Supplemental Material: Theory of Subradiant States of a One-Dimensional Two-Level Atom Chain

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A. Energy Levels of the Subradiant States

We derived the decay rates of the subradiant states in the one-excitation sector. As a byproduct, their energy levels are given as following: for the subradiant states with $k_\xi \approx \xi\pi/(Nd)$ ($\xi \ll N$), we have

$$\omega_\xi = \frac{\Gamma_{1D}}{2}\cot(\frac{k_{1D}}{2}d) + \Gamma_{1D}\frac{\cos(k_{1D}d/2)}{\sin^3(k_{1D}d/2)}(\frac{\xi\pi}{2N})^2; \quad (1)$$

for those with $k_\xi \approx -\pi/d + \xi\pi/(Nd)$ ($\xi \ll N$), we have

$$\omega_\xi = -\frac{\Gamma_{1D}}{2}\tan(\frac{k_{1D}}{2}d) - \Gamma_{1D}\frac{\sin(k_{1D}d/2)}{\cos^3(k_{1D}d/2)}(\frac{\xi\pi}{2N})^2. \quad (2)$$

Apart from the constant part, the above expressions show that the subradiant states have Lamb shifts proportional to ξ^2/N^2. The band is parabolic and flat around the extreme point $k = 0$ or $k = \pm\pi/d$.

B. Hamiltonian of 1D Atom Chain in 3D Free-Space

The effective atom-atom coupling Hamiltonian is expressed as

$$H_{3D,\text{eff}} = -\mu_0\omega_0^2 \sum_{i,j=1}^{N} \mathbf{d}_i^* \cdot \mathbf{G}(\mathbf{r}_i, \mathbf{r}_j, \omega_0) \cdot \mathbf{d}_j \; \sigma_i^\dagger \sigma_j, \quad (3)$$

where $\mathbf{d}_i$ and $\mathbf{r}_i$ is the dipole moment and the position of the ith atom, respectively, and μ_0 is the vacuum permeability. For the case of 3D free-space, the dyadic Green's tensor $\mathbf{G}(\mathbf{r}_i, \mathbf{r}_j, \omega) = \mathbf{G}(\mathbf{r}_i - \mathbf{r}_j, \omega)$ is given as

$$\mathbf{G}(\mathbf{r}, \omega_0) = \frac{e^{ik_0 r}}{4\pi k_0^2 r^3}\Bigg[(k_0^2 r^2 + ik_0 r - 1)\mathbf{I} + (-k_0^2 r^2 - 3ik_0 r + 3)\frac{\mathbf{rr}}{r^3}\Bigg] \quad (4)$$

where $k_0 = \omega_0/c$. This expression can be transformed to the wave number presentation and yields $H_{3D,\text{eff}}$ presented as Eq. (6) of the main text.

C. Transformation to Continuous Limit

Equation (9) of the main text is written in a discrete notation. The continuous expression can be obtained from the discrete notation by the mapping

$$\sum_{i=1}^{N} \to \frac{1}{d}\int_0^{Nd} dx, \qquad b_i \to \sqrt{d}\, b_x. \quad (5)$$

The bosonic commutation relation changes from $[b_i, b_j^\dagger] = \delta_{i,j}$ to $[b_x, b_y^\dagger] = \delta(x - y)$.

D. Direct construction

To see why the fermionic ansatz has the N^{-3} decay rate, we introduce the two-excitation state $|k_1, k_2\rangle = \sum_{m<n} e^{ik_1 z_m + ik_2 z_n}|e_m, e_n\rangle$, and evaluate

$$H_{\text{eff}}^I |k_1, k_2\rangle = \frac{N\Gamma_{1D}}{4}\sum_{\epsilon=\pm}\Bigg[g_{k_1,\epsilon}|b_{k_2,\epsilon k_{1D}}\rangle - h_{k_2,\epsilon}|b_{k_1,\epsilon k_{1D}}\rangle + c_{k_1,k_2,\epsilon}|b_{k_1+k_2-\epsilon k_{1D},\epsilon k_{1D}}\rangle\Bigg], \quad (6)$$

where

$$g_{k,\epsilon} = \frac{1}{N}\frac{1}{e^{-i(k-\epsilon k_{1D})d} - 1}, \qquad h_{k,\epsilon} = \frac{1}{N}\frac{e^{i(k-\epsilon k_{1D})Nd}}{e^{-i(k-\epsilon k_{1D})d} - 1}, \quad (7)$$

and $c_{k_1,k_2,\epsilon} = g_{k_1,\epsilon}^* + g_{k_2,\epsilon}$ and $|b_{k,k'}\rangle = |k, k'\rangle + |k', k\rangle$. All these state amplitudes scale as N^{-1} rather than the desired N^{-2}-scaling, which is required to obtain the $1/N^3$-scaling of the decay rates.

As in the one-excitation sector, we may proceed and construct superpositions of four degenerate states $|k_1, k_2\rangle$, $|-k_1, k_2\rangle$, $|k_1, -k_2\rangle$ and $|-k_1, -k_2\rangle$, to $|\phi_{k_1}, \phi_{k_2}\rangle = \sum_{m<n}\phi_{k_1}(z_m)\phi_{k_2}(z_n)|e_m, e_n\rangle$, where $\phi_k(z_m) = \langle e_m|\phi_k\rangle$. Then in the expression of $H_{\text{eff}}^I|\phi_{k_1}, \phi_{k_2}\rangle$, the state amplitudes on $|b_{k_{1(2)},\epsilon k_{1D}}\rangle$, but not on $|b_{k_1+k_2-\epsilon k_{1D},\epsilon k_{1D}}\rangle$, are successfully reduced to the N^{-2}-scaling.

To also suppress the latter, we form the superposition with the permuted state $|\phi_{k_2}, \phi_{k_1}\rangle$. The suitable superposition turns out to be "fermionic", i.e., $|F_{k_1,k_2}\rangle \propto |\phi_{k_1}, \phi_{k_2}\rangle - |\phi_{k_2}, \phi_{k_1}\rangle$. Different from what we have seen in the one-excitation sector, here the "tails" cannot be erased completely by the superposition. It means that the Fermionic ansatz is only the leading order solution.

With reference to our concluding remarks in the main text on the full treatment of H_{eff}^R and H_{eff}^I, the same construction can be applied for the full Hamiltonian H_{eff}, where we obtain

$$\begin{aligned}H_{\text{eff}}|k_1,k_2\rangle =&(\omega_{k_1}+\omega_{k_2})|k_1,k_2\rangle\\ &-i\frac{N\Gamma_D}{2}\Big(g_{k_1,+}|b_{k_D,k_2}\rangle - h_{k_2,-}|b_{-k_D,k_1}\rangle\\ &+c_{k_1,k_2,-}|-k_D,k_1+k_2+k_D\rangle\\ &+c_{k_1,k_2,+}|k_1+k_2-k_D,k_D\rangle\Big).\end{aligned} \tag{8}$$

While Eq. (8) features "tails" that are neither symmetric nor anti-symmetric, its main features, and hence the applicability of the fermionic ansatz, are captured by H^I_{eff} given in Eq. (6). As we show in the main text, the "tails" $|b_{k_1+k_2-\epsilon k_{1D},\epsilon k_{1D}}\rangle$ of Eq. (6) arise via the second order terms in the Holstein-Primakoff (HP) transformation, which in turn establish the Tonks-Girardeau limit of the Lieb-Liniger model, and hence the fermionic solutions.