{\Gamma}_{p,q}(z) = \Gamma_{\frac{q}{p}}(z) = \frac{\left(1-\frac{q}{p}\right)_{q/p}^{\infty}}{\left(1-\left(\frac{q}{p}\right)^z\right)_{q/p}^{\infty}}\left(1-\frac{q}{p}\right)^{1-z}, \tag{4.3}$$

Proposition 4.1: for positive integer z we have:

$$\tilde{\Gamma}_{p,q}(z) = p^{2z-1}\Gamma_{p,q}(z)\,, \quad 0 < p < q < 1. \tag{4.4}$$

Proof:

$$\tilde{\Gamma}_{p,q}(z) = \Gamma_{\frac{q}{p}}(z) = \frac{\left(1-\frac{q}{p}\right)_{\frac{q}{p}}^{\infty}}{\left(1-\left(\frac{q}{p}\right)^z\right)_{\frac{q}{p}}^{\infty}}\left(1-\frac{q}{p}\right)^{1-z} = \frac{\prod_{j=1}^{\infty}(1-(q/p)^j)}{\prod_{j=0}^{\infty}(1-(q/p)^{z+j})}\left(1-\frac{q}{p}\right)^{1-z},$$

$$= \frac{\prod_{j=1}^{\infty}\left(\frac{p^j-q^j}{p^j}\right)}{\prod_{j=0}^{\infty}\left(\frac{p^{j+z}-q^{j+z}}{p^{j+z}}\right)}\frac{(p-q)^{1-z}}{p^{1-z}} = p^{2z-1}\frac{\prod_{j=1}^{\infty}(p^j-q^j)}{\prod_{j=0}^{\infty}(p^{j+z}-q^{j+z})}(p-q)^{1-z},$$

$$= p^{2z-1}\frac{(p\ominus q)_{p,q}^{\infty}}{(p^z\ominus q^z)_{p,q}^{\infty}}(p-q)^{1-z} = p^{2z-1}\Gamma_{p,q}(z), \qquad 0<\frac{q}{p}<1.$$

Definition 5: For $s,t>0$, then:

$$\tilde{B}_{p,q}(s,t) = B_{\frac{q}{p}}(s,t) = \int_0^1 z^{s-1}\left(1-\frac{q}{p}\right)_{q/p}^{t-1} d_{q/p}z,$$

Also we can write the following expression as:

$$\tilde{B}_{p,q}(s,t) = B_{\frac{q}{p}}(s,t) = \frac{\Gamma_{\frac{q}{p}}(s)\Gamma_{\frac{q}{p}}(t)}{\Gamma_{\frac{q}{p}}(s+t)}, \tag{4.5}$$

Proposition 4.2: For $s,t>0$, we have:

$$\tilde{B}_{p,q}(s,t) = \frac{1}{p}B_{p,q}(s,t), \tag{4.6}$$

Proof: By using the equation (4.4) we get:

$$\tilde{B}_{p,q}(s,t) = B_{\frac{q}{p}}(s,t) = \frac{\Gamma_{\frac{q}{p}}(s)\Gamma_{\frac{q}{p}}(t)}{\Gamma_{\frac{q}{p}}(s+t)} = \frac{p^{2s-1}\Gamma_{p,q}(s)p^{2t-1}\Gamma_{p,q}(t)}{p^{2s+2t-1}\Gamma_{p,q}(s+t)},$$

$$= \frac{1}{p}\frac{\Gamma_{p,q}(s)\Gamma_{p,q}(t)}{\Gamma_{p,q}(s+t)} = \frac{1}{p}B_{p,q}(s,t),$$

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