The Beautiful Cubit System

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Abstract

An analysis of the Egyptian Royal cubit, presenting some research and opinions flowing from that research, into what I believe was the original cubit, and how it was corrupted. I show various close arithmetic approximations and multiple ways of getting the divisions of the cubit, as well as some related measures. The cubit also encapsulates the basic components for the metric system.

Keywords: *Egyptology, metrology, royal cubit, cubit, metre, foot, metric system*

Contents

1. Introduction

The cubit is a well-know ancient measure of length, used around various places in the Middle East and Mediterranean region in the distant past.

It is allegedly based on the length of a human (male) fore-arm. It is typically measured from the back of the elbow to some point between the wrist and the end of the outstretched middle finger, or in some variants, a point beyond that.

The problem with this approach is that everyone's arm is a different length. If the heights of the dynastic Egyptians is taken as representative, then their arms would have been too short to justify the accepted lengths. There is also the issue of a whole range of different cubit lengths, not only between different cultures, but even within the same culture.

So I propose a different origin, based on mathematics, and dating back to a much earlier time.

2. Summary of current understanding

Mark Stone's overview [1] covers the different cubit lengths in different cultures, as well as issues regarding human anatomy as the basis for the cubit and other measures.

Quentin Leplat [2] analysed the Turin cubit, noting that it is 0.5236m long, and consists of 24 digits of 18.5mm, and 4 of 19.75mm.

If I can summarize the current consensus regarding the cubit, it would be something like this:

1. The cubit was based on the length of a forearm, from the back of the elbow to some point from the wrist to the end of the extended middle finger, or possibly further.

2. Different cubits exist because different communities each made their own.

3. The standard may actually have been the arm of some king, at some point in time.

4. The divisions are similarly based on and named after various other body parts, like palm, span or digit.

There are several problems with this consensus.

1. Measuring from the elbow: Depending on how hard the arm is pressed against some "zero point" backstop, you can change the measured length by a few millimetres. Given that cubit lengths are usually quoted down to fractions of a millimetre, this alone will give varying results.

2. If we take the height of the Egyptians as typical for populations in the area (or in any event, as a sample), then their heights do not support the standard short cubit of about 45cm. Consider:

"The average height of the male population varied between 161 cm (5.28 feet) in the New Kingdom (about 1550–1070 BC) and 169.6 cm (5.56 feet) in the Early Dynastic period (about 2925–2575 BC), making an average of 165.7 cm (5.43 feet) for all time periods." [3]

3. If we compare the short cubit of 45cm, or as is more usually stated, 18", with the royal cubit of say 20.6", then we have a different problem. The difference is 2.6 inches or 66mm. However, the royal cubit is a short cubit plus a palm, with a palm normally given as 75mm. So this does not work either.

3. An alternative origin

As discussed in my other two papers [4] and [5], the cubit may be very much older than we think. So I propose that instead of saying it was based on the length of a forearm, we look at the number more closely, and at how a highly logical population would derive it.

We start with the royal cubit rather than the regular cubit. I am convinced that those who say that the royal cubit was based on a circle with diameter one metre, are correct. The royal cubit would then be $\pi/6$ metres, or 0.5236 (to 4 places) long.

The population that invented this disappeared a long time ago. Some time on this side of the last ice age, our forefathers in the middle east found one or more surviving cubit rods and adopted it, perhaps as "given by the gods."

This found cubit was copied and spread around. Bad copies led to varying lengths. At some point, people noticed it was "about" the length of their forearm plus hand, and back-named the length accordingly, as well as the subdivisions.

I can't answer the question of how they had the metre to start, but they clearly did. Perhaps the answer will surface in due course.

4. Different ways of approximating the royal cubit

If we start with $\pi/6$, then there are two well-known approximations that produce values close to this, both based on π and/or φ, the golden ratio.

These are $\varphi^2/5$, and $\pi - \varphi^2$. However, there are other formulas that I either figured out or rediscovered, that give better approximations. These are listed in Tables 2, 3, 4 and 5 in decreasing order of closeness to $\pi/6$.

First we put the differences in perspective in Table 1, using values supplied by Wikipedia. [6]

Microns	**Metres**	**Less than / about**
0.04	0.00000004	Length of a lysosome
1.5	0.00000150	Anthrax spore
2	0.00000200	Length of an average E. coli bacteria
3.5	0.00000350	Size of a typical yeast cell
5	0.00000500	Length of a typical human spermatozoon's head
7	0.00000700	Diameter of human red blood cells
10	0.00001000	Transistor width of the Intel 4004
17	0.00001700	Minimum width of a strand of human hair
30	0.00003000	Length of a human skin cell
50	0.00005000	Typical length of a human liver cell
60	0.00006000	Length of a sperm cell
100	0.00010000	The smallest distance that can be seen with the naked eye
181	0.00018100	Maximum width of a strand of human hair
200	0.00020000	Typical length of Paramecium caudatum, a ciliate protist
500	0.00050000	Typical length of Amoeba proteus, an amoeboid protist

Table 1: Putting small distances in perspective

Table 2 has very close approximations for the Royal Cubit (henceforth ₢).

Method	**Value**	**Abs difference from π/6**	**Rounded**
π/6	0.523598776	0.000000000	0.5236
$((6\sqrt{2}/10)^2 + (6/100)^2+(8\sqrt{2}/10000)^2)^2$	0.523598812	0.000000037	0.5236
cube roots *(see below)*	0.523600350	0.000001575	0.5236
$((6\sqrt{2}/10)^2 + (6/100)^2)^2$	0.523596960	0.000001816	0.5236
$(7\pi/5e)^2/5$	0.523596637	0.000002138	0.5236
28φπ/100e	0.523601717	0.000002942	0.5236
φe/8.4	0.523603856	0.000005080	0.5236
ln(4) *(see below)*	0.523591499	0.000007277	0.5236
((1 + π)/e) - 1	0.523606791	0.000008015	0.5236
$\varphi^2/5$	0.523606798	0.000008022	0.5236

Table 2: Formulas giving approximations very close to π/6

The "cube roots" formula is $$\frac{1}{\sqrt[3]{7-\left(\frac{\sqrt[3]{2}\cdot(\sqrt[3]{5}-\sqrt[3]{3})}{10}\right)}}$$

The ln(4) formula is the solution to the equation $\ln(4)+x=\frac{1}{x}$

We should also point out that thanks to Euler and the Zeta function, we can also write π/6 as $\frac{\zeta(2)}{\pi}$ or $\frac{\sum_{n=0}^{\infty}\frac{1}{n^2}}{\pi}$ as both equal to ₲ precisely.

The formulas are easier to follow when shown in conventional form:

$$\text{₠} = \frac{\pi}{6} = \frac{\zeta(2)}{\pi} = \frac{\sum_{n=0}^{\infty} \frac{1}{n^2}}{\pi} \approx \left(\left(\frac{6\sqrt{2}}{10} \right)^2 + \left(\frac{6}{100} \right)^2 + \left(\frac{8\sqrt{2}}{10000} \right)^2 \right)^2$$

$$\approx \frac{1}{\sqrt[3]{7 - \left(\frac{\sqrt[3]{2} \cdot (\sqrt[3]{5} - \sqrt[3]{3})}{10} \right)}} \approx \left(\left(\frac{6\sqrt{2}}{10} \right)^2 + \left(\frac{6}{100} \right)^2 \right)^2$$

$$\approx \frac{1}{5} \left(\frac{7\pi}{5e} \right)^2 \approx \frac{7\varphi\pi}{25e} \approx \frac{28\varphi\pi}{100e} \approx \frac{\varphi e}{8.4} \approx \left(\ln(4) + x = \frac{1}{x} \right)$$

$$\approx \left(\frac{1+\pi}{e} - 1 \right) \approx \frac{\varphi^2}{5}$$

Next are formulas giving close values in Table 3.

Method	Value	Abs difference from π/6	Rounded
$\pi - (7\pi/5e)^2$	0.523609468	0.000010692	0.5236
$e\varphi^3/7\pi$	0.523611878	0.000013103	0.5236
$(10\varphi)/(11e + 1)$	0.523616953	0.000018177	0.5236
$\tan(2\varphi\pi e) = \tan(\tau\varphi e)$	0.523569002	0.000029774	0.5236
$(\varphi^2/(e-1))-1$	0.523634799	0.000036024	0.5236
$\pi - \varphi^2$	0.523558665	0.000040111	0.5236
$\sqrt{(\sqrt{5}/(3e))}$	0.523642193	0.000043417	0.5236
$e(2\sqrt{2} - \varphi)/2\pi$	0.523649949	0.000051174	0.5236

Table 3: Formulas giving close approximations of π/6

The conventional formulas are like this:

$$\text{₠} \approx \pi - \left(\frac{7\pi}{5e} \right)^2 \approx \frac{e\varphi^3}{7\pi} \approx \frac{10\varphi}{11e+1} \approx \tan(2\pi\varphi e) \approx \tan(\tau\varphi e)$$

$$\approx \left(\frac{\varphi^2}{e-1} - 1 \right) \approx \pi - \varphi^2 \approx \sqrt{\frac{\sqrt{5}}{3e}} \approx \frac{e(2\sqrt{2}-\varphi)}{2\pi} \approx \frac{e(2\sqrt{2}-\varphi)}{\tau}$$

Table 4 has less-close approximations of ₠, but still better than 0.5250.

Method	**Value**	**Abs difference from π/6**	**Rounded**
$10e/(36^3\sqrt{3})$	0.523542042	0.000056733	0.5235
$\ln(10)/\varphi e$	0.523520348	0.000078427	0.5235
$e/(3\sqrt{3})$	0.523133582	0.000465194	0.5231
square roots *(see below)*	0.523403737	0.000195039	0.5234
$1/(\sqrt{(\ln 365.25/\varphi)})$	0.523656373	0.000057597	0.5237
$2\sqrt{5}/\pi e$	0.523685613	0.000086838	0.5237
$1/\log(\varphi^2\pi^3)$	0.523717901	0.000119125	0.5237
$(10/\varphi\pi e)^2$	0.523764441	0.000165665	0.5238
$(10\sqrt{2})/27$	0.523782801	0.000184025	0.5238
$\cos(\pi(4\varphi e+1))$	0.523808794	0.000210019	0.5238
$\pi.10^8/2c$	0.523961255	0.000362479	0.5240
$(\sqrt[3]{3}\ \sqrt[3]{5}\ \sqrt[3]{7})/9$	0.524188220	0.000589444	0.5242
$(\sqrt[3]{5}\varphi^2)/\pi e$	0.524228861	0.000630086	0.5242
$\sin((\varphi^2\pi e\sqrt{2})$	0.524253715	0.000654940	0.5243
$(\varphi)/(e + 1/e)$	0.524286921	0.000688145	0.5243
$(\sqrt{2}+\sqrt{3})/6$	0.524377395	0.000778619	0.5244
$\sqrt{(7\sqrt{2})}/6$	0.524391047	0.000792272	0.5244
$\pi/(\varphi^3\sqrt{2})$	0.524411195	0.000812419	0.5244

Table 4: Less-close approximations of π/6

The "square roots" formula is $\frac{1}{\sqrt{(\sqrt{2}+\sqrt{5})}}$

Here are these conventionally:

$$Ɛ = \frac{\pi}{6} \approx \frac{10e}{36\sqrt[3]{3}} \approx \frac{10e}{2^2 3^2 \sqrt[3]{3}} \approx \frac{\ln(10)}{\varphi e} \approx (\sqrt{2}+\sqrt{5})^{-\frac{1}{2}} \approx \frac{e}{3\sqrt{3}}$$

$$\approx \left(\frac{\ln(365.25)}{\varphi}\right)^{-\frac{1}{2}} \approx \frac{2\sqrt{5}}{\pi e} \approx \frac{1}{\log_{10}(\varphi^2 \pi^3)} \approx \left(\frac{10}{\pi \varphi e}\right)^2 \approx \frac{10\sqrt{2}}{3^3}$$

$$\approx \cos(\pi(4\varphi e+1)) \approx \frac{\pi 10^8}{2c} \approx \frac{\sqrt[3]{3}\sqrt[3]{5}\sqrt[3]{7}}{3^2} \approx \frac{\sqrt[3]{5}\varphi^2}{\pi e}$$

$$\approx \sin(\sqrt{2}\varphi^2 \pi e) \approx \frac{\varphi}{e+e^{-1}}$$

$$\approx \frac{\sqrt{2}+\sqrt{3}}{2\times 3} \approx \frac{\sqrt{7\sqrt{2}}}{6} \approx \frac{\pi}{\varphi^3\sqrt{2}}$$

4. Different ways of getting the cubit divisions

4.1 The different extant lengths

It appears that apart from making bad copies, the ancients decided to “improve” the cubit subdivisions, in both directions. They did this by changing the length of the digit. One change was from 18.7mm to 18.75mm, leading to the 45cm short cubit and 52.5 royal cubit.

This digit size also produces the 30cm Egyptian Foot, as well as other measures based around 7.5cm intervals.

The other change was a move to a digit of 18.5mm, which led to further complications, resulting in the curious Turin cubit with its two digit sizes.

18.7 x 24 = 448.8mm = short cubit (original).
18.7 x 28 = 523.6mm = royal cubit (original).

18.75 x 24 = 450mm = short cubit (variant 1).
18.75 x 28 = 525mm = royal cubit (variant 1).
18.75 x 16 = 300mm = Egyptian foot (variant 1).

18.5 x 24 = 444mm = short cubit (variant 2).
18.5 x 28 = 518mm = royal cubit (variant 2), which doesn’t work, hence they had to do
18.5 x 24 = 444mm, plus 4 x 19.75 = 79mm, giving 523mm.

This is an explanation for the various cubit lengths ranging from 523 to 525mm.

In truth, it is difficult for modern students with sharp pencils and accurate rulers, to differentiate between a line of 18.7 and 18.75mm. You need to use micrometer-style or slide-rule techniques as discussed by Monnier et al. [7]

Figure 0 shows two lines, one 18.5mm and the other 18.7mm, to demonstrate how subtle the difference is. Obviously the difference between 18.7 and 18.75mm will be even harder to see.

This is a screenshot of a drawing done with SVG and may print out slightly differently.

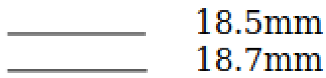

Figure 0: 18.5mm vs. 18.7mm

4.2 The π method

I first heard that the royal cubit was π/6 from Robert Bauval, but have seen references to someone back in the 1800's who first proposed it, possibly Karl Richard Lepsius.

The thinking is that you take a circle with diameter of 1 metre, which gives a circumference of πm. You then take one sixth of this (i.e. a 60° arc) and that is the royal cubit ₢.

This division matches nicely with a six-spoked chariot wheel, and some Egyptian chariots had six spokes and a diameter of close to 1 metre. [8]

If we accept that π/6 from a circle of diameter one metre was the origin of the ₢, then it is simple to generate the divisions of the cubit following the same pattern.

These are compared to the "reference values" taken from Wikipedia [9], which we can use as "currently accepted" even though I disagree with them. They are similar to the figures from "*The Cadastral Survey of Egypt" [10].*

Table 5 has values for the divisions of the cubit, using π and τ, where τ is 2π.

For the π values, we can use a divisor of 168, and a divisor of 336 for τ. We just need to multiply by the number of digits in each division to get the answer.

There appears to be conflicting opinions about the remen, one based on it being 20 digits, and the other setting it as half the diagonal of a square of 1₢ side, which is also the height measured from the diagonal.

This method shows the beauty of the relationship between the short cubit (henceforth C) and ₢. The ₢ is π/6, and the C is π/7. That is the origin of this paper's title.

The Nby-rod, a measure used by builders, has its own special beauty in referencing π.

Digits	Length	Reference Value	Formula π	Formula τ	Value
1	Digit	0.01875m	$\frac{1\pi}{168}=\frac{\pi}{168}$	$\frac{\tau}{336}$	0.0187m
4	Palm	0.0750m	$\frac{4\pi}{168}=\frac{\pi}{42}$	$\frac{4\tau}{336}=\frac{\tau}{84}$	0.0748m
5	Hand	0.0938m	$\frac{5\pi}{168}$	$\frac{5\tau}{336}$	0.0935m
			$\frac{2\pi}{67}$	$\frac{\tau}{67}$	0.0938m
6	Fist	0.1125m	$\frac{6\pi}{168}=\frac{\pi}{28}$	$\frac{6\tau}{336}=\frac{\tau}{56}$	0.1122m
8	Double Handbreadth	0.1500m	$\frac{8\pi}{168}=\frac{\pi}{21}$	$\frac{8\tau}{336}=\frac{\tau}{42}$	0.1496m
12	Small span	0.2250m	$\frac{12\pi}{168}=\frac{\pi}{14}$	$\frac{12\tau}{336}=\frac{\tau}{28}$	0.2244m
14	Great span	0.2600m	$\frac{14\pi}{168}=\frac{\pi}{12}$	$\frac{\tau}{24}$	0.2618m
16	Foot	0.3000m	$\frac{16\pi}{168}=\frac{2\pi}{21}$	$\frac{\tau}{21}$	0.2992m
	Remen	0.3702m	$\frac{\pi}{6\sqrt{2}}=\frac{₠}{\sqrt{2}}$	$\frac{\tau}{12\sqrt{2}}$	0.3702m
20	Remen	0.3750m	$\frac{20\pi}{168}$	$\frac{5\tau}{84}$	0.3740m
24	Cubit (standard)	0.4500m	$\frac{24\pi}{168}=\frac{\pi}{7}$	$\frac{\tau}{14}$	0.4488m
28	Cubit (royal) ₠	0.523m or 0.525m	$\frac{28\pi}{168}=\frac{\pi}{6}$	$\frac{\tau}{12}$	0.5236m
32	Pole	0.6000m	$\frac{32\pi}{168}=\frac{4\pi}{21}$	$\frac{4\tau}{42}$	0.5984m
36	Nby-rod (not on Wikipedia)	0.67 – 0.68m	$\frac{36\pi}{168}=\frac{3\pi}{14}$	$\frac{3\tau}{28}$	0.6732m
64	Double pole (not on Wikipedia)	1.2000m	$\frac{64\pi}{168}=\frac{8\pi}{21}$	$\frac{8\tau}{42}$	1.1968m

Table 5: Divisions of the cubit based on π or τ

In their book *The Lost Science of Measuring the Earth [11],* Heath and Michell refer to a 'sacred' cubit of 2.057142857 feet, which converts to 0.627017m. This value slots into the above table nicely at $\pi/5 = 0.62832$m. The term 'sacred cubit' may be confusing as others use it as a synonym for the royal cubit. There is also Isaac Newton's version at 25.025 British inches, which is supposed to give a 25 "pyramid inch" sacred cubit.

I'm going to show alternative ways of dividing the cubit using famous mathematical constants, mostly π, φ, e, $\sqrt{2}$ and $\sqrt{5}$. First up is a version that produces values very close to Table 5, just a fraction larger as we get to the bigger lengths because the digit is fractionally larger. It is based on $\sqrt{5}/\pi e$.

Digits	Length	Reference Value	Formula	Value
1	Digit	0.01875m	$\frac{1\sqrt{5}}{14\pi e}$	0.0187m
4	Palm	0.0750m	$\frac{4\sqrt{5}}{14\pi e}$	0.0748m
5	Hand	0.0938m	$\frac{5\sqrt{5}}{14\pi e}$	0.0935m
6	Fist	0.1125m	$\frac{6\sqrt{5}}{14\pi e}$	0.1122m
8	Double Handbreadth	0.1500m	$\frac{8\sqrt{5}}{14\pi e}$	0.1496m
12	Small span	0.2250m	$\frac{12\sqrt{5}}{14\pi e}$	0.2244m
14	Great span	0.2600m	$\frac{14\sqrt{5}}{14\pi e}=\frac{\sqrt{5}}{\pi e}$	0.2618m
16	Foot	0.3000m	$\frac{16\sqrt{5}}{14\pi e}$	0.2992m
	Remen	0.3702m	$\frac{\sqrt{2}\sqrt{5}}{\pi e}$	0.3703m
20	Remen	0.3750m	$\frac{20\sqrt{5}}{14\pi e}$	0.3741m
24	Cubit (standard)	0.4500m	$\frac{24\sqrt{5}}{14\pi e}$	0.4489m
28	Cubit (royal) ₢	0.523m or 0.525m	$\frac{28\sqrt{5}}{14\pi e}=\frac{2\sqrt{5}}{\pi e}$	0.5237m
32	Pole	0.6000m	$\frac{32\sqrt{5}}{14\pi e}$	0.5985m
36	Nby-rod (not on Wikipedia)	0.67 – 0.68m	$\frac{36\sqrt{5}}{14\pi e}$	0.6733m
64	Double pole (not on Wikipedia)	1.2000m	$\frac{64\sqrt{5}}{14\pi e}$	1.1970m

Table 6: Divisions of the cubit based on √5/πe.

We then look at the problematic version where the digit is 18.5mm. This is based on $\frac{\pi}{\sqrt{2}}$, or by using a divisor of 120. Of necessity, the ₢, its half-value the great span, and one of the remen do not fit the digit-multiplier pattern.

Digits	Length	Value	Formula	Value
1	Digit	0.01875m	$\frac{\pi}{120\sqrt{2}}$	0. 0185m
4	Palm	0.0750m	$\frac{4\pi}{120\sqrt{2}}=\frac{\pi}{30\sqrt{2}}$	0.0741m
5	Hand	0.0938m	$\frac{5\pi}{120\sqrt{2}}=\frac{\pi}{24\sqrt{2}}$	0.0926m
6	Fist	0.1125m	$\frac{6\pi}{120\sqrt{2}}=\frac{\pi}{20\sqrt{2}}$	0.1111m
8	Double Handbreadth	0.1500m	$\frac{8\pi}{120\sqrt{2}}=\frac{\pi}{15\sqrt{2}}$	0.1481m
12	Small span	0.2250m	$\frac{12\pi}{120\sqrt{2}}=\frac{\pi}{10\sqrt{2}}$	0.2221m
	Great span	0.2618m	$\frac{\pi}{2\sqrt{18}\sqrt{2}}=\frac{\pi}{2\sqrt{36}}=\frac{\pi}{12}$	0.2618m
16	Foot	0.3000m	$\frac{16\pi}{120\sqrt{2}}=\frac{2\pi}{15\sqrt{2}}$	0.2962m
	Remen	0.3702m	$\frac{20\pi}{120\sqrt{2}}=\frac{\pi}{6\sqrt{2}}$	0.3702m
20	Remen	0.3750m		
24	Cubit (standard)	0.4500m	$\frac{24\pi}{120\sqrt{2}}=\frac{\pi}{5\sqrt{2}}$	0.4443m
	Cubit (royal) Ǥ	0.5236m	$\frac{\pi}{\sqrt{18}\sqrt{2}}=\frac{\pi}{\sqrt{36}}=\frac{\pi}{6}$	0.5236m
32	Pole	0.6000m	$\frac{32\pi}{120\sqrt{2}}=\frac{4\pi}{15\sqrt{2}}$	0.5924m
36	Nby-rod	0.67 – 0.68m	$\frac{36\pi}{120\sqrt{2}}=\frac{3\pi}{10\sqrt{2}}$	0.6664m
64	Double pole	1.2000m	$\frac{64\pi}{120\sqrt{2}}=\frac{8\pi}{15\sqrt{2}}$	1.1848m

Table 7: Poor divisions of the cubit based on π/√2

We can now look at the various ways of getting the divisions of the other slightly larger cubit, of 0.525m, based on a digit of 18.75mm. The first version uses π and φ^2. These formulas handle both versions of the remen, great span and Ǥ rather elegantly.

Digits	Length	Value	Formula	Value
1	Digit	0.01875m	$\frac{1\pi}{64\varphi^2}$	0.01875m
4	Palm	0.0750m	$\frac{4\pi}{64\varphi^2}$	0.0750m
5	Hand	0.0938m	$\frac{5\pi}{64\varphi^2}$	0.0938m
6	Fist	0.1125m	$\frac{6\pi}{64\varphi^2}$	0.1125m
8	Double Handbreadth	0.1500m	$\frac{8\pi}{64\varphi^2}$	0.1500m
12	Small span	0.2250m	$\frac{12\pi}{64\varphi^2}$	0.2250m
	Great span	0.2618m	$\frac{\pi-\varphi^2}{2}$	0.2618m
14	Great span	0.2625m	$\frac{14\pi}{64\varphi^2}$	0.2625m
16	Foot	0.3000m	$\frac{16\pi}{64\varphi^2}$	0.3000m
	Remen	0.3702m	$\frac{\pi-\varphi^2}{\sqrt{2}}$	0.3702m
20	Remen	0.3750m	$\frac{20\pi}{64\varphi^2}$	0.3750m
24	Cubit (standard)	0.4500m	$\frac{24\pi}{64\varphi^2}$	0.4500m
	Cubit (royal) Cr	0.5236m	$\pi-\varphi^2$	0.5236m
28	Cubit (royal) Cr	0.5250m	$\frac{28\pi}{64\varphi^2}$	0.5250m
32	Pole	0.6000m	$\frac{32\pi}{64\varphi^2}$	0.6000m
36	Nby-rod	0.67 – 0.6 m	$\frac{36\pi}{64\varphi^2}$	0.6750m
64	Double pole	1.2000m	$\frac{64\pi}{64\varphi^2}$	1.2000m

Table 8: Formulas for the large royal cubit using π and φ²

The next set of formulas are based on π, φ and e. The general form uses multiples of 3/224 of φe/π, except for the remen, great span and 𝔊, which flip the irrationals slightly and use πφ/e instead.

Digits	Length	Value	Formula	Value
1	Digit	0.01875m	$\frac{3}{224}\frac{\varphi e}{\pi}=1\frac{3}{224}\frac{\varphi e}{\pi}$	0.01875m
4	Palm	0.0750m	$\frac{3}{56}\frac{\varphi e}{\pi}=4\frac{3}{224}\frac{\varphi e}{\pi}$	0.0750m
5	Hand	0.0938m	$\frac{15}{224}\frac{\varphi e}{\pi}=5\frac{3}{224}\frac{\varphi e}{\pi}$	0.0938m
6	Fist	0.1125m	$\frac{18}{224}\frac{\varphi e}{\pi}=6\frac{3}{224}\frac{\varphi e}{\pi}$	0.1125m
8	Double Handbreadth	0.1500m	$\frac{3}{28}\frac{\varphi e}{\pi}=8\frac{3}{224}\frac{\varphi e}{\pi}$	0.1500m
12	Small span	0.2250m	$\frac{9}{56}\frac{\varphi e}{\pi}=12\frac{3}{224}\frac{\varphi e}{\pi}$	0.2250m
	Great span	0.2618m	$\frac{7}{50}\frac{\varphi\pi}{e}$	0.2618m
14	Great span	0.2625m	$\frac{3}{16}\frac{\varphi e}{\pi}=14\frac{3}{224}\frac{\varphi e}{\pi}$	0.2625m
16	Foot	0.3000m	$\frac{3}{14}\frac{\varphi e}{\pi}=16\frac{3}{224}\frac{\varphi e}{\pi}$	0.3000m
	Remen	0.3702m	$\frac{7\varphi\pi}{25e\sqrt{2}}$	0.3702m
20	Remen	0.3750m	$\frac{15}{56}\frac{\varphi e}{\pi}=20\frac{3}{224}\frac{\varphi e}{\pi}$	0.3750m
24	Cubit (standard)	0.4500m	$\frac{9}{28}\frac{\varphi e}{\pi}=24\frac{3}{224}\frac{\varphi e}{\pi}$	0.4500m
	Cubit (royal) 𝔊	0.5236m	$\frac{7}{25}\frac{\varphi\pi}{e}$	0.5236m
28	Cubit (royal)	0.5250m	$\frac{3}{8}\frac{\varphi e}{\pi}=28\frac{3}{224}\frac{\varphi e}{\pi}$	0.5250m
32	Pole	0.6000m	$\frac{3}{7}\frac{\varphi e}{\pi}=32\frac{3}{224}\frac{\varphi e}{\pi}$	0.6000m
36	Nby-rod	0.67 – 0.68m	$\frac{27}{56}\frac{\varphi e}{\pi}=36\frac{3}{224}\frac{\varphi e}{\pi}$	0.6750m
64	Double pole	1.2000m	$\frac{6}{7}\frac{\varphi e}{\pi}=64\frac{3}{224}\frac{\varphi e}{\pi}$	1.2000m

Table 9: Formulas for the large cubit divisions using π, e and φ.

We now show formulas based on π, e and $\sqrt[3]{3}$. These formulas are also starting to drift from the "accepted" values as per Wikipedia. The classic values for 𝕮, great span and remen can not be handled.

Digits	Length	Value	Formula	Value
1	Digit	0.01875m	$\frac{1e}{32\pi\sqrt[3]{3}}$	0. 01875m
4	Palm	0.0750m	$\frac{4e}{32\pi\sqrt[3]{3}}$	0.0750m
5	Hand	0.0938m	$\frac{5e}{32\pi\sqrt[3]{3}}$	0.0937m
6	Fist	0.1125m	$\frac{6e}{32\pi\sqrt[3]{3}}$	0.1125m
8	Double Handbreadth	0.1500m	$\frac{8e}{32\pi\sqrt[3]{3}}$	0.1500m
12	Small span	0.2250m	$\frac{12e}{32\pi\sqrt[3]{3}}$	0.2250m
	Great span	0.2618m		
14		0.2625m	$\frac{14e}{32\pi\sqrt[3]{3}}$	0.2625m
16	Foot	0.3000m	$\frac{16e}{32\pi\sqrt[3]{3}}$	0.3000m
	Remen	0.3702m		
20	Remen	0.3750m	$\frac{20e}{32\pi\sqrt[3]{3}}$	0.3750m
24	Cubit (standard)	0.4500m	$\frac{24e}{32\pi\sqrt[3]{3}}$	0.4500m
	Cubit (royal) ₢	0.5236m		
28		0.5250m	$\frac{28e}{32\pi\sqrt[3]{3}}$	0.5249m
32	Pole	0.6000m	$\frac{32e}{32\pi\sqrt[3]{3}}$	0.5999m
36	Nby-rod	0.67 – 0.68m	$\frac{36e}{32\pi\sqrt[3]{3}}$	0.6749m
64	Double pole	1.2000m	$\frac{64e}{32\pi\sqrt[3]{3}}$	1.1999m

Table 10: Formulas for the large cubit divisions using π, e and $\sqrt[3]{3}$.

The next formulas are more complicated, using π^2, φ^2 and e. They are also slightly more inaccurate.

Digits	Length	Value	Formula	Value
1	Digit	0.01875m	$\frac{1}{16}\left(\frac{\sqrt{\pi^2+\varphi^2}}{e}-1\right)$	0.01875m
4	Palm	0.0750m	$\frac{4}{16}\left(\frac{\sqrt{\pi^2+\varphi^2}}{e}-1\right)$	0.0750m
5	Hand	0.0938m	$\frac{5}{16}\left(\frac{\sqrt{\pi^2+\varphi^2}}{e}-1\right)$	0.09375m
6	Fist	0.1125m	$\frac{6}{16}\left(\frac{\sqrt{\pi^2+\varphi^2}}{e}-1\right)$	0.1125m
8	Double Handbreadth	0.1500m	$\frac{8}{16}\left(\frac{\sqrt{\pi^2+\varphi^2}}{e}-1\right)$	0.1500m
12	Small span	0.2250m	$\frac{12}{16}\left(\frac{\sqrt{\pi^2+\varphi^2}}{e}-1\right)$	0.2250m
14	Great span	0.2618m	$\frac{1\,\varphi^2}{3}\left(\frac{\sqrt{\pi^2+\varphi^2}}{e}-1\right)$	0.2618m
16	Foot	0.3000m	$\frac{16}{16}\left(\frac{\sqrt{\pi^2+\varphi^2}}{e}-1\right)$	0.3000m
	Remen	0.3702m		
20	Remen	0.3750m	$\frac{20}{16}\left(\frac{\sqrt{\pi^2+\varphi^2}}{e}-1\right)$	0.3750m
24	Cubit (standard)	0.4500m	$\frac{24}{16}\left(\frac{\sqrt{\pi^2+\varphi^2}}{e}-1\right)$	0.4500m
	Cubit (royal) ₲	0.5236m	$\frac{2\,\varphi^2}{3}\left(\frac{\sqrt{\pi^2+\varphi^2}}{e}-1\right)$	0.5236m
28	Cubit (royal) ₲	0.5250m	$\frac{28}{16}\left(\frac{\sqrt{\pi^2+\varphi^2}}{e}-1\right)$	0.5250m
32	Pole	0.6000m	$\frac{32}{16}\left(\frac{\sqrt{\pi^2+\varphi^2}}{e}-1\right)$	0.6000m
36	Nby-Rod	0.67 – 0.68m	$\frac{36}{16}\left(\frac{\sqrt{\pi^2+\varphi^2}}{e}-1\right)$	0.3750m
64	Double pole	1.2000m	$\frac{64}{16}\left(\frac{\sqrt{\pi^2+\varphi^2}}{e}-1\right)$	1.2000m

Table 11: Formulas for the large cubit divisions using π^2, φ^2 and e.

The last set of formulas are the most inaccurate, and based on √2/π.

Digits	Length	Value	Formula	Value
1	Digit	0.01875m	$\frac{1\sqrt{2}}{24\pi}$	0.01876m
4	Palm	0.0750m	$\frac{4\sqrt{2}}{24\pi}$	0.0750m
5	Hand	0.0938m	$\frac{5\sqrt{2}}{24\pi}$	0.0938m
6	Fist	0.1125m	$\frac{6\sqrt{2}}{24\pi}$	0.1125m
8	Double Handbreadth	0.1500m	$\frac{8\sqrt{2}}{24\pi}$	0.1501m
12	Small span	0.2250m	$\frac{12\sqrt{2}}{24\pi}$	0.2251m
	Great span	0.2618m	$\frac{2\sqrt{2}\varphi^2}{9\pi}$	0.2619m
14		0.2625m	$\frac{14\sqrt{2}}{24\pi}$	0.2626m
16	Foot	0.3000m	$\frac{16\sqrt{2}}{24\pi}$	0.3001m
	Remen	0.3702m		
20	Remen	0.3750m	$\frac{20\sqrt{2}}{24\pi}$	0.3751m
24	Cubit (standard)	0.4500m	$\frac{24\sqrt{2}}{24\pi}$	0.4502m
	Cubit (royal) G	0.5236m	$\frac{4\sqrt{2}\varphi^2}{9\pi}$	0.5238m
28		0.5250m	$\frac{28\sqrt{2}}{24\pi}$	0.5252m
32	Pole	0.6000m	$\frac{32\sqrt{2}}{24\pi}$	0.6002m
36	Nby-Rod	0.67 – 0.68m	$\frac{36\sqrt{2}}{24\pi}$	0.6752m
64	Double pole	1.2000m	$\frac{64\sqrt{2}}{24\pi}$	1.2004m

Table 12: Formulas for the large cubit divisions using √2/π

This demonstrates that the divisions of the cubit can be calculated arithmetically in multiple different ways, with varying degrees of accuracy. The divisions do not need to have been based on actual measurements of some random, average or specific person.

Table 13 has a few formulas that don't slot in anywhere else. Foot and cubit are the "long" versions at 30cm and 45cm respectively.

Length	Value	Formula	Value
Nby-rod	0.67 – 0.68m	Foot x $\sqrt{\pi\varphi}$	0.6764m
		Cubit x $\frac{2\sqrt{\pi\varphi}}{3}$	0.6764m

Table 13: Other assorted interesting formulas

The last set of formulas I want to demonstrate is based on what I call the "Grand Metre" (symbol $\mathcal{M}$) for lack of a better name. It is 1 metre plus $\mathfrak{C}$, totalling 1.5236m to 4 digits.

I have no evidence that this was ever used, but it has popped up on occasion when I was doing research for this paper, and the formulas are interesting.

The curious thing is that we can approximate it rather well and easily, using the favourite π, φ and e, as follows:

$$\mathcal{M} = 1+\mathfrak{C} \approx \frac{1+\pi}{e} \approx \frac{\varphi^2}{e-1} \approx \pi-\varphi \approx 1.5236m$$

The value is also very close to 5 English feet (1.524m), or correct to 3 digits.

Digits	Length	Value	Formula	Value
1	Digit	0.01875m	$\frac{\mathscr{M}}{16\,\pi\,\varphi}$	0.01873m
4	Palm	0.0750m	$\frac{\mathscr{M}}{4\,\pi\,\varphi}$	0.0749m
			$\frac{\mathscr{M}}{9\,\sqrt{\pi\,\varphi}}$	0.0751m
5	Hand	0.0938m	$\sqrt{\frac{\mathscr{M}}{100\,\sqrt{3}}}$	0.0938m
6	Fist	0.1125m	$\frac{\mathscr{M}}{6\,\sqrt{\pi\,\varphi}}$	0.1126m
8	Double Handbreadth	0.1500m	$\frac{\mathscr{M}}{2\,\pi\,\varphi}$	0.1499m
			$\frac{2\,\mathscr{M}}{9\,\sqrt{\pi\,\varphi}}$	0.1502m
12	Small span	0.2250m	$\frac{\mathscr{M}}{3\,\sqrt{\pi\,\varphi}}$	0.2253m
14	Great span	0.2618m	$\frac{\mathscr{M}\;\varphi}{3\,\pi}$	0.2616m
16	Foot	0.3000m	$\frac{\mathscr{M}}{\pi\,\varphi}$	0.2997m
			$\frac{4\,\mathscr{M}}{9\,\sqrt{\pi\,\varphi}}$	0.3003m
	Remen	0.3702m		
20	Remen	0.3750m	$\frac{5\,\mathscr{M}}{4\,\pi\,\varphi}$	0.3747m
24	Cubit (standard)	0.4500m	$\frac{3\,\mathscr{M}}{2\,\pi\,\varphi}$	0.4496m
			$\frac{2\,\mathscr{M}}{3\,\sqrt{\pi\,\varphi}}$	0.4505m
	Cubit (royal) Cr	0.5236m	$\frac{2\,\mathscr{M}\;\varphi}{3\,\pi}$	0.5231m
32	Pole	0.6000m	$\frac{2\,\mathscr{M}}{\pi\,\varphi}$	0.5995m
			$\frac{\mathscr{M}}{2\,\sqrt{\varphi}}$	0.5989m
36	Nby-Rod	0.67 – 0.68m	$\frac{4\,\mathscr{M}}{9}$	0.6772m

Digits	Length	Value	Formula	Value
64	Double pole	1.2000m	$\frac{4\mathcal{M}}{\pi\varphi}$	1.1989m

Table 14: Formulas for the large cubit divisions using $\mathcal{M}$

Then there are a few formulas that produce interesting values, they have no name but round well to four decimal places.

Length	Value	Formula	Value
1 metre	1.0000m	$\frac{5\,\varphi e}{7\,\pi}=\frac{10\,\varphi e}{7\,\tau}$	1.000m
4 “Egyptian Feet”	1.2000m	$\frac{\pi}{\varphi^2}$	1.2000m
?	1.3000m	$\frac{\sqrt{\pi^2+\varphi^2}}{e}$	1.3000m
?	1.4000m	$\frac{\varphi e}{\pi}$	1.4000m

Table 15: Interesting lengths using famous irrationals.

Table 16 has some assorted formulas, either related to the Ꞡ, digit, $\mathcal{M}$, or other ancient units. At some point there were either “bad copies” or people actually using their shoes, or feet of a statue, as the basis for some official unit of length, which we can’t easily approximate mathematically. Official standards vary over time and complicate the problem, especially when standards get set by decree based on opinion rather than science.

Nevertheless, some relationships are interesting.

Length	Value	Formula	Value
English inch	0.0254m	digit x e/2 $\frac{\pi}{(6\,x\,28)} \times \frac{e}{2} = \frac{\pi e}{336}$	0.0254m
English foot	0.3048m	$\frac{\mathcal{M}}{5}$	03047m
		$\frac{1.524}{5}$	0.3048m
Five English feet	60” = 1.5240m	$1 + \frac{\pi}{6} = \mathcal{M}$	1.5236m
Persian foot	0.32004m	$\frac{\mathcal{M}}{(\pi + \varphi)}$	0.32011m
Doric order foot	±0.324m	$\frac{\pi}{6\,\varphi} = \frac{\text{₢}}{\varphi}$	0.3236m
		$\frac{\pi}{\sqrt{2}\,\varphi^4}$	0.3241m
Luwian foot	±0.323m	$\frac{\pi}{6\,\phi} = \frac{\text{₢}}{\phi}$	0.3236m
Attic foot	0.3084m	$\sqrt{\frac{\mathcal{M}}{16}}$	0.3086m ?
Minoan foot	+-0.304m	$\frac{\mathcal{M}}{5}$	0.3047m
Athenian foot	±0.315m	$\frac{\pi}{10}$	0.3142m
Phoenician foot	0.3000m	$\frac{\pi}{4\,\varphi^2} = \frac{3\,\varphi e}{14\,\pi}$	0.3000m
Megalithic yard	0.8275m 0.8297m	remen x √5	0.8279m
	0.8275m 0.8297m	₢ + foot	0.8284m
Nautical mile	1852m (*currently*)	$100\,\pi\,\varphi\left(\frac{1}{\text{₢}}\right)^2$	1854.1m
		$100\,\pi\,\varphi\left(\frac{1}{0.524}\right)^2$	1851.3m
		$\frac{3600\,\varphi}{\pi}$	1854.1m
		$\frac{5040}{e} = \frac{7\,!}{e}$	1854.1 m

Table 16: Assorted interesting formulas

6. Geometry, the 𐀀 and the metric system

One thing that has bothered me for a long time is the answer to the sceptic's question, "If they had the metre, why didn't they use it instead of the 𐀀?"

I'm purposefully vague about who "they" were.

I still don't have an answer for that, but trying to find it led to something else.

I received guidance that it was connected to the radian. About the same time, YouTube was constantly suggesting that I watch videos about the unit circle. I don't think much of the traditional unit circle done with π, because the τ version is much better and more logical. In the end I gave in and watched part of one, mainly because it was by the very talented NancyPi.

Little did I know that these were strong hints to the answer, which eventually came when I saw a website that pointed out that 30° in radians is π/6. Then the pennies started to fall into place.

The usual way of describing the 𐀀 is as one-sixth of the circumference of a circle with diameter one metre, as in Figure 1.

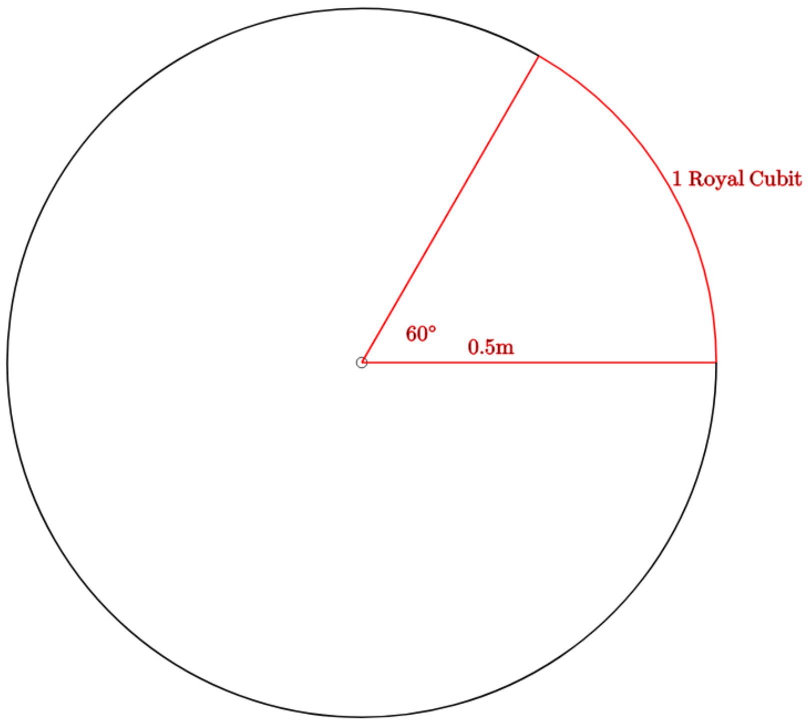

Figure 1: The usual way of showing the 𐀀

Drawing one radian on that diagram does not help, because 1 radian is 57.295°, which is almost 60° and it's hard to see any relationship.

However, if we switch to using a unit circle, with a radius (instead of diameter) of one metre as in Figure 2, then suddenly things work much better, and I rediscovered the elegance.

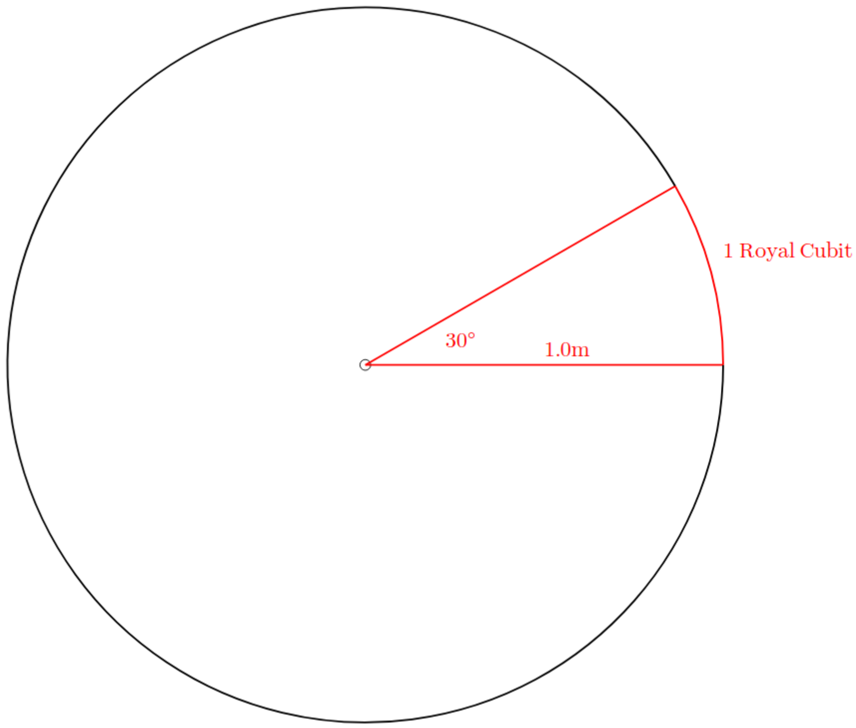

Figure 2: The 𝔊 based on a 1 metre radius circle.

As an aside, this divides the circle in 12, which may connect to things like the zodiac.

So we have a radius of 1 metre, and an arc length of 1 𝔊.

The angle of the arc is 30°, which we can convert to radians:

$$30° = \frac{30\pi}{180} radians = 0.5235987756\, radians$$

We can restate that as:

The angle is $\frac{\pi}{6}$ radians.

The arc length is $\frac{\pi}{6}$ metres.

The radius is 1 metre.

The 𝔊 segment can be viewed as defining a pendulum, with a length of 1 metre, and a swing of 30°.

This is (extremely close to) the seconds pendulum [12], where each swing takes 1 second for a period of 2 seconds. The arc of swing should not exceed 30°. I note the official length at 45° is actually slightly under 1 metre, this may imply that the force of gravity at Giza, or wherever the cubit originated, was slightly different a long time ago.

[To be fair, I rechecked some videos I had watched previously about the seconds pendulum, and the presenter did mention that 30° in radians was numerically the same as the Ͼ, but didn't join the rest of the dots. Nor did it trigger things for me at that time.]

So Figure 2 has the metre and the second. From the metre and some water, we can get the kilogram. This is the basis of the metric system, all encapsulated in a circle showing the royal cubit. We can thus relabel Figure 2 as Figure 3:

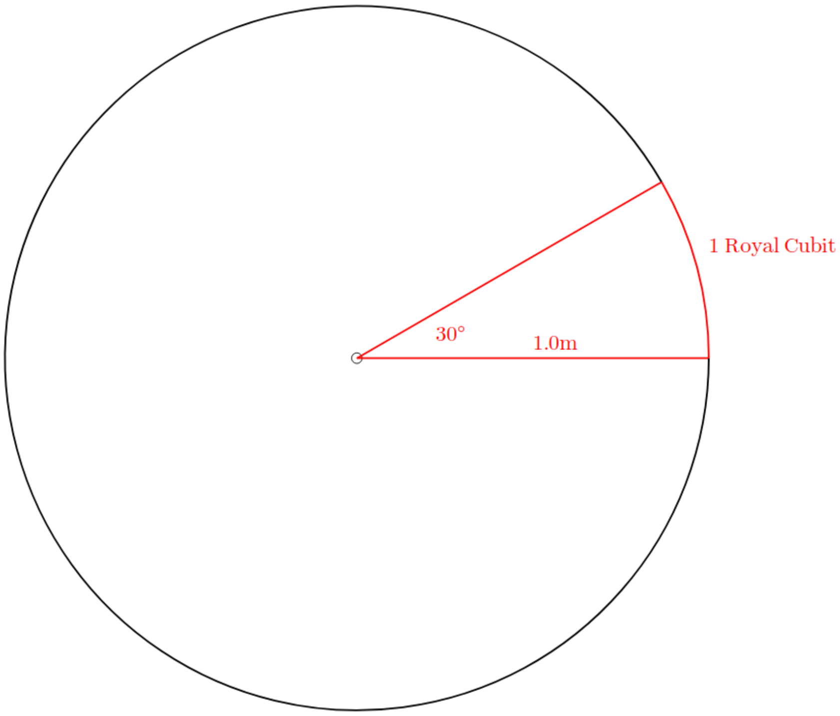

Figure 3: The metric system, summarised.

Welcome to the beautiful cubit system.

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