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Original Research Article

A concept map for teaching-learning logic and methods of proof: Enhancing students' abilities in constructing mathematical proofs

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Abstract

The study aimed at describing university students' ability in constructing mathematical proofs using logical arguments and constructing an effective teaching-learning map. The study adopted a correlational research design. An open-ended mathematics assessment involving questions on mathematical logic and construction of mathematical proofs was administered to fifteen University students of mathematics (control group) at Mountains of the Moon university. The students' responses were analyzed and their ability to construct valid mathematical proofs was measured and correctness of their solutions was discussed. A concept map for teaching-learning mathematical logic and methods of proof was developed and shared with a different group of fifteen students of mathematics (assumed to have same level of mathematical knowledge). The group was assessed using the similar assessment tool. The mean level of knowledge of mathematical logic and method of proof was from to be 2.2 (between moderate and average) for the first assessment teat and 3.4 (between average and high level), while the mean score for test 1 and test 2 were 48 and 70 respectively. The Chi square test at 0.01 level indicated a statistically significant difference between the two sets of results with p<0.05. The students' knowledge level increased as they made use of the teaching-learning map. The teaching-learning concept map was evaluated to be an effective teaching-learning guide for logic and methods of proof.

Keywords: Mathematical proof, logic, arguments, abilities and concept map.

INTRODUCTION

Mathematics and science are based on valid relationships and laws, they are also characterized by deductive construction of knowledge. Mathematics is built on arguments and individual opinions; this has made the ability to reason and attitudes and opinions of individuals an integral part of everyday life. Correct and proper reasoning, argumentation and information processing is increasingly becoming a requirement for all mathematics scholars. Mathematical thinking, reasoning and use of arguments are necessary skills for the construction of mathematical proofs as one of the high-level skills showing the level of mathematical literacy. For deeper understanding of mathematical concepts and principles, mathematical proofs are necessary (Bulkova et al., 2018).

Proofs are certainly the foundation of mathematics; this makes it indispensable for every university mathematics student be able to learn and understand the concept and process of proof writing.

Mathematics students are expected to be able to produce logical arguments and present formal proofs that successfully explain their reasoning. This requires that teachers of mathematics should be prepared to understand and construct mathematical proofs, in order to pass on the learnt knowledge to the subsequent generation of people.

Research has shown that a large percentage of undergraduate mathematics students have difficulties in constructing, understanding, and validating proofs (Martin and Harel, 1989), (Moore, 1994), and (Epp, 2003). Since proofs form the underlying structure of mathematics, the lack of ability presents a major problem.

Claims have been made that teachers are not teaching students in the way that mathematicians actually construct a proof, that they give them no insight into the processes and errors that occur along the way (Alibert and Thomas, 1991; Almeida, 2003).

The strategies to strengthen this have been described by a number of researchers such as in (Polya, 1973; Schoenfeld, 1985; Yerushalmy,2000; Pape and Wang, 2003; Carlson and Bloom, 2005). However, little is known about concept map for teaching-learning of logic and mathematical proofs. This study was aimed at finding students ability to construct proofs as a basis for developing a teaching-learning concept map for logic and mathematical methods of proof.

According to (Bulkova et al., 2018), provision of skills is built upon argumentation and reasoning. They also added that more mathematically competent students are able to provide notably high-level argumentation. There is a relationship between the ability to formulate conclusion and applicability of the conclusions.

Concept Maps

Concept maps are graphical tools for organizing and representing knowledge. They include concepts (usually framed) and links among concepts that are represented with lines. Words on the lines are called linking phrases, and they represent relationships between two concepts.

A concept is defined as a noticeable regularity in events and objects or as a record of events or objects, named with labels (in word of symbolic form). For example, a proposition contains two or more concepts, linked with linking words of phrase to form a meaningful claim. Sometimes there are links between concepts in different segments or domain of knowledge on the map, they help in understanding how these domains are related to one another. Concept maps are presented in a hierarchical form, following the level of generality of concepts. (Novak and Canas, 2008)

The use of concept maps and their application in mathematics teaching, learning and knowledge assessment was discussed in (Marija et al., 2011).

Logical argument and Proofs

An argument is an attempt to demonstrate the truth of an assertion called conclusion, based on the truth of a set of assertions called premises.

The word logic is derived from the Greek word logos which may be interpreted to mean reason or discourse. Most of the study of logic revolves around the idea of a statement.

Importance of mathematical proofs

The various functions of proof in mathematics and mathematics education have been discussed by researchers during many years and they have gained a wide consensus in the mathematics education research community (Bell, 1976; De Villiers, 1990; Hanna, 2000). Especially the functions of conviction and explanation have been the focus in the field (e.g. de Villiers, 1991; Hanna, 2000; Hersh, 1993). However, Weber (2002) states that besides proofs that convince or/and explain, there are proofs that justify the use of definitions or an axiomatic structure and proofs that illustrate proving techniques useful in other proving situations. Lucast (2003) studied the relation between problem solving and proof and found support for the importance of proofs rather than theorems in mathematics and mathematics education for example from Rav's (1999) philosophical article. Lucast considers proof and methods for problem solving as in principal the same and states that proving is involved in the cognitive processes needed for problem solving.

We need proofs in mathematics, first, because we want to be sure that what we do is right. In mathematics, unlike science or any other field, we can prove that what we do is absolutely right. That is because mathematics is not dependent on partially known physical laws or unpredictable human behavior, but simply on reason.

Problem solving and techniques for constructing mathematical proofs

In this section the steps of mathematical problem solving according to Polya, 1973 are discussed and the four techniques or methods of proofs are presented with examples.

Problem solving is a process by which a choice of an appropriate trick enables one to proceed with what is in the problem to its solution. It is the process by which one combines previously learnt elements (experience) and concepts to provide a solution to a given problem. It is important to note that what is important in a problem solving is the process but not the answer.

According Polya (1973) there are for steps involved in problem solving, these are illustrated on Figure 1.

Step 1 – Understand the problem: Here one has to understand the problem what is the unknowns, what are the data, what is the conditions, is it possible to satisfy

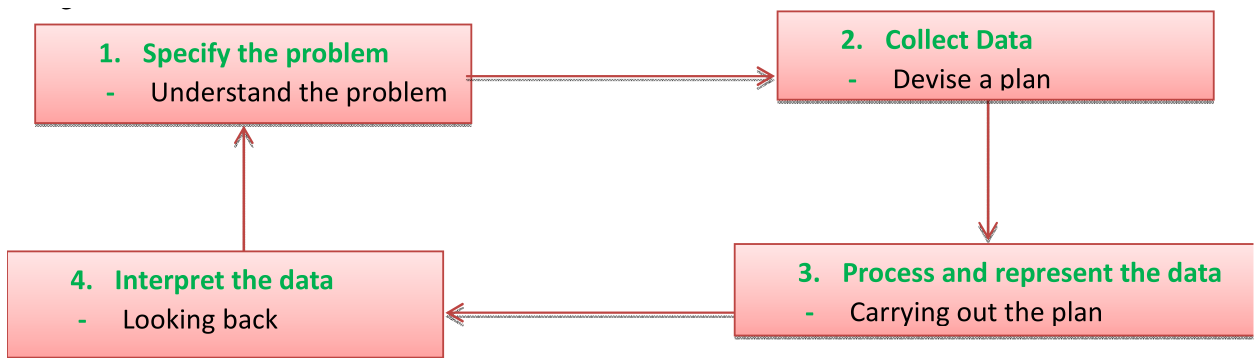

Figure 1. Steps for Problem solving

the conditions? Are the conditions sufficient to determine the unknown or are they insufficient? Draw a figure if required or introduce suitable notations. Separate the various parts of the conditions and write them down.

Step 2 – Devise a plan: Here find the connection between the data and the unknown. One maybe obliged to consider an auxiliary problem if an immediate connection can be found. One should eventually obtain a plan of the solution. Have you seen it before or you have seen the same problem but in a slightly different form? Do you know a related problem? Do you know any theorem that can be useful? Look at the unknown and try to think of a familiar problem having the same or similar unknown. If there is a problem related to yours and solved for, could you use it, could you use its result or methods? Should you introduce auxiliary elements in order to make its use possible, could you state the problem in a different way? Then go back to definitions- if you cannot solve the proposed problem try to solve first some related problems.

Step 3 - Carrying out the plan of the solutions: Carrying the plan of the solution and check its steps. Can you clearly see that each step is correct?

Step 4 - Looking back: Examine the solution obtained, can you check the result, can you check the arguments, can you derive the solution differently, can you use the result of the method for some other problems?

Methods of Proof

A proof is a process of establishing the truth of an assertion, it is a sequence of logical sound arguments which establishes the truth of a statement in question. It is an argument that convinces other people that something is true. The four methods of proof are discussed below.

Direct Method

Suppose $P \Rightarrow Q$, in this method we assume that P is true and proceed through a sequence of logical steps to arrive at the conclusion that Q is also true.

Example 1.1

Show that if m is an even integer and n is an odd integer then m+n is an odd integer.

Solution

Assume that m is an even integer and n is an odd integer.

Then m = 2k and n = $2\ell+1$ for some integer k and l.

Therefore $m+n = 2k+2\ell+1 = 2(k+\ell)+1 = 2d+1$ for some integer d = $k+\ell$. Since d is an integer then m+n is an odd integer.

Contrapositive Method

Associated with the implication $P \Rightarrow Q$ is the logical equivalent statement $\neg Q \Rightarrow \neg P$, the contrapositive of the conditional statement $P \Rightarrow Q$. So, one way to prove the conditional statement $P \Rightarrow Q$ is to give a direct proof of the contrapositive statement $\neg Q \Rightarrow \neg P$. The first step is to write down the negation of the conclusion then follow it by a series of logical steps that this leads to the negation of the hypothesis of the original conditional statement.

Example 1.2

Show that if $n^2$is even integer then n is an even integer.

Solution

We will show that if n is odd then n^2 is odd. Assume that n is odd. Then $n = 2k+1$for some integer k.

Now $n^2 = (2k+1)^2 = 4k^2+4k+1 = 2(2k^2+2k)+1 = 2d+1$ for some integer $d = 2k^2+2k$. Since d is an integer then n^2 is an odd integer. Thus, by contrapositive if n^2 is even then n is itself even.

Contradiction Method

Assume that P is true and Q is false i.e. $P \wedge \neg Q$ is true.

Table 1. Codes description of the nature of students work

Code	Description of nature of students' work
1	**Low knowledge** Generate one or more examples to verify the implication but also: provides a rationale for the choices ofexamples by considering examples belonging todifferent cases of the antecedent shows evident use of at least one extreme instance uses mathematical properties inferred fromgenerated examples to make conclusions
2	**Moderate knowledge** Deduce relevant mathematical properties for proving the implication but missing one or two key inferences to deduce the implication. Deduce the implication to be true for some cases of the antecedent but leave some others out
3	**Average knowledge** Generate logical deductions to justify conclusions but one or two inferences may be interpreted as inductive due to insufficient substantiation Generate logical deductions to justify conclusions but contain minor reasoning errors that may be interpreted as writing error from the context Inferences made are not organized into a chain of logical inferences
4	**High level Knowledge** Generate logically coherent and mathematically valid proofs with inferences derived through use of mathematical symbols and notations

Then show a series of logical steps that leads to a contradiction or impossibility or absurdity. This will mean that the statement $P\wedge\neg Q$ must have been fallacious and therefore, its negation $\neg[P\wedge\neg Q]$ must be true. Since $\neg[P\Rightarrow Q]\equiv[P\wedge\neg Q]$, it follows that $[P\Rightarrow Q]\equiv\neg[P\wedge\neg Q]$ and hence $P\Rightarrow Q$ must be true.

Example 1.3

Show that $\sqrt{2}$ is irrational. That is there is no integer p and q such that $\sqrt{2}=\frac{p}{q}$ or there is no rational number p such that $p^2=2$.

Solution

By contradiction, we assume that $\sqrt{2}$ is rational.

So, we have $\sqrt{2}=\frac{p}{q}$. And so $2=\frac{p^2}{q^2}$ thus $2q2 = p^2$ …………………..(i)

From (i), observe that p^2 is even, so p is even and can be written as p = 2c for some integer c.

So (i) becomes: $2q^2 = (2c)^2 = 4c^2$ and so $q^2 = 2c^2$………………….. (ii)

Similarly, from (ii) it is observed that q^2 is even and so q is itself even. Therefore, both p and q are even integers; implying that they both have a common factor of 2. This contradict the fact that the GCD [p, q] =1 and thus $\sqrt{2}$ is not rational but rather it is irrational.

Example 1.4

If r is rational and x is irrational then prove that r+x and rx are irrational.

Solution

Let r+x be rational (by contradiction), then r+x $=\frac{p}{q}$ with $q\neq 0$, GCD [p, q] = 1.

Since r is rational, then r $=\frac{n}{m}$ and then x $=\frac{p}{q}-\frac{n}{m}=\frac{pm-qn}{qm}$ which is rational. This indicates that x is both rational and irrational which is absurd and so r+x is irrational.

Similarly, suppose rx is rational, then rx $=\frac{p}{q}$ for some integers p, q; $q\neq 0$ and GCD [p, q] =1. Since r is rational then r $=\frac{n}{m}$ and then rx $=\frac{pn}{qm}$ which is rational. This indicates that x is both rational and irrational; which is impossible and thus the assumption that rx is ration is false so rx is irrational.

Mathematical induction Method

It is a method that proves statements of the type $\forall n \in N$, $A(n)$ holds. Here A(n) is statement depending on n. e.g. $1+2+3+4+\cdots+n=\frac{1}{2}n(n+1)$.

Definition

Let $n_o \in N$ and let A(n) be statement depending on $n\in N$ such that:

(1) A(n) is true.

(2) For every $k\geq n_o$ if A(k) is true then A(k+1) is also true. That is $\forall k \geq n_o$, $A(k)\Rightarrow A(k+1)$.

Then A(n) is true for $n\geq N$.

Steps for mathematical induction

1- Initial stage: verify that $A(n_o)$ is true
2- Induction hypothesis: A(n) is true for some n say n = k
3- Induction step: deduce (from step1 and 2) that A(n) is true for n = k+1
4- Conclusion: by the PMI, A(n) is true for all integers $n\geq N$

Table 2. Statistics for group 1 and group 2

Statistics	Group 1	Group 2
Mean	47.9	70.0
Standard Error	3.5	4.3
Median	44.0	70.0
Mode	37.0	75.0
Standard Deviation	13.6	16.5
Skewness	0.9	-0.1
Confidence Level (99.0%)	10.5	12.7

Table 3. Students' Knowledge level for test 1 and test 2 in the various concepts

	Group 1	Group 2
Concepts tested	**Mean**	**Mean**
Open Sets	2.3	3.0
Cauchy Sequence	2.5	3.2
uniqueness of limits	2.5	3.6
Algebra of limits	2.2	3.7
Logical statement	1.9	3.5
Proof by induction method	2.1	3.1
Schwarz's inequality	1.5	3.5
Continuity of functions	1.7	3.4
Convergence of sequences	2.5	3.3
Differentiation of functions	2.7	3.8
Average percentage score	2.2	3.4

Example 1.5

Prove that "7 is a divisor of $3^{2n} - 2^{n}$"

Solution

Step 1: For n =1 we have $3^2 - 2^1 = 9 - 2 = 7$ True

Step 2: $3^{2n} - 2^{n}$is divisible by 7 for some n = k

Step 3: set n = k+1, step 2 implies that ∃ j: $3^{2n} - 2^{n}$ =7j

$3^{2(k+1)} - 2^{(k+1)}$=$9 * 3^{2k} - 2 * 2^{k}$

=$9 * (7j + 2^{k}) - 2 * 2^{k}$

= 9 (7j) +(9-2) *2^{k}

= 7(9 j+2^{k}) which is divisible by 7

Step 4: It follows from the PMI that 7 is a divisor of $3^{2n} - 2^{n}$ for all n ≥ 1

RESEARCH METHODS AND MATERIALS

A correlational research design was used and a quantitative approach was adopted during the study. Two groups, each consisting of fifteen students were randomly selected from the mathematics class at Mountains of the Moon University to participate in the study. An assessment (Questionnaire) consisting of basic questions on mathematical proof were administered to each group.

After the first assessment, students' ability to construct valid mathematical proofs was analyzed and correctness of students' solutions was discussed at four levels as indicated in Table 1. A concept map for teaching-learning mathematical logic and methods of proof was developed and shared with the second group of fifteen students of mathematics. The second group was then assessed using the same assessment tool in order to ascertain if there is any difference between the scores of first (control) and the second (real) group.

The student's scripts were analyzed and the nature of their work was coded following Blooms taxonomy of knowledge as low knowledge (1), Moderate knowledge (2), Average knowledge (3) and high-level knowledge (4) as described in Table 1.

DISCUSSION OF RESULTS

An assessment test consisting of ten short proof questions was administered to fifteen mathematics students of Mountains of the Moon University. After analyzing the students' responses, a concept map for teaching and learning was developed and discussed with a different group of fifteen students of mathematics. The descriptive statistics for the two assessment tests are indicated in Table 2, the students' knowledge levels are indicated in Table 3 while the significance test is illustrated in Table 4.

According to Table 2, it was observed that the students mean score for the first group is 47.8 with a standard error of 3.5, median of 44, mode of 37 and standard deviation of 13.6. The distribution of the scores

Table 4. Test results to measure difference between students' scores between the two tests

Df	14
t Stat	-5.52
P(T<=t) one-tail	0.00004
t Critical one-tail	1.76131
P(T<=t) two-tail	0.00007
t Critical two-tail	2.14479
α	0.01

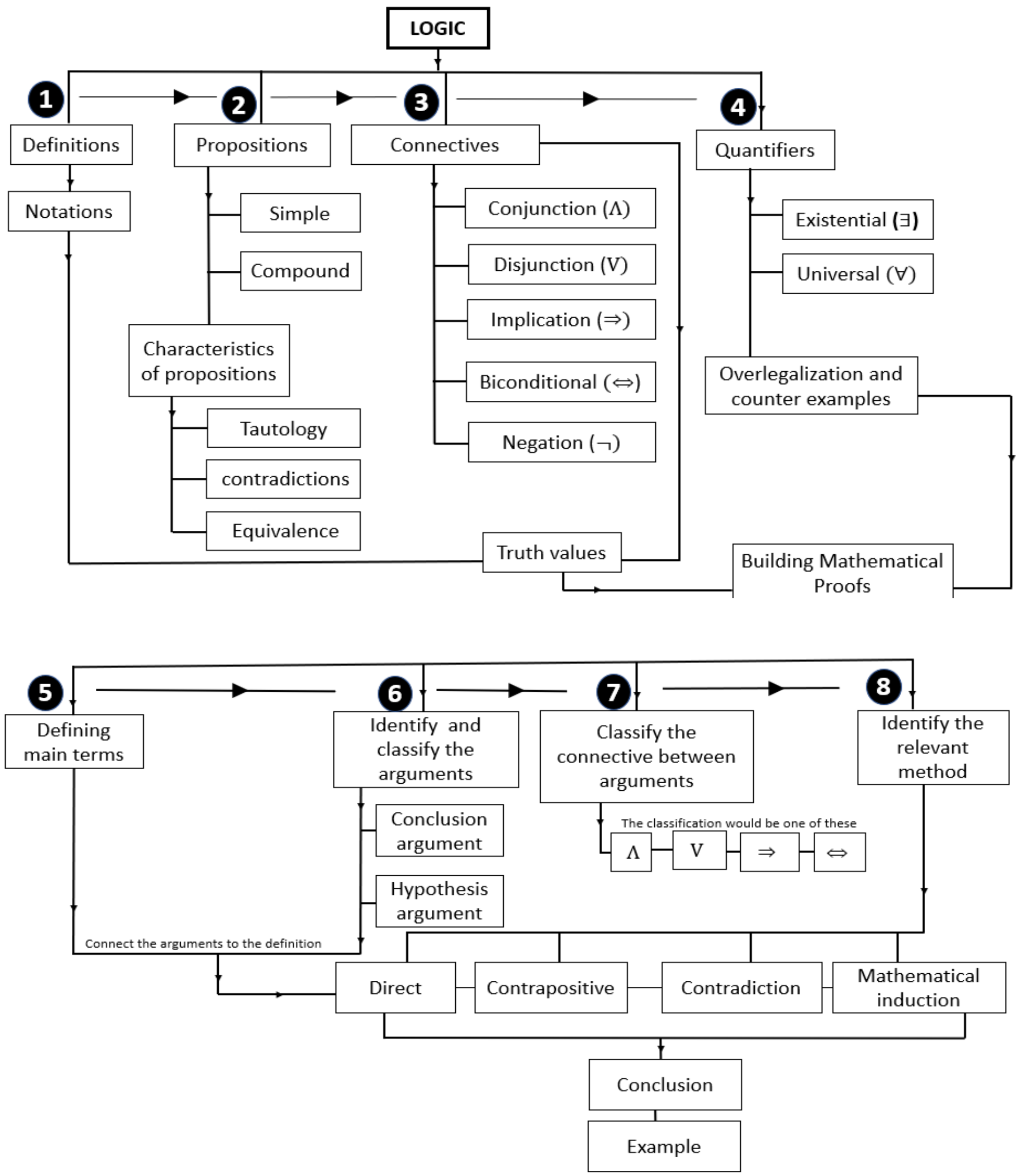

Figure 2. Concept Map for Teaching and Learning of Logic and Mathematical Proofs

from group one is positively skewed with a Pearson coefficient of skewness of 0.9. This indicates that majority of the students in group had scores laying blow the average. This is line with (Martin and Harel, 1989), (Moore, 1994), and (Epp, 2003) who noted that a large percentage of undergraduate mathematics students have difficulties in constructing, understanding, and validating proofs.

The results from test 2, administered to the second group indicated the mean score as 70, the median as 70 and the mode as 75. The standard error was 4.3 while the standard deviation was 16.5. The distribution of the scores from this group was negatively skewed with the coefficient of skewness as -0.1. This indicate the majority of the students in the second group had scores above the mean.

Generally, there was a difference between the statistics of the scores from the first group and the second group. This difference is assumed to have been as a result of the use of the teaching-learning concept map that the second group interacted with before they were tested.

The students' level of knowledge was measure and described using four levels as 1. Low knowledge level, 2. moderate knowledge level, 3. average knowledge level and 4. high knowledge level. Ten concepts were tested, these were; Open sets, Cauchy sequences, uniqueness of limits, algebra of limits, logical statements, proof by contradiction, Schwarz's inequality, continuity of functions, convergence of sequences and differentiation of functions. The mean level of knowledge on each concept was obtained and the results are presented in Table 2.

The mean level of knowledge for the first group range from 1.5 to 2.7. While the mean ranged from 3.0 to 3.8 for the second group. This indicates that the level of knowledge of the second group was at a high level than that of the first group. This difference is attributed to the fact that, the second group had interacted with the teaching-learning concept map for logic and methods of proof. The average of the means for group one was 2.2 indicating that the level of knowledge was between moderate level and average level. While the average of the means for the second group was 3.4 indicating that the level of knowledge for students in the second group was between average and high. It was generally found out that no student in any two groups that showed a low level of knowledge this is because the students in both groups were mathematics students with some knowledge of logic and construction of mathematics proofs.

The results of the Chi square test with Df of 14 and α of 0.01 are indicated in table 4. This table suggests that there is a statistically significant difference between the scores of group 1 and group 2. This means that group two students substantially showed a high level of knowledge on logic and methods of proof compared to their counterparts in group1.

Concept Map for teaching-learning of logic and methods proofs

Figure 2 shows the proposed concept map that can be used during the teaching of logic and method of mathematical proofs. The map is aimed at enhancing the students' ability to construct mathematical proofs at universities.

The flow of the concepts on the map is indicated by the direction arrows; from left to right and top to bottom. The map suggests that the teaching and learning of logic and methods of proof should state with defining and notation of important concepts. This is followed by teaching and learning subsequent concepts as indicated by the direction arrows.

CONCLUSION

The process of teaching and learning mathematical concepts requires a well-organized and systematics approach. Teaching -learning concepts map are important tools for teaching of the concepts in mathematics and should be shared with students during teaching. Students level of knowledge in mathematics is influences by the available resources that students interact with. Mathematics educators are required to ensure that students are provided with relevant tools of leaning different concepts. Similarly, mathematics learners are required to make use of learning resource in order to conceptualize and internalize the learnt concepts.

RECOMMENDATIONS FOR FURTHER RESEARCH

The teaching of logic should follow the teaching map discussed in this paper in order to improve the students' ability to construct proofs. Further research can be conducted on other variables that may affect students' ability to construct mathematical proofs other content arrangement.

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