Introduction to Latent Class Analysis

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Outline

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Heterogeneity

- Suppose we observe a *set of response variables* for n statistical units, with $\boldsymbol{Y}_i$ denoting the corresponding random vector for the i-th unit

- A statistical model allows us to account for the *heterogeneity* among the statistical units, which may be of two types:
  - *observed*: it may be explained on the basis of the observed covariates collected in vectors $\boldsymbol{X}_i$
  - *unobserved*: it cannot be explained on the basis of the observed covariates (it depends on factors that are not observed/observable)

- *Standard statistical/econometric models* (in particular when one response variable is observed) only account for the observed heterogeneity

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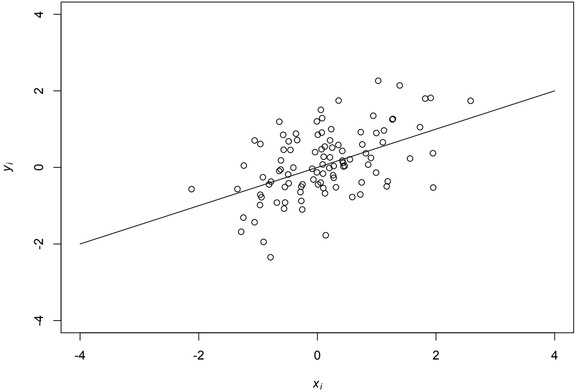
- This is the case of the *linear regression model* (for a single response variable):

$$Y_i = \boldsymbol{x}_i'\boldsymbol{\beta} + \varepsilon_i$$

  - $\boldsymbol{x}_i$: observed vector of covariates
  - $\boldsymbol{\beta}$: vector of regression coefficients
  - ε_i: error term



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- When $\boldsymbol{Y}_i$ is *a vector of more response variables* for each statistical unit, it is possible to account for the unobserved heterogeneity; typical situations:
  - *different response variables* are considered at the same time (e.g., different performance indicators)
  - *repeated observations of the same response variable* at different time occasions (longitudinal/panel data)
  - *(1st level) units are grouped in clusters*, which are 2nd level units (multilevel data)
- Almost all statistical/econometric models that account for unobserved heterogeneity may be cast in the class of *Latent Variable Models (LVMs)*, among which the *Latent Class (LC) model* is very important
- LVMs may also be used to *account for measurement errors* or *summarizing different measurements*
- *Main references*: Skrondal & Rabe-Hesketh (2004), Bartholomew *et al.* (2011)

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Latent Variable Models

- *Latent variables* ($\boldsymbol{U}_i$) are unobservable variables supposed to exist and to affect $\boldsymbol{Y}_i$; these may be correlated with $\boldsymbol{X}_i$

- An LVM formulates *assumptions* on:
  - the conditional distribution of $\boldsymbol{Y}_i$ given $\boldsymbol{U}_i$ and $\boldsymbol{X}_i$, $f(\boldsymbol{y}_i|\boldsymbol{u}_i, \boldsymbol{x}_i)$ (*measurement model*)
  - the conditional distribution of $\boldsymbol{U}_i$ given $\boldsymbol{X}_i$, $f(\boldsymbol{u}_i|\boldsymbol{x}_i)$ (*structural model*)

- A common assumption of LVMs is that of *local independence (LI)*, according to which the response variables are conditionally independent given the latent variables and the covariates:

$$f(\boldsymbol{y}_i|\boldsymbol{u}_i, \boldsymbol{x}_i) = \prod_{j=1}^{J} f(y_{ij}|\boldsymbol{u}_i, \boldsymbol{x}_i)$$

  - J: number of response variables
  - y_{ij}: single element of $\boldsymbol{y}_i$

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- By marginalizing out the latent variables we obtain the *manifest distribution*:

$$f(\boldsymbol{y}_i|\boldsymbol{x}_i) = \int f(\boldsymbol{y}_i|\boldsymbol{u}, \boldsymbol{x}_i) f(\boldsymbol{u}|\boldsymbol{x}_i) d\boldsymbol{u}$$

- By the Bayes theorem we obtain the *posterior distribution*

$$f(\boldsymbol{u}_i|\boldsymbol{x}_i, \boldsymbol{y}_i) = \frac{f(\boldsymbol{y}_i|\boldsymbol{u}_i, \boldsymbol{x}_i) f(\boldsymbol{u}_i|\boldsymbol{x}_i)}{f(\boldsymbol{y}_i|\boldsymbol{x}_i)}$$

  that is used for predicting the latent variables on the basis of the manifest variables

- *Estimation* is typically based on the maximum likelihood approach relying on numerical algorithms, among which the Newton-Raphson and particularly the Expectation-Maximization (EM) are very popular

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- LVMs may be *classified* according to:
  - *type of response variables* (discrete, continuous, categorical, etc.)
  - *type of latent variables* (discrete or continuous)
  - *presence or absence of covariates* (that may be included in different ways)
- LVMs based on discrete latent variables are of particular interest as they permit:
  - to naturally *group units in homogeneous latent clusters*, also named latent groups or latent classes (model-based clustering)
  - to account for the *unobserved heterogeneity* in a nonparametric way (it is necessary to specify a parametric distribution for the latent variables)
- The *LC model* is one of the most important LVMs based on discrete latent variables and may be seen as a *Finite Mixture (FM) model* for categorical response variables

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Expectation-Maximization algorithm

- This is a general approach for maximum likelihood estimation in the presence of *missing data* (Dempster *et al.*, 1977)
- In our context, missing data correspond to the latent variables; then:
  - *incomplete (observable) data*: covariates and response variables $(\boldsymbol{X}, \boldsymbol{Y})$
  - *complete (unobservable) data*: incomplete data + latent variables $(\boldsymbol{U}, \boldsymbol{X}, \boldsymbol{Y})$
- The corresponding *log-likelihood functions* are:

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{n} \log f(\boldsymbol{y}_i|\boldsymbol{x}_i)$$

$$\ell^*(\boldsymbol{\theta}) = \sum_{i=1}^{n} \log[f(\boldsymbol{y}_i|\boldsymbol{U}_i, \boldsymbol{x}_i) f(\boldsymbol{U}_i|\boldsymbol{x}_i)]$$

  - $\boldsymbol{\theta}$: overall vector of model parameters

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- The EM algorithm maximizes $\ell(\boldsymbol{\theta})$ by alternating *two steps* until convergence (h=iteration number):
  - *E-step*: compute the expect value of $\ell^*(\boldsymbol{\theta})$ given the current parameter value $\boldsymbol{\theta}^{(h-1)}$ and the observed data, obtaining

$$Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(h-1)}) = E[\ell^*(\boldsymbol{\theta})|\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{\theta}^{(h-1)}]$$

  - *M-step*: maximize $Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(h-1)})$ with respect to $\boldsymbol{\theta}$ obtaining $\boldsymbol{\theta}^{(h)}$

- *Convergence is* checked on the basis of the difference

$$\ell(\boldsymbol{\theta}^{(h)}) - \ell(\boldsymbol{\theta}^{(h-1)}) \quad \text{or} \quad \|\boldsymbol{\theta}^{(h)} - \boldsymbol{\theta}^{(h-1)}\|$$

- The algorithm is usually *easier to implement and much more stable* with respect to the Newton-Raphson algorithm, but it may be much slower

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Finite Mixture Model

- *Main references*: Lindsay (1995), McLachlan & Peel (2000), Bouveyron *et al.* (2019)
- The underlying idea is that statistical units come from different groups, where the grouping is unobserved (*latent groups or clusters*)
- *Model assumptions*:
    - there exist unit-specific *discrete latent variables* U_i, $i = 1, \dots, n$, with the same finite distribution with k levels defining the groups
    - the groups have *prior probabilities (weights)* $\pi_u = p(U_i = u)$, $u = 1, \dots, k$
    - for each group we have a specific *conditional response distribution* $f(\boldsymbol{y}_i|u) = f(\boldsymbol{y}_i|U_i = u)$, $u = 1, \dots, k$
- An FM model may include or not *individual covariates*; these may directly affect the measurement model or the structural model

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- An advantage of FM models with respect to LVMs based on continuous latent variables is that manifest and posterior distributions may be *explicitly computed*

- The *manifest distribution* is a weighted average of density or probability mass functions:

$$f(\boldsymbol{y}_i) = \sum_{u=1}^{k} f(\boldsymbol{y}_i|u)\pi_u$$

- By the Bayes theorem we obtain the *posterior distribution*

$$p(U_i = u|\boldsymbol{y}_i) = \frac{f(\boldsymbol{y}_i|u)\pi_u}{f(\boldsymbol{y}_i)}, \quad u = 1, \dots, k,$$

that is used to assign each unit to a specific latent group on the basis of the manifest variables

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- *Example*: for a single response variable, $k = 2$ components exist with Normal distribution with specific means and variances and different weights:
  - $U_i = 1 \; \rightarrow \; Y_i \sim N(0, 1)$
  - $U_i = 2 \; \rightarrow \; Y_i \sim N(3, 4)$
  - $\pi_1 = 0.25$, $\pi_2 = 0.75$

- Through the general rule for LVMs we obtain the *manifest distribution* of Y_i:

$$f(y_i) = 0.25\,\phi(y_i; 0, 1) + 0.75\,\phi(y_i; 3, 4)$$

  - $\phi(y_i; \mu, \sigma^2)$: density function of the Normal distribution with mean μ and variance σ^2

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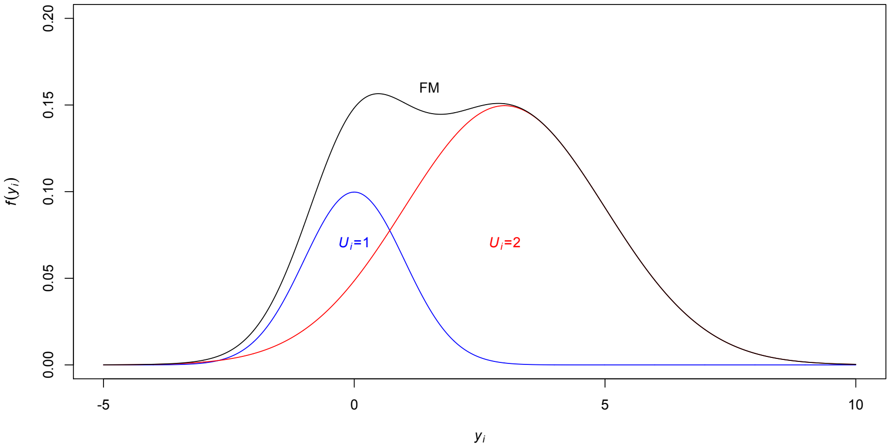

- Apart from model-based clustering, an FM model is a valid approach for *density estimation* given its flexibility (able to easily reproduce skewed and multimodal distributions)

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- The *posterior distribution* of U_i may be found by the Bayes theorem:

$$p(U_i = 1|y_i) = \frac{0.25\,\phi(y_i;0,1)}{0.25\,\phi(y_i;0,1) + 0.75\,\phi(y_i;3,4)}$$

$$p(U_2 = 1|y_i) = \frac{0.75\,\phi(y_i;3,4)}{0.25\,\phi(y_i;0,1) + 0.75\,\phi(y_i;3,4)}$$

- Units are assigned to the latent groups on the basis of the *Maximum A-Posterior (MAP) rule*; example:

	U_i		Assigned
y_i	1	2	group
-3	0.400	**0.600**	2
0	**0.673**	0.327	1
2	0.093	**0.907**	2
4	0.000	**1.000**	2

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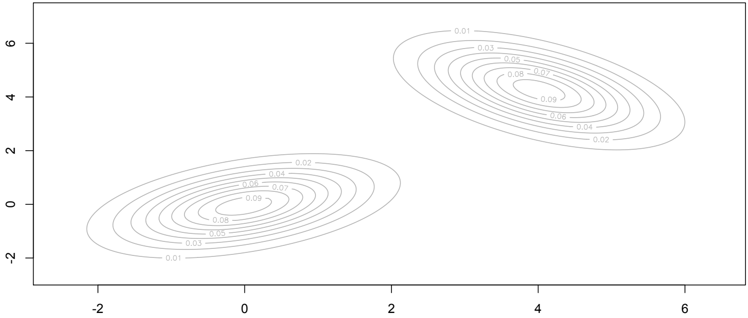
- The term *finite mixture model* may be used for the general discrete LV approach, although it is typically used for continuous data

- For *continuous data*, $f(\boldsymbol{y}_i|u)$ are typically multivariate Normal distributions (*FM of Normal distributions*) with specific mean vectors $\boldsymbol{\mu}_u$ and variance-covariance matrices $\boldsymbol{\Sigma}_u$



- For *categorical data* we obtain the LC model that has peculiarities in terms of assumptions and its estimation

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Estimation of FM models

- *Estimation* of an FM model is based on the maximum likelihood approach that is easily carried out through the Expectation-Maximization (EM) algorithm

- To introduce the EM algorithm it is convenient to substitute each latent variable U_i with the *(binary) indicator variables* $Z_{i1}, \dots, Z_{ik}$, where $U_i = u$ iff $Z_{iu} = 1$ with all other variables equal to 0

- *Example* with $k = 4$:

U_i	Z_{i1}	Z_{i2}	Z_{i3}	Z_{i4}
1	1	0	0	0
2	0	1	0	0
3	0	0	1	0
4	0	0	0	1

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- In this way the *complete data log-likelihood* is expressed as

$$\ell^*(\boldsymbol{\theta}) = \sum_{i=1}^{n}\sum_{u=1}^{k}[Z_{iu}\log f(\boldsymbol{y}_i|u) + Z_{iu}\log \pi_u]$$

- The EM algorithm consists in alternating two steps:
  - *E-step*: compute the posterior expect value of each indicator variable Z_{iu} by the Bayes theorem:

$$\hat{z}_{iu} = E(Z_{iu}|\boldsymbol{y}_i) = p(Z_{iu} = 1|\boldsymbol{y}_i) = p(U_i = u|\boldsymbol{y}_i)$$

  - *M-step*: maximize function $\ell^*(\boldsymbol{\theta})$ with each indicator variables Z_{iu} substituted by $\hat{z}_{iu}$ obtained at the E-step

- At the M-step, an *explicit solution* exists for the class weights:

$$\pi_u = \frac{1}{n}\sum_{i=1}^{n}\hat{z}_{iu}, \quad u = 1,\dots,k$$

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- For the *FM of Normal distributions*, an explicit solution exists for the other parameters ($u = 1, \dots, k$):

$$\boldsymbol{\mu}_u = \frac{1}{\sum_{i=1}^n \hat{z}_{iu}} \sum_{i=1}^n \hat{z}_{iu} \boldsymbol{y}_i$$

$$\boldsymbol{\Sigma}_u = \frac{1}{\sum_{i=1}^n \hat{z}_{iu}} \sum_{i=1}^n \hat{z}_{iu}(\boldsymbol{y}_i - \boldsymbol{\mu}_u)(\boldsymbol{y}_i - \boldsymbol{\mu}_u)' \quad \text{(homoskedastic)}$$

$$\boldsymbol{\Sigma} = \frac{1}{n} \sum_{i=1}^n \sum_{u=1}^k \hat{z}_{iu}(\boldsymbol{y}_i - \boldsymbol{\mu}_u)(\boldsymbol{y}_i - \boldsymbol{\mu}_u)' \quad \text{(heteroskedastic)}$$

- In general, different starting values must be used in order to face the problem of the *multimodality of the log-likelihood function*

- The EM algorithm is *implemented* in several softwares such as `R` (package `mclust`) and `Stata`

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- In order to *select the number of components* (k) two criteria can be adopted when there is no precise idea based on substantial reasons:

$$\begin{aligned}\text{Akaike Information Criterion (AIC)} &= -2\ell(\hat{\boldsymbol{\theta}}_k) + 2 \times \#\text{par.}\\ \text{Bayesian Information Criterion (BIC)} &= -2\ell(\hat{\boldsymbol{\theta}}_k) + \log(n) \times \#\text{par.}\end{aligned}$$

- These criteria rely on *penalized versions* of the log-likelihood function (Akaike, 1973; Schwarz,1978): the selected model is that with the minimum value of AIC (or BIC), corresponding to the best compromise between goodness-of-fit and model parsimony

- Certain authors prefer to report AIC (or BIC) with *reversed sign* (search for the maximum)

- BIC often selects a *more parsimonious model* with respect to AIC

- Many *other selection criteria* are available and these may be used in general (not only to select k)

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Example 1: simulated data

- Consider an FM model for $n = 100$ *bivariate responses* with $k = 2$ components:
  - $U_i = 1 \rightarrow \boldsymbol{Y}_i \sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$
  - $U_i = 2 \rightarrow \boldsymbol{Y}_i \sim N(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2)$
  - $\pi_1 = \pi_2 = 0.5$

$$\boldsymbol{\mu}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{\Sigma}_1 = \begin{pmatrix} 1.0 & 0.5 \\ 0.5 & 1 \end{pmatrix}$$

$$\boldsymbol{\mu}_2 = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \quad \boldsymbol{\Sigma}_2 = \begin{pmatrix} 1.0 & -0.5 \\ -0.5 & 1 \end{pmatrix}$$

- This is a *heterockedatic FM model* because the two variance-covariance matrices are different

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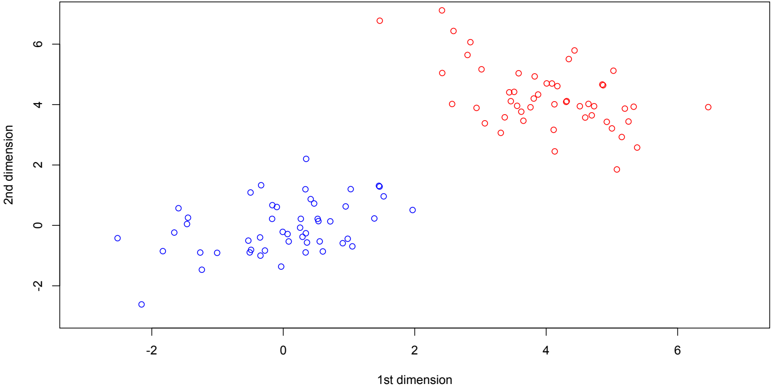
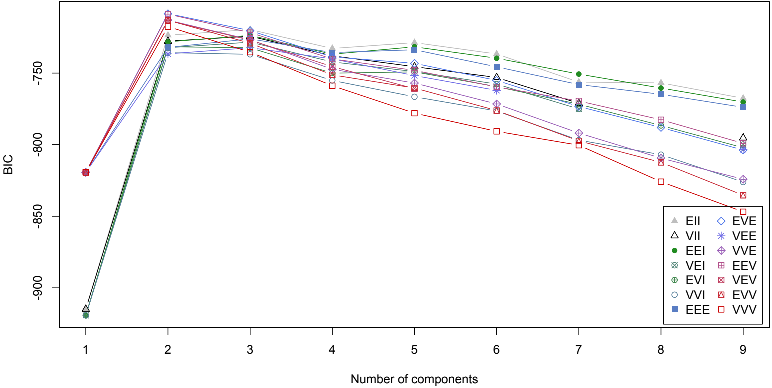
- *Data representation* (blue = 1st component, red = 2nd component):



- These data are analyzed by package `mclust` *of* R

- *Model selection* based on BIC (with sign reversed) in terms of k and structure of the variance-covariance matrices (EII, VII,...):



- *Two groups* are indeed selected, with different variance-covariance matrices

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- *Estimated parameters*:

$$\hat{\boldsymbol{\mu}}_1 = \begin{pmatrix} -0.015 \\ -0.058 \end{pmatrix}, \quad \hat{\boldsymbol{\Sigma}}_1 = \begin{pmatrix} 1.006 & 0.403 \\ 0.403 & 0.835 \end{pmatrix}$$

$$\hat{\boldsymbol{\mu}}_2 = \begin{pmatrix} 4.012 \\ 4.251 \end{pmatrix}, \quad \hat{\boldsymbol{\Sigma}}_2 = \begin{pmatrix} 0.869 & -0.519 \\ -0.519 & 1.089 \end{pmatrix}$$

$$\hat{\pi}_1 = \hat{\pi}_2 = 0.5$$

- *Posterior probabilities and clustering*:

i	$\hat{z}_{i1}$	$\hat{z}_{i2}$	Assigned group	True group
1	1.0000	0.0000	1	1
⋮	⋮	⋮	⋮	⋮
70	0.0000	1.0000	2	2
⋮	⋮	⋮	⋮	⋮
74	0.0027	0.9973	2	2

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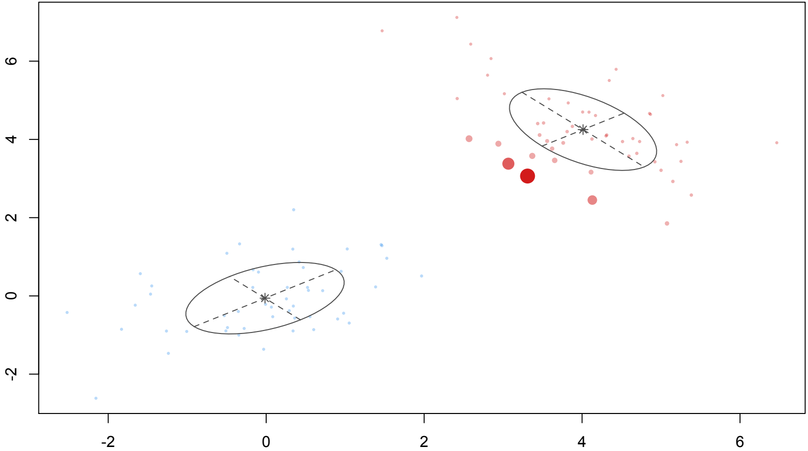
- *Clustering representation* (blue = 1st component, red = 2nd component) with measure of uncertainty:



Example 2: real data

- Data about all world countries regarding certain *macro-economic and demographic indicators* for 2016 (source: World Bank):

i	Name	Code	GDP	Ages	Life expect	School enp	School ens
1	Afghanistan	AFG	1793.89	43.86	65.02	NA	51.75
2	Albania	ALB	11355.62	17.72	80.45	109.78	94.98
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
85	Italy	ITA	34655.26	13.61	84.90	100.36	102.83
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
170	Switzerland	CHE	57421.55	14.83	85.10	104.41	102.29
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
186	United States	USA	53399.36	19.03	81.20	101.36	98.77
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
236	Sub-Saharan Africa (IDA & IBRD)	TSS	3482.87	42.88	62.09	97.18	42.81

- Countries with at least one *missing value* are eliminated so that $n = 162$ countries are considered

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- A model with $k = 3$ components and unequal variance-covariance matrices (VEV structure) are *selected by BIC*

- *Estimated means and weights*:

$$\hat{\boldsymbol{\mu}}_1 = \begin{pmatrix} 3428.67 \\ 39.38 \\ 64.18 \\ 104.13 \\ 49.33 \end{pmatrix}, \quad \hat{\boldsymbol{\mu}}_2 = \begin{pmatrix} 16661.98 \\ 24.01 \\ 77.34 \\ 102.93 \\ 93.48 \end{pmatrix}, \quad \hat{\boldsymbol{\mu}}_3 = \begin{pmatrix} 51140.18 \\ 16.67 \\ 83.23 \\ 102.22 \\ 113.31 \end{pmatrix}$$

$$\hat{\pi}_1 = 0.265, \quad \hat{\pi}_2 = 0.529, \quad \hat{\pi}_3 = 0.205$$

- The *estimated variances* have a size that tend to increase with the mean and, generally, with a negative correlation between variable Ages and the other variables

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- *Posterior probabilities and clustering*:

i	Code	$\hat{z}_{i1}$	$\hat{z}_{i2}$	$\hat{z}_{i3}$	Assigned group
2	ALB	0.006	0.988	0.007	2
⋮	⋮	⋮	⋮	⋮	⋮
85	ITA	0.000	0.199	0.801	3
⋮	⋮	⋮	⋮	⋮	⋮
170	CHE	0.000	0.000	1.000	3
⋮	⋮	⋮	⋮	⋮	⋮
186	USA	0.000	0.000	1.000	3
⋮	⋮	⋮	⋮	⋮	⋮
236	TSS	1.000	0.000	0.000	1

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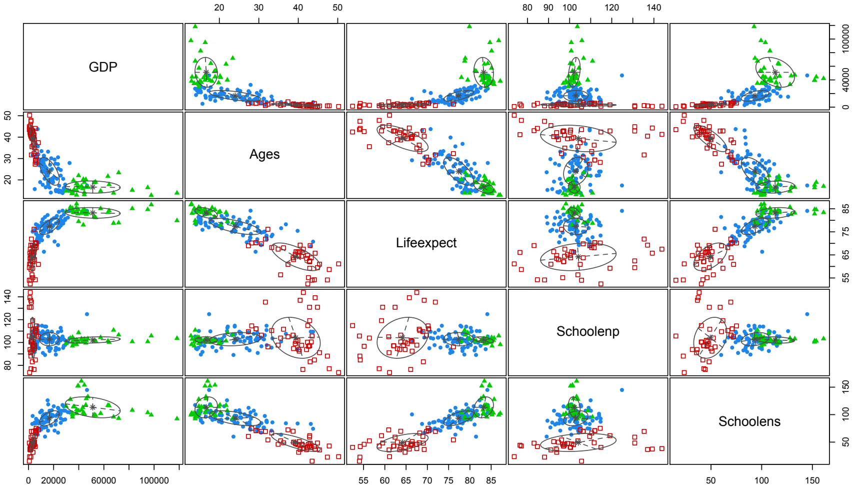
- *Clustering representation* (red = 1st component, blue = 2nd component, green = 3rd component):



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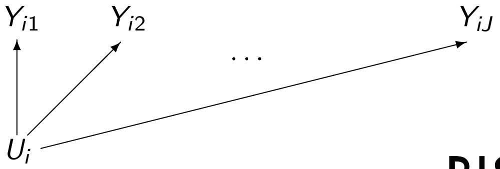
Latent Class Model

- *Main references*: Lazarfeld (1968), Goodman (1974)
- It follows an FM approach that:
  - is suitable for *categorical response variables* Y_{ij} with categories labeled from 0 to $c_j - 1$, $j = 1, \ldots, J$
  - *assumes LI* so that the latent variable is the only explanatory factor of the responses



- The *conditional probability of a response configuration* given the latent class is obtained by a single product:

$$p(\boldsymbol{y}_i|u) = \prod_{j=1}^{J} \eta_{j,y_{ij}|u}$$

  - $\eta_{j,y|u}$: probability that $Y_{ij} = y$ given $U_i = u$

- In the *binary case* it may be expressed using the Bernoulli probability mass function:

$$p(\boldsymbol{y}_i|u) = \prod_{j=1}^{J} \lambda_{j|u}^{y_{ij}} (1 - \lambda_{j|u})^{1-y_{ij}}$$

  - $\lambda_{j|u}$: probability that $Y_{ij} = 1$ given $U_i = u$ corresponding to $\eta_{j,1|u}$

- The *manifest distribution* becomes

$$p(\boldsymbol{y}_i) = \sum_{u=1}^{k} \left(\prod_{j=1}^{J} \eta_{j,y_{ij}|u} \right) \pi_u$$

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- The *posterior distribution* becomes

$$p(U_i = u|\boldsymbol{y}_i) = \frac{\sum_{u=1}^{k}\left(\prod_{j=1}^{J}\eta_{j,y_{ij}|u}\right)\pi_u}{p(\boldsymbol{y}_i)}, \quad u = 1,\ldots,k$$

- The *number of free parameters* is in general

$$\#\mathrm{par} = \underbrace{k-1}_{\pi_u} + \underbrace{\prod_{j=1}^{J}(c_j - 1)}_{\eta_{j,y|u}}$$

that in the binary case becomes

$$\#\mathrm{par} = \underbrace{k-1}_{\pi_u} + \underbrace{kJ}_{\lambda_{j|u}}$$

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Estimation

- Estimation is based on an *EM algorithm* having the same structure of that for FM models; it may be easily implemented using the binary indicator variable representation (Z_{iu} vs U_i)

- The *complete data log-likelihood* is expressed as

$$\ell^*(\boldsymbol{\theta}) = \sum_{i=1}^{n}\sum_{u=1}^{k}\left[Z_{iu}\sum_{j=1}^{J}\log(\eta_{j,y_{ij}|u}) + Z_{iu}\log(\pi_u)\right]$$

- At the M-step an *explicit solution* exists for the $\eta_{j,y|u}$ parameters:

$$\eta_{j,y|u} = \frac{1}{\sum_{i=1}^{n}\hat{z}_{iu}}\sum_{i=1}^{n}\hat{z}_{iu}I(y_{ij}=y)$$

that in the binary case becomes

$$\lambda_{j|u} = \frac{1}{\sum_{i=1}^{n}\hat{z}_{iu}}\sum_{i=1}^{n}\hat{z}_{iu}y_{ij}$$

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- A crucial issue is still that of *multimodality of the log-likelihood function* that requires the use of different starting points

- Typically, apart from a deterministic initialization (depending on the observed data), several *random initializations* are tried with each probability π_u and $\eta_{j,y|u}$ drawn from a uniform distribution from 0 to 1 and then suitably renormalized

- A crucial point after estimation is that of *assigning individuals to the latent classes*, which is still based on the posterior probabilities $\hat{z}_{iu}$ using the MAP rule

- The same *model selection criteria* as for FM models (in particular AIC and BIC) are typically adopted for model selection

- The EM algorithm is *implemented* in several softwares such as `R` (package `MultiLCIRT`) and `Stata`

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Example

- *Data* are collected on 216 subjects who responded to $J = 4$ items concerning the behavior in certain role conflict situations (Goodman, 1974)
- Each binary *response variable* is equal to 1 if the interviewed individual has a universalistic behavior and 0 if he/she has a particularistic behavior
- Data may be represented by a 2^4-dimensional *vector of frequencies* for all the response configurations:

Y_{i1}	Y_{i2}	Y_{i3}	Y_{i4}	Frequency
0	0	0	0	42
0	0	0	1	23
0	0	1	0	6
⋮	⋮	⋮	⋮	⋮
1	1	1	1	20

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- *Selection of the number of classes*:

k	$\ell(\hat{\boldsymbol{\theta}})$	#par	AIC	BIC
1	-543.65	4	1095.30	1108.80
2	-504.47	9	1026.94	1057.31
3	**-503.30**	**14**	**1034.60**	**1081.86**
4	-503.11	19	1044.22	1108.35

- Both AIC and BIC select *$k = 3$ latent classes*

- *Parameter estimates*:

	$\hat{\lambda}_{j\|u}$				
Class (u)	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$\hat{\pi}_u$
1	0.003	0.023	0.006	0.101	0.200
2	0.164	0.519	0.563	0.807	0.601
3	0.548	0.922	0.736	0.928	0.199

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- *Posterior probability* for each possible response configuration:

y_{i1}	y_{i2}	y_{i3}	y_{i4}	$\hat{z}_{i1}$	$\hat{z}_{i2}$	$\hat{z}_{i3}$	Assigned class
0	0	0	0	**0.894**	0.105	0.001	1
0	0	0	1	0.183	**0.801**	0.016	2
0	0	1	0	0.040	**0.946**	0.013	2
0	0	1	1	0.001	**0.957**	0.042	2
0	1	0	0	0.150	**0.793**	0.057	2
0	1	0	1	0.004	**0.815**	0.180	2
0	1	1	0	0.001	**0.865**	0.134	2
0	1	1	1	0.000	**0.676**	0.324	2
1	0	0	0	0.105	**0.860**	0.035	2
1	0	0	1	0.003	**0.887**	0.110	2
1	0	1	0	0.001	**0.919**	0.080	2
1	0	1	1	0.000	**0.787**	0.213	2
1	1	0	0	0.002	**0.693**	0.305	2
1	1	0	1	0.000	0.423	**0.577**	3
1	1	1	0	0.000	**0.512**	0.488	2
1	1	1	1	0.000	0.253	**0.747**	3

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Inclusion of covariates

- Two possible *choices to include individual covariates* collected in $\boldsymbol{x}_i$
- The first is in the *measurement model* so that we have random intercepts; for instance, for the LC model for binary variables we could assume:

$$\lambda_{ij|u} = p(Y_{ij} = 1 | U_i = u, \boldsymbol{x}_i),$$

$$\log \frac{\lambda_{ij|u}}{1 - \lambda_{ij|u}} = \alpha_u + \boldsymbol{x}_i' \boldsymbol{\beta}, \quad i = 1, \ldots, n, \; j = 1, \ldots, J, \; u = 1, \ldots, k$$

  - α_u: random intercepts
  - $\boldsymbol{\beta}$: vector of logistic regression parameters

- The latent variables are used to account for the *unobserved heterogeneity* in addition to the observed heterogeneity; the model may be seen as a "discrete version" of the random-effects logit model

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- The second is in the *structural model* governing the distribution of the latent variables (via a multinomial logit parametrization):

$$\pi_{iu} = p(U_i = u | \boldsymbol{x}_i),$$
$$\log \frac{\pi_{iu}}{\pi_{i1}} = \boldsymbol{x}'_{i1}\gamma_u, \quad u = 2, \ldots, k$$

  - γ_u: vectors of regression coefficients specific for each latent class

- The *main interest is in the latent variable* that is measured through the observable response variables (e.g., health status) and on how this latent variable depends on the covariates

- Both extensions lead to FM/LC models that may be *estimated* by an EM algorithm having a structure similar to that of the corresponding models without covariates; particular care is necessary to obtain the *standard errors* for the regression coefficients

- Usual criteria may be used for *model selection* in terms of number of components (k) and other assumptions

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Extension to longitudinal data

- The extension of FM/LC models to the analysis of longitudinal data is known as *Latent Markov (ML) model*

- *Main references*: Wiggins (1973), Bartolucci *et al.* (2013), Zucchini *et al.* (2016)

- For individual sequences of response variables at T time occasions, $\boldsymbol{Y}_i = (Y_{i1}, \ldots, Y_{iT})'$, $i = 1, \ldots, n$, the *basic version of the LM model* assumes that:
  - (*LI*) the response variables in $\boldsymbol{Y}_i$ are conditionally independent given a latent process $\boldsymbol{U}_i = (U_{i1}, \ldots, U_{iT})'$
  - every latent process $\boldsymbol{U}_i$ follows a *first-order Markov chain* with state space $\{1, \ldots, k\}$, initial probabilities π_u, and transition probabilities $\pi_{v|u}$, $u, v = 1, \ldots, k$

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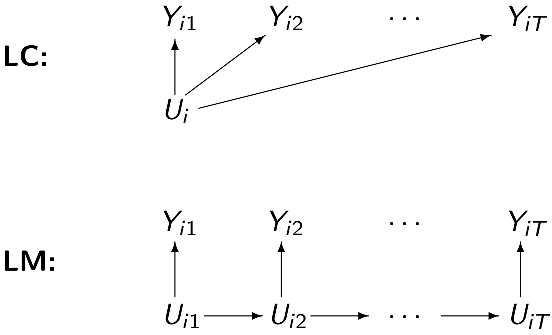
Possible interpretation

- The LM model may be seen as a *generalization of the LC model* in which subjects are allowed to move between latent classes



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Model parameters

- Each latent state u ($u = 1, \dots, k$) corresponds to a *class of subjects* (or latent state) in the population, and is characterized by:
  - *initial probability*:
  $$\pi_u = p(U_{i1} = u)$$
  - *transition probabilities* (which may also be time-specific in the non-homogenous case):
  $$\pi_{v|u} = p(U_{it} = v|U_{i,t-1} = u), \quad t = 2, \dots, T, \; v = 1, \dots, k$$
  - *distribution of the response variables* (with categorical responses with c categories):
  $$\psi_{y|u} = p(Y_{it} = y|U_{it} = u), \quad t = 1, \dots, T, \; y = 0, \dots, c-1$$
- The transition probabilities are collected in the *transition matrix* $\mathbf{\Pi}$ of size $k \times k$

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- LI implies that the *conditional distribution* of $\boldsymbol{Y}_i$ given $\boldsymbol{U}_i$ is:

$$p(\boldsymbol{y}_i|\boldsymbol{u}_i) = p(\boldsymbol{Y}_i = \boldsymbol{y}_i|\boldsymbol{U}_i = \boldsymbol{u}_i) = \prod_{t=1}^{T} \psi_{y_{it}|u_{it}}$$

- *Distribution* of $\boldsymbol{U}_i$: $$p(\boldsymbol{u}_i) = p(\boldsymbol{U}_i = \boldsymbol{u}_i) = \pi_{u_{i1}} \prod_{t>1} \pi_{u_{it}|u_{i,t-1}}$$

- *Manifest distribution* of $\boldsymbol{Y}_i$: $$p(\boldsymbol{y}_i) = p(\boldsymbol{Y}_i = \boldsymbol{y}_i) = \sum_{\boldsymbol{u}} p(\boldsymbol{y}_i|\boldsymbol{u})p(\boldsymbol{u})$$

- This may be *efficiently computed* through suitable recursions known in the hidden Markov literature (Baum *et al.*, 1970, Welch 2003)

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- The same tools available for FM/LC models may be used for *model estimation and selection*, although the EM algorithm requires particular care in the implementation based on certain recursions (Baum *et al.*, 1970, Welch 2003); package `LMest` in `R` may be used for applications

- After estimation, an important analysis is that of the *prediction of the latent states* for every unit and time occasion (dynamic clustering) that requires particular recursions (Viterbi, 1967; Juang & Rabiner, 1991)

- The LM model has been extended in several directions:
  - *multivariate longitudinal data* when more response variables are available at each time occasion
  - *inclusion of covariates* in the measurement or structural model with different interpretations and types of analysis
  - *multilevel longitudinal data* when units are clustered and so have a hierarchical structure

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