On Inertial Frames

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Abstract

The transformation of coordinates and time from an inertial frame to another inertial frame is obtained without using rigid measuring-rods and clocks as primitive entities. The obtained transformation is applied to some cases.

1 Introduction

In section 3 of his work on electrodynamics of moving bodies [1] Einstein derives the Lorentz transformation using rigid measuring-rods and clocks both in the "stationary" system as in the system in uniform motion of translation relatively to the former. These two types of physical things, that is, measuring rods and clocks were treated as primitive entities.

In this work we avoid the use of rods and clocks as in [1]. Instead we use hypothetical objects called AM which have very small volume, internal clock and communication capacity with others AM's.

Using the AM's, a system of coordinates, S, is constructed as well another system, S', moving with constant speed with respect to S. The transformation of coordinates and time from S to S' is obtained. The resultant transformation differs from Lorentz' transformation.

In section 2 the transformation found in this work is given. The sections 2 to 8 the proposed transformation is applied to position, speed, Doppler effect, aberration of light, moment, mass, force, energy, electromagnetic field, wave equation and Sagnac's effect. The Schrödinger equation is invariant when the obtained transformation is used. In section 10 some comparisons with the special relativity and a consideration on space properties are made.

2 The Transformations

2.1 Lorentz Transformation

Let S' be a reference system moving with respect to the reference system S with constant speed v in the direction $x+$. The axis-y' and axis-z' are parallel to axis-y and axis-z, respectively. Let us suppose that the zero of t' coincides with the zero of t and the origin of x', y', z' coincides with x, y, z when $t = 0$. The Lorentz transformation is

$$x' = \frac{x - vt}{(1 - \frac{v^2}{c^2})^{1/2}} \tag{1}$$

$$y' = y \tag{2}$$

$$z' = z \tag{3}$$

and

$$t' = \frac{t - \frac{v}{c^2}x}{(1 - \frac{v^2}{c^2})^{1/2}} \tag{4}$$

2.2 Proposed Transformation

Let us assume that the universe is flat, that is, the curvature k is zero and the light velocity c is constant.

Let there be a set of AM's (Automobile Modules). Each AM has a light clock, very small volume and can send light communication signals to any other AM. Two AM's can determine if one is moving with respect to the other or at rest using Doppler Effect. The movement of a AM does not affect the movement or rest of a second AM localized in the vicinity of the first. When an AM sends a signal in a given direction it also emits light in the opposite direction so that the action-reaction effect on the AM is zero. The same action-reaction effect zero due to the emission of light by an AM module also occurs when the module absorbs or reflects a light signal.

2.2.1 The Inertial System S

Let there be two AM's identified by O and X. A light signal a_1 with frequency ν_o was sent from O to X and reflected by X to O with the same frequency ν_0. The position of X in relation to O is being monitored with this signal and ν_0 remains constant. This means that X is not moving with respect to O. Let a_2 be another light signal, this signal is sent by O to X at the instant t_x measured with the clock of O which is immediately reflected back by X and returns to O at the instant $t_x + \Delta t_x$ measured with the clock of O. The distance between O and X is

$$x_u = c\Delta t_x/2 > 0 \tag{5}$$

Let us consider an AM identified by Y. Two light monitoring signals a_3 and a_4 with frequencies ν_1 and ν_2 were sent from O and X to Y, respectively, and they were reflected back to O and X with the same frequencies that were sent. The distances of Y in relation to O and X are being monitored with these signals and the frequencies ν_1 and ν_2 remain constant. Let a_5 be a light signal, this signal is sent by O to Y at the instant t_y measured with the clock of O, it is reflected back by Y, and returns to O at the instant $t_y + \Delta t_y$ measured with the clock of O. The distance between O and Y is

$$y_u = c\Delta t_y/2 > 0 \tag{6}$$

Let a_6 be a light signal with a frequency ν_3, this signal is sent by X to Y at the instant t_x measured with the clock of X, it is reflected back by Y, and returns to X at the instant $t_x + \Delta t_{xy}$ measured with the clock of X. The distance between X and Y is

$$d_{XY} = c\Delta t_{xy}/2 > 0 \tag{7}$$

The distance d_{XY} is communicated by X to an observer in O which verifies that

$$d_{XY} < x_u + y_u \tag{8}$$

that is, Y is not in the straight line that passes by O and X.

Let us consider an AM identified by Z. Three light monitoring signals a_7, a_8, and a_9 with frequencies ν_4, ν_5 and ν_6 were sent from O, X, and Y to Z and were reflected back to O, X, and Y with the same frequencies, respectively. These signals show that Z is not moving with respect to O, X, and Y. Let a_{10} be a light signal sent by O to Z at the instant t_z measured with the clock of O which is reflected back by Z and returns to O at the instant $t_z + \Delta t_z$. The distance between O and Z is

$$z_u = c\Delta t_z/2 > 0 \tag{9}$$

Using the information on the light signals among X, Y, and Z provided to O the observer in O verifies that the straight line determined by the positions of O and Z is not in the plane that contains the positions of O, X, and Y.

Let the positions of O, X, Y, and Z be such that

$$y_u = \frac{1}{\sqrt{2}}\Big(c\Big(\frac{\Delta t_{XY}}{2}\Big)\Big) \tag{10}$$

$$z_u = \frac{1}{\sqrt{2}}\Big(c\frac{\Delta t_{XZ}}{2}\Big) = \frac{1}{\sqrt{2}}\Big(c\frac{\Delta t_{YZ}}{2}\Big) \tag{11}$$

and

$$x_u = y_u = z_u \tag{12}$$

where (i) Δt_{XY} is the time interval for a signal to be sent from X to Y and reflected back by Y to X, (ii) Δt_{XZ} is the time interval for a signal to be sent from X to Z and reflected back by Z to X,(iii) Δt_{YZ} is the time interval for a signal to be sent from Y to Z and reflected back by Z to Y and (iv) x_u is the length unit. The position of O is the origin of the inertial system S. The axis-x is the straight line determined by the positions of O and X. Analogously the axis-y and axis-z are the straight lines determined by the positions of O and Y, and O and Z.

Let P an AM be at rest with respect to O, X, Y and Z. The distances from P to O, X, Y and Z are measured using the same procedure used to determine the distance from O to X above. Then the distances from the position P to axis-x, axis-y and axis-z can be calculated. The position of P can be put in the form $(a_x x_u,\, a_y y_u,\, a_z z_u)$, where a_x, a_y, and a_z are real numbers.

The synchronization of the clocks of O and P, where P is at rest in S, is obtained considering that the light speed is constant and needs the time interval equal to the distance from O to P divided by c to move from O to P. Let g be a light signal sent by P to O which is reflected back to P and Δt_g the time interval that g spent to go to O and return to P. When g arrives at P the time of the clock of P is t_2. As soon as the signal g returns from P to O the module O sends a signal h to P informing $(t_1 + \Delta t_g/2)$, where t_1, the time measured in O when g leaves O. The synchronization is obtained setting the clock of P such that the value t_2 is changed to the value when h is received.

$$t_1 + \Delta t_g/2 \tag{13}$$

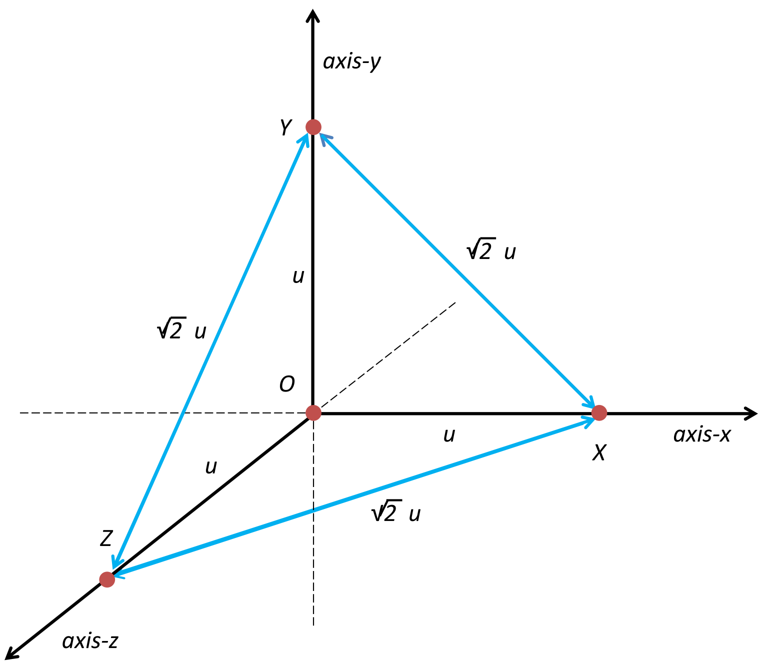

Figure 1: System S

Consider the point p_x on the axis-x, the point p_y on the axis-y and the point p_z on the axis-z. The distances from these three points to the origin of S are equal. Be an observer in p_z looking at the plane-xy. If, for this observer, the rotation of p_x in the plane-xy of $\pi/2$ radians in the left-hand direction results in the coincidence of the p_x with p_y, then p_x is on the positive part of the axis-x, p_y and is on the positive part of the axis-y, and p_z is in the part of the axis-z.

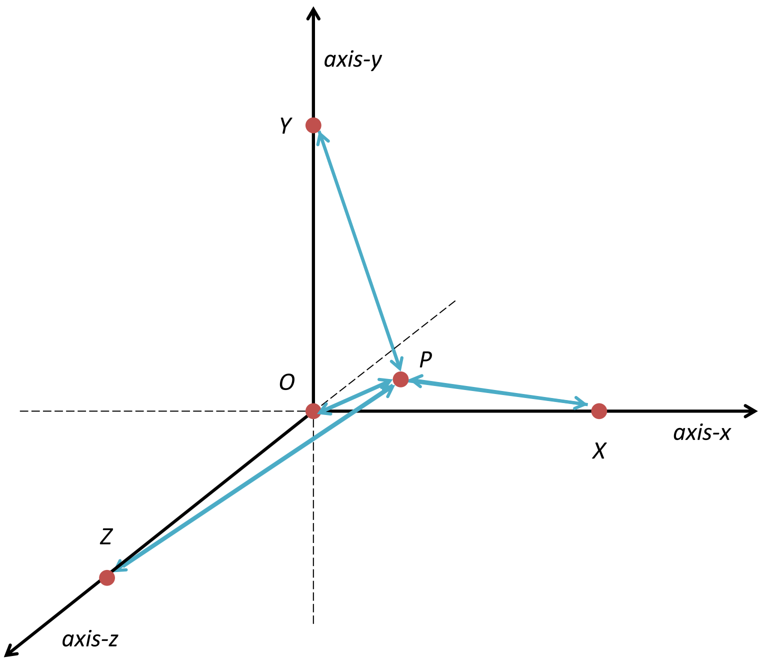

Figure 2: Point p in the system S

2.2.2 The Inertial System S'

We can assume that the origins of S and S' are coincident: when $(x = 0, y = 0, z = 0, t = 0)$ we have $(x' = 0, y' = 0, z' = 0, t' = 0)$. Let there be three planes: (i) the plane *mirror1* such that $z = 0$; (ii) the plane *mirror2* such that $z = z_0$ with $z_0 > 0$; and (iii) the plane *trajectory* such that $y = 0$. Let us consider a photon emitted by a source at the origin of the system S. The trajectory of the photon is in *trajectory* and the angle that the photon form with the axis-x when it leaves the source is θ, where $0 < \theta < \pi/2$.

Now let (i) be a AM identified by A which is moving in system S along axis-x in the positive direction with constant speed v, (ii) an AM identified by B which is moving on plane *trajectory* at speed v along a straight line parallel to axis-x in the positive direction with $z = z_0$, (iii) a AM identified by C at rest in position $(0, 0, z_0)$, and (iv) two AM's identified by D and E both at rest in positions $(x_0/2, 0, z_0)$ and $(x_0, 0, 0)$ with $x_0 > 0$, respectively. When A reaches the origin of S the position of B is $(0, 0, z_0)$ and the clocks of O, A, B, and C are synchronized at $t = t' = 0$.

At $t = t_0$ the module A emits a photon forming a positive angle θ with the axis-x less than $\pi/2$. Consider the angle θ such that, when the photon is reflected back by *mirror2* it arrives at the *mirror1* and the module A in the position of the axis-x equal to

$$v\Delta t \tag{14}$$

In this case the angle is given by

$$\theta = \arctan\left(\frac{z_0}{v(\Delta t/2)}\right) \tag{15}$$

Δt is the time interval between two successive reflections of the photon in the *mirror1*.

The time interval spent for light emitted from O at $t = 0$ in direction given by the angle $\pi/2$ to returns to O after have been reflect back by *mirror2* and the module C is

$$\Delta t_1 = \frac{2z_0}{c} \tag{16}$$

The time interval observed in S for the light emitted from A at the time $t = 0$ and θ given by (15) to arrives at the *mirror2* and B and be reflected back to A is Δt_2 given in

$$\frac{2\sqrt{(v\Delta t_2/2)^2 + z_0^2)}}{c} = \Delta t_2 \tag{17}$$

Substituting (16) into (17) we obtain

$$\Delta t_2 = \gamma \Delta t_1 \tag{18}$$

where

$$\gamma = \frac{1}{\sqrt{1 - (v/c)^2}} \tag{19}$$

For the subsystem formed by A, B and the photon, the light is emitted from A directly to B and reflected back directly to A. This subsystem is a light clock, let us call it *clock-AB*. Let *clock-OC* be the subsystem formed by O, C, and a photon emitted by O at $t = 0$ at the angle $\pi/2$. If we consider (i) the reflection of the photon by the *mirror2* in the position of B as the "tick" of the *clock-AB* and the reflection of the photon by the *mirror1* in the position of A as its "tock" (ii) the reflection of the photon by the *mirror2* in the position of C as the "tick" of the *clock-OC* and the reflection of the photon by the *mirror1* in the position of O as its "tock", then the "tick-tock" in the *clock-AB* flows more slowly than the "tick-tock" of the *clock-OC* by the factor γ.

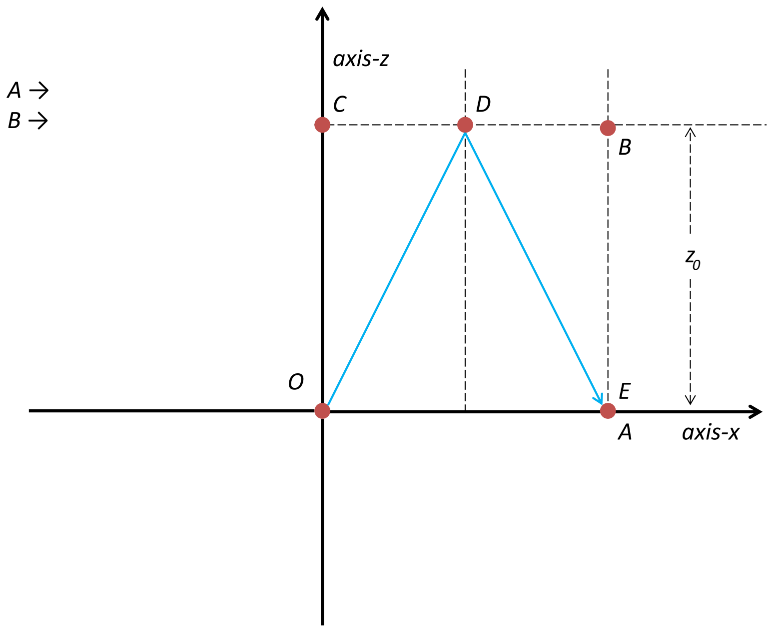

Figure 3: $Clock - AB$

Let x_0 be such that (i) when the first "tick" occurs in the *clock-AB* the module B coincides with D and at this instant B communicates to D the "tick" occurrence and (ii) when the first "tock" occurs in the *clock-AB* the module A coincides with E and at this instant A communicates to E the "tock" occurrence. Therefore for a number n of "tick-tock"

$$t' = \frac{n\Delta t_1}{n\Delta t_2} t = \frac{t}{\gamma} \tag{20}$$

where $n >>> 1$, t is the time of S and t' is the time of A.

Let us rename as O' and Y' the modules A and B, respectively. Let in O' be the origin of S' and the AM modules X' and Z' of S' be moved in such way that the distances from X' and Z' to O' stay constant with the time t' and

(a) X' and O' determine the axis-x', the axis-x and axis-x' are in the same straight line with the same positive direction

(b) Y' and O' determine the axis-y' and the axis-y' is in the plane-xy

(c) Z' and O' determine the axis-z' and the axis-z' is in the plane-xz

(d) when $t' = t = 0$ the axis-y' coincides with axis-y and they have the same positive direction

(e) when $t' = t = 0$ the axis-z' coincides with axis-z and they have the same positive direction.

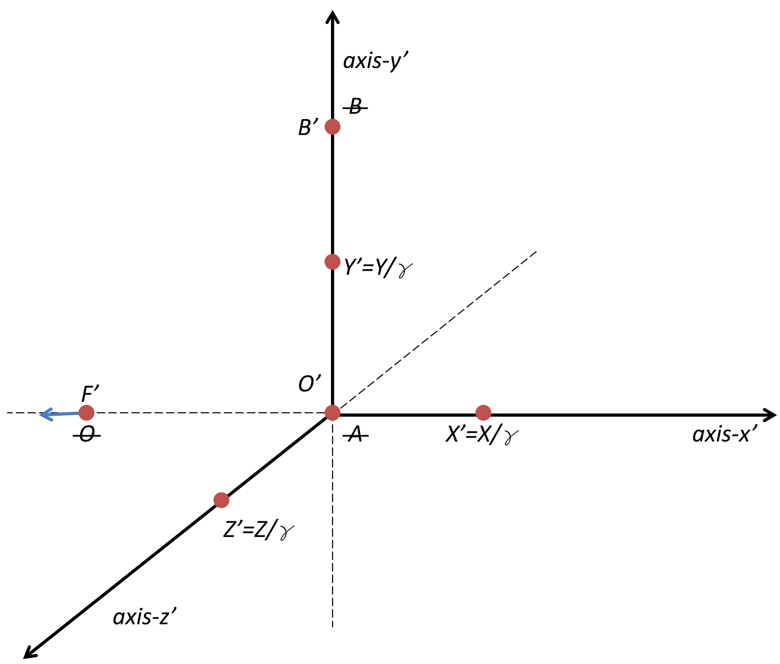

Figure 4: System S'

The same procedure used to synchronize the clocks in the system S is used in S'. We have that the time is the same in all positions of the system S'. The time of O' is the time of S'.

Similarly to S the distance between two AM's in S' is the time (t' in this case) that the light takes to go from one AM to the other multiplied by the speed of light.

The time interval that the light spent to go from (x_i, y, z) to (x, y, z) in S is

$$t - t_i = \frac{x - x_i}{c} \tag{21}$$

From (20) this time interval when measured in S' becomes

$$t' - t'_i = \frac{x - x_i}{\gamma c} \tag{22}$$

and the distance in S' corresponding to $t' - t'_i$ is

$$x' - x'_i = c(t' - t'_i) = \frac{x - x_i}{\gamma} \tag{23}$$

for $x_i = 0$ we have

$$x' = \frac{x}{\gamma} \tag{24}$$

The time interval that the light spent to go from (x, y_j, z) to (x, y, z) in S is

$$t - t_j = \frac{y - y_j}{c} \tag{25}$$

This time interval when measured in S' becomes

$$t' - t'_j = \frac{y - y_j}{\gamma c} \tag{26}$$

and the distance in S' corresponding to $t' - t'_j$ is

$$y' - y'_j = c(t' - t'_j) = \frac{y - y_j}{\gamma} \tag{27}$$

for $y_j = 0$ we have

$$y' = \frac{y}{\gamma} \tag{28}$$

In the same way the distance in S' corresponding to the distance in S from (x, y, z_k) to (x, y, z) is

$$z' - z'_k = c(t' - t'_k) = \frac{z - z_k}{\gamma} \tag{29}$$

for $z_k = 0$ we have

$$z' = \frac{z}{\gamma} \tag{30}$$

in the inertial system S'.

3 Speed and Movement in S'

Let p be a particle moving in S with uniform speed v. The components v'_x, v'_y, v'_z of the speed of the particle relatively to the referential S' are

$$v'_x = \frac{dx'}{dt'} = \frac{\frac{dx}{\gamma}}{\frac{dt}{\gamma}} = \frac{dx}{dt} = v_x \tag{31}$$

$$v'_y = \frac{dy'}{dt'} = \frac{\frac{dy}{\gamma}}{\frac{dt}{\gamma}} = \frac{dy}{dt} = v_y \tag{32}$$

$$v'_z = \frac{dz'}{dt'} = \frac{\frac{dz}{\gamma}}{\frac{dt}{\gamma}} = \frac{dz}{dt} = v_z \tag{33}$$

where v_x, v_y, v_z are the components of the speed of the particle relatively to the referential S.

Let the position of a particle along the time in S be given by

$$p_x(t) = x_0 + v_x t \tag{34}$$

$$p_y(t) = y_0 + v_y t \tag{35}$$

$$p_z(t) = z_0 + v_z t \tag{36}$$

where v_x, v_y, v_z and x_0, y_0, z_0 are constants. The corresponding coordinates of the particle along the time in S' are

$$p_{x'}(t') = x'_0 + v'_x t' \tag{37}$$

$$p_{y'}(t') = y'_0 + v'_{y'} t' \tag{38}$$

$$p_{z'}(t') = z'_0 + v'_{z'} t' \tag{39}$$

Considering (20), (24), (28), (30), (37), (38) and (39) we obtain

$$p_{x'}(t') = \frac{x_0 + v_x t}{\gamma} \tag{40}$$

$$p_{y'}(t') = \frac{y_0 + v_y t}{\gamma} \tag{41}$$

$$p_{z'}(t') = \frac{z_0 + v_z t}{\gamma} \tag{42}$$

4 Doppler Effect

Let s_r be a source of light at rest in the system S and emitting with the frequency ν_r. Let us consider another source of light s_v moving with uniform speed v in S. The monochromatic light emitted by s_v originates from the same process that produces the light of s_r. An observer o_r is at rest in the origin of S and at the instant t_a the observed position of s_v is (s_{vx}, s_{vy}, s_{vz}). The source s_v is at rest with respect to the system S'.

In S' the light is emitted with the frequency ν_v, using (20) this frequency is transformed to

$$\nu'_v = \frac{1}{\Delta t_v} = \frac{\gamma}{\Delta t_r} = \gamma \nu_r \tag{43}$$

where Δt_v and Δt_r are the periods measured in S' an S, respectively.

Let us consider, without loss of generality, that $s_{vy} = s_{vz} = 0$. At the instant t_a the component of v in the straight line which passes by by s_v and o_r is

$$v_{os} = v \cos\theta \tag{44}$$

where

$$\cos\theta = \frac{v_x}{\sqrt{v_x^2 + v_y^2 + v_z^2}} = \frac{v_x}{v} \tag{45}$$

The frequency ν_o observed by o_r is such that

$$\frac{c - v_{os}}{\nu_o} = \frac{c}{\nu_v} \tag{46}$$

or

$$\nu_o = \nu_v \Big(1 - \frac{v_{os}}{c}\Big) \tag{47}$$

Substituting (43) and (44) into (47) we obtain

$$\nu_o = \gamma \nu_r \Big(1 - \frac{v\cos\theta}{c}\Big) \tag{48}$$

The longitudinal Doppler occurs when $\theta = 0$, that is,

$$\nu_o = \gamma \nu_r \Big(1 - \frac{v}{c}\Big) \tag{49}$$

When $\theta = \pi/2$ we have the transversal Doppler

$$\nu_o = \gamma \nu_r \tag{50}$$

5 Aberration of Light

Let s_r be a source of light at rest in S which at instant $t_i = 0$ begins to emit a pulse of light. The spatial position of s_r is $(0, p_y, 0)$ with $p_y > 0$.

The pulse of light spends

$$t_f - t_i = \frac{p_y}{c} \tag{51}$$

to reach the origin of S. In the system S' the time interval for the pulse emitted by the source to reach the axis-x' is

$$t'_f - t'_i = \frac{t_f - t_i}{\gamma} = \frac{p_y}{\gamma c} \tag{52}$$

In S' the position corresponding to the origin of S for $t'_i = 0$ moves with speed v in the direction x'^-. The pulse reaches the axis-x' at position

$$p'_x = -(t'_f - t'_i)v \tag{53}$$

An observer o'_r located in $(p'_x, 0, 0)$ of S' will see the arriving light forming an angle α with axis-x'. Since for $\theta = \frac{\pi}{2} - \alpha$ the adjacent side is

$$\sqrt{\Big((t'_f - t'_i)c\Big)^2 - \Big((t'_f - t'_i)v\Big)^2} \tag{54}$$

we can write

$$\tan\theta = -\frac{(t'_f - t'_i)v}{\sqrt{\Big((t'_f - t'_i)c\Big)^2 - \Big((t'_f - t'_i)v\Big)^2}} \tag{55}$$

or

$$\tan\theta = -\gamma\frac{v}{c} \tag{56}$$

6 Moment, Mass, Force, and Energy

6.1 Moment

Now let us consider the moment of a particle in the form

$$p = \frac{h}{\lambda} \tag{57}$$

where h is the Planck constant and λ is length of the wave associated with the particle. Let us assume that h is invariant under the transformation from S to S'. Therefore we have

$$p' = \frac{h}{\lambda'} = \frac{h}{\frac{\lambda}{\gamma}} = \gamma p \tag{58}$$

6.2 Mass

The moments p and p' are

$$p = m(v)\ v \tag{59}$$

where $m(v)$ denotes that m is a function of v and

$$p' = m'(v')\ v' \tag{60}$$

Substituting (59) and (60) into (58) we obtain

$$m'(v')\ v' = \gamma m(v)\ v \tag{61}$$

Seeing that $v' = v$ we have

$$m'(v) = \gamma m(v) \tag{62}$$

From (62) we note that the rest mass m in S increases to the rest mass γm in S'.

The dimension of the Planck constant is ML^2T^{-1}. Seeing that the part corresponding to L^2T^{-1} is divided by γ when transformed from S to S' and that h is invariant under the transformation, we conclude that the part corresponding to M is multiplied by γ when transformed from S to S', that is,

$$m' = \gamma m \tag{63}$$

6.3 Force

In system S' let v' be equal to v'_x. The component x of the force is given by

$$F'_x = \frac{d}{dt'}p'_x \tag{64}$$

and the component x' of the moment is

$$p'_x = \gamma m v_x \tag{65}$$

Since v'_x changes with the time we have

$$\frac{d\gamma}{dt} = \frac{d}{dt}\left(\frac{1}{\sqrt{1-\frac{v_x^2}{c^2}}}\right) = \frac{\gamma^3 v_x}{c^2}\frac{dv_x}{dt} \tag{66}$$

considering that

$$dt' = \frac{dt}{\gamma} \tag{67}$$

we find

$$F'_x = \frac{d}{dt'}\Big(\gamma m v_x\Big) = m\left(v_x\frac{d\gamma}{dt'} + \gamma\frac{dv_x}{dt'}\right) \tag{68}$$

$$F'_x = m\frac{dv_x}{dt'}\left(\frac{\gamma^2 v_x^2}{c^2} + \gamma\right) \tag{69}$$

or

$$F'_x = F_x\left(\frac{\gamma^3 v_x^2}{c^2} + \gamma^2\right) \tag{70}$$

6.4 Energy

The work realized by a force F to accelerate a particle from rest to speed v is the kinetic energy E_c of the particle. For simplicity let us consider F in direction x^+,

$$E_c = \int_0^v F ds = \int_0^v \frac{d(m(v)v)}{dt}ds = \int_0^v \frac{d(\gamma m v)}{dt}ds = \int_0^v v d(\gamma m v) \tag{71}$$

where

$$d(\gamma m v) = \left(\frac{v^2}{c^2\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} + \gamma\right)dv \tag{72}$$

so

$$E_c = \int_0^v v\left(\frac{v^2}{c^2\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} + \gamma\right)dv = mc^2(\gamma - 1) \tag{73}$$

mc^2 is the rest energy and the total energy of the free particle is

$$\gamma mc^2 = E_c + mc^2 \tag{74}$$

6.5 Example with a star binary system

Let be a binary system composed of two stars. One of the stars has mass m, the other has mass M and $m <<< M$. The Figure 5 shows the star system in S.

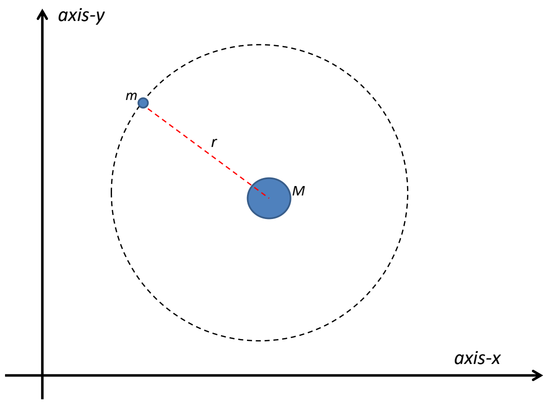

Figure 5: A star binary system

The force exerted on the smaller star is

$$F = G\frac{Mm}{r^2} \tag{75}$$

where G is the gravitational constant and r is the distance between the centers of the two stars. The orbital period is

$$T = \sqrt{\frac{2\pi r^3}{GM}} \tag{76}$$

As usual the system S' noves with speed u, where $u = u_x$, and using the equation (70) the force is

$$F'_x = \gamma^2 F_x \tag{77}$$

and the radius

$$r' = \frac{r}{\gamma} \tag{78}$$

The force variation with the inverse of the square of the distance holds in S'. The unity of G is $N.m^2kg^{-2}$ or $[L^3T^{-2}kg^{-1}]$ and

$$G' = \frac{G}{\gamma^2} \tag{79}$$

As expected the orbital period of the smaller star is

$$T' = \sqrt{\frac{2\pi\left(\frac{r}{\gamma}\right)^3}{\left(\frac{G}{\gamma^2}\right)\left(\gamma M\right)}} = \frac{T}{\gamma} \tag{80}$$

7 Electromagnetic Field

The differential forms of the Maxwell's microscopic equations are

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \tag{81}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = \frac{\partial \boldsymbol{B}}{\partial t} \tag{82}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{83}$$

$$\nabla \times \boldsymbol{B} = \epsilon_0 \mu_0 \frac{\partial \boldsymbol{E}}{\partial t} + \mu_0 \boldsymbol{J} \tag{84}$$

Let us notice that ϵ_0 has dimension $[L^{-3}.M^{-1}.T^2.C^2]$ and μ_0 has the dimension $[L.M.C^2]$. It is easy to see that ϵ_0 and μ_0 are invariant under the proposed transformation.

The integral form of (81) is

$$\oint_s \boldsymbol{E} \cdot d\boldsymbol{A} = \int_V \frac{\rho}{\epsilon_0} dV \tag{85}$$

Let us consider in S an electrical charge Q confined inside the spherical surface s given by

$$x^2 + y^2 + z^2 = r^2 \quad \text{with} \quad r > 0 \tag{86}$$

In this case

$$\int_V \frac{\rho}{\epsilon_0} dV = \frac{Q}{\epsilon_0} \tag{87}$$

where $V = \frac{4}{3}\pi r^3$. For the system S' the spherical surface s is transformed to

$$(x' - vt')^2 + y'^2 + z'^2 = r'^2 \quad \text{with} \quad r' > 0 \tag{88}$$

The integral

$$\oint_{s'} \boldsymbol{E'} \cdot d\boldsymbol{A'} \tag{89}$$

over the spherical surface s' given by (88) is

$$\oint_{s'} \boldsymbol{E'} \cdot d\boldsymbol{A'} = \frac{1}{4\pi\epsilon_0} \frac{Q'}{r'^2} \oint_{s'} dA' = \frac{Q'}{\epsilon_0} \tag{90}$$

Since

$$\int_{v'} \frac{\rho'}{\epsilon_0} dV' = \int_v \frac{(\gamma^3 \rho)}{\epsilon_0} \Big(\frac{dV}{\gamma^3}\Big) = \frac{Q}{\epsilon_0} \tag{91}$$

we conclude that

$$\oint_{s'} \boldsymbol{E'} \cdot d\boldsymbol{A'} = \oint_s \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{Q}{\epsilon_0} \tag{92}$$

The charge is invariant under the proposed transformation.

The action of the electromagnetic field on a charged particle q is given by

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{93}$$

From section 6.3 we have

$$\boldsymbol{F'} = \Big(\frac{\gamma^3 v^2}{c^2} + \gamma^2\Big)\boldsymbol{F} \tag{94}$$

therefore

$$\boldsymbol{F'} = q\bigg(\Big(\frac{\gamma^3 v^2}{c^2} + \gamma^2\Big)\boldsymbol{E} + \boldsymbol{v} \times \Big(\frac{\gamma^3 v^2}{c^2} + \gamma^2\Big)\boldsymbol{B}\bigg) \tag{95}$$

Provided that

$$\boldsymbol{F'} = q(\boldsymbol{E'} + \boldsymbol{v'} \times \boldsymbol{B'}) = q(\boldsymbol{E'} + \boldsymbol{v} \times \boldsymbol{B'}) \tag{96}$$

we conclude

$$\boldsymbol{E'} = \Big(\frac{\gamma^3 v^2}{c^2} + \gamma^2\Big)\boldsymbol{E} \tag{97}$$

and

$$\boldsymbol{B'} = \Big(\frac{\gamma^3 v^2}{c^2} + \gamma^2\Big)\boldsymbol{B} \tag{98}$$

As example, let be the case in the system S shown in Figure 6. The electrical current is

$$I = 2(\eta_+ - \eta_-)v \tag{99}$$

and

$$v = v_+ = -v_- \tag{100}$$

where

η_+ - positive charge per unit length

η_- - negative charge per unit length [a negative value]

v_+ - drift speed of the positive electrical charges on the conductor

v_- - drift speed of the negative electrical charges on the conductor

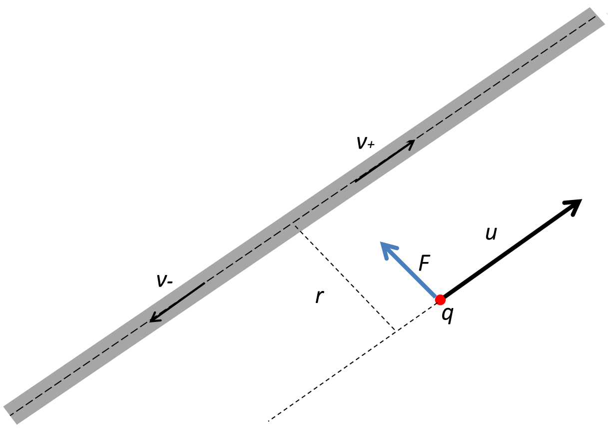

Figure 6: A charge q moving parallel to a conductor with current I

The electric and magnetic fields and the force are

$$\boldsymbol{E} = (\eta_+ + \eta_-)\frac{\boldsymbol{r}}{2\epsilon_0 r} = 0 \tag{101}$$

$$\boldsymbol{B} = \mu_0 I \frac{\boldsymbol{\varphi}}{2\pi r} \tag{102}$$

and

$$\boldsymbol{F} = -q\boldsymbol{u} \times \boldsymbol{B} \tag{103}$$

where

q - positive electric charge

u - speed of the charge q

r - distance from the charge q to the central axis of the conductor

For system S' moving in relation to S with speed $u_x = u$, $u_y = 0$ and $u_z = 0$ we have

$$\gamma = \frac{1}{\sqrt{1-(\frac{u}{c})^2}} \tag{104}$$

$$v'_+ = v_+ \tag{105}$$

$$v'_- = v_- \tag{106}$$

$$\eta'_+ = \gamma\eta_+ \tag{107}$$

$$\eta'_- = \gamma\eta_- \tag{108}$$

and

$$u' = u \tag{109}$$

therefore

$$\boldsymbol{E'} = \gamma(\eta_+ + \eta_-)\frac{(\boldsymbol{r}/\gamma)}{2\epsilon_0(r/\gamma)} = 0 \tag{110}$$

$$\boldsymbol{B'} = \mu_0(\gamma I)\frac{\boldsymbol{\varphi}}{2\pi(r/\gamma)} = \gamma^2\boldsymbol{B} \tag{111}$$

and

$$\boldsymbol{F'} = -q\boldsymbol{u'} \times \boldsymbol{B'} = -\gamma^2 q\boldsymbol{u} \times \boldsymbol{B} = \gamma^2\boldsymbol{F} \tag{112}$$

8 Wave Equation

Let us represent the plane wave by

$$\Psi(\boldsymbol{x}, t; \boldsymbol{k}) = A\exp\left(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t\right) \tag{113}$$

where A is the amplitude of the wave, $\boldsymbol{k}$ the wave vector, and ω the angular speed. For a particle associated to the wave (113) and moving in the system S with speed v we have

$$v = \frac{d\omega}{dk} \tag{114}$$

For simplicity, let the particle be at rest in the origin of the system S'. De Broglie proposes that the relation $E = \hbar\omega$ holds for material particles, that is,

$$\gamma mc^2 = \hbar\omega \tag{115}$$

Let us assume

$$\omega = vk \tag{116}$$

From (115) and (116) we can obtain

$$k\frac{v^2\hbar}{c^2} = p \tag{117}$$

The energy and the angular speed can be written as

$$E = \gamma mc^2 = (\gamma mv)\frac{c^2}{v} = p\frac{c^2}{v} \tag{118}$$

and

$$\omega = \frac{E}{\hbar} = p\frac{c^2}{\hbar v} \tag{119}$$

Substituting the $\boldsymbol{k}$ of the vetorial form of (117) and ω from (119) into (113) we have

$$\Psi(\boldsymbol{x}, t; \boldsymbol{p}) = A\exp\Big(\frac{ic^2}{\hbar v^2}\boldsymbol{p}\cdot\boldsymbol{x} - \frac{ipc^2}{\hbar v}t\Big) \tag{120}$$

Using

$$\varphi(\boldsymbol{x};\boldsymbol{p}) = A\exp\Big(\frac{ic^2}{\hbar v^2}\boldsymbol{p}\cdot\boldsymbol{x}\Big) \tag{121}$$

and

$$\phi(t) = \exp\Big(-\frac{ipc^2}{\hbar v}t\Big) \tag{122}$$

the equation (120) becomes

$$\Psi(\boldsymbol{x},t;\boldsymbol{p}) = \varphi(\boldsymbol{x};\boldsymbol{p})\phi(t) \tag{123}$$

The Laplacian of $\varphi(\boldsymbol{x};\boldsymbol{p})$ and the second derivative of $\phi(t)$ are

$$\nabla^2\varphi(\boldsymbol{x};\boldsymbol{p}) = -A\Big(\frac{pc^2}{\hbar v^2}\Big)^2\exp\Big(\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{x}\Big) = -\Big(\frac{pc^2}{\hbar v^2}\Big)^2\varphi(\boldsymbol{x};\boldsymbol{p}) \tag{124}$$

and

$$\frac{d^2\phi(t)}{dt^2} = -\Big(\frac{pc^2}{\hbar v}\Big)^2\exp\Big(-\frac{ipc^2}{\hbar v}t\Big) = -\Big(\frac{pc^2}{\hbar v}\Big)^2\phi(t) \tag{125}$$

Since

$$\frac{d^2\Psi(\boldsymbol{x},t;\boldsymbol{p})}{d\boldsymbol{x}^2} = [\nabla^2\varphi(\boldsymbol{x};\boldsymbol{p})]\phi(t) \tag{126}$$

and

$$\frac{d^2\Psi(\boldsymbol{x},t;\boldsymbol{p})}{dt^2} = \varphi(\boldsymbol{x};\boldsymbol{p})\frac{d^2\phi(t)}{dt^2} \tag{127}$$

we have

$$\frac{d^2\Psi(\boldsymbol{x},t;\boldsymbol{p})}{d\boldsymbol{x}^2} = -\Big(\frac{pc^2}{\hbar v^2}\Big)^2\Psi(\boldsymbol{x},t;\boldsymbol{p}) \tag{128}$$

and

$$\frac{d^2\Psi(\boldsymbol{x},t;\boldsymbol{p})}{dt^2} = -\Big(\frac{pc^2}{\hbar v}\Big)^2\Psi(\boldsymbol{x},t;\boldsymbol{p}) \tag{129}$$

From (128) and (129) we find the wave equation

$$\frac{d^2\Psi(\boldsymbol{x},t;\boldsymbol{p})}{d\boldsymbol{x}^2} = \frac{1}{v^2}\frac{d^2\Psi(\boldsymbol{x},t;\boldsymbol{p})}{dt^2} \tag{130}$$

In the system S' the equation (130) is

$$\frac{d^2\Psi'(\boldsymbol{x'},t';\boldsymbol{p'})}{d\boldsymbol{x'}^2} = \frac{1}{v'^2}\frac{d^2\Psi'(\boldsymbol{x'},t';\boldsymbol{p'})}{dt'^2} \tag{131}$$

with the solution

$$\Psi'(\boldsymbol{x'},t';\boldsymbol{p'}) = A\exp\Big(\frac{ic^2}{\hbar v'^2}\boldsymbol{p'}\cdot\boldsymbol{x'} - \frac{ip'c^2}{\hbar v'}t'\Big) \tag{132}$$

Substituting

$$\boldsymbol{x'} = \frac{\boldsymbol{x}}{\gamma} \tag{133}$$

$$t' = \frac{t}{\gamma} \tag{134}$$

$$v' = v \tag{135}$$

and

$$\boldsymbol{p'} = \boldsymbol{k'}\frac{\boldsymbol{v'}^2\hbar}{c^2} = \boldsymbol{k'}\frac{\boldsymbol{v}^2\hbar}{c^2} = (\gamma\boldsymbol{k})\frac{\boldsymbol{v}^2\hbar}{c^2} = \gamma\boldsymbol{p} \tag{136}$$

into (132) we find

$$\Psi'(\boldsymbol{x'},t';\boldsymbol{p'}) = \Psi(\boldsymbol{x},t;\boldsymbol{p}) \tag{137}$$

On (137) it is worth mentioning John von Neumann [2]

"First of all we must admit that this objection points at an essential weakness which is, in fact, the chief weakness of quantum mechanics: its non-relativistic character, which distinguishes the time t from the three space coordinates x, y, z, and presupposes an objective simultaneity concept. In fact, while all other quantities (especially those x, y, z, closely connected with t by the Lorentz transformation) are represented by operators, there corresponds to the time an ordinary number-parameter t, just as in classical mechanics."

9 Sagnac Effect

Consider a rigid Sagnac interferometer shown in Figure 7. The two counter-propagating light beams share a common optical circular path with radius r for the interferometer at rest ($\omega = 0$). In the system S' the angular speed of the interferometer is ω'.

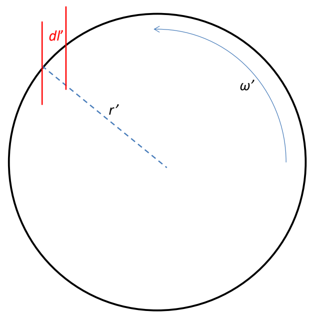

Figure 7: A rigid Sagnac interferometer

Using the proposed transformation we have

$$\omega' = \frac{v'}{r'} = \frac{\frac{dl/\gamma}{dt/\gamma}}{\frac{r}{\gamma}} = \gamma \frac{\frac{dl}{dt}}{r} \tag{138}$$

Let us assume that the speed u of S' in relation to S is

$$v = \frac{dl}{dt} \tag{139}$$

therefore

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \tag{140}$$

Let us denote $\frac{v}{r}$ by ω. In the system S' the time intervals for the two light beams cross dl' are

$$\frac{dl'}{c + v'} \tag{141}$$

and

$$\frac{dl'}{c - v'} \tag{142}$$

The difference between (142) and (141) is

$$\delta t' = \left(\frac{1}{c - v'} - \frac{1}{c + v'}\right) dl' = \frac{2v'}{c^2 - v'^2} dl' \tag{143}$$

for the complete path the difference is

$$\Delta t' = \frac{2v'}{c^2 - v'^2} 2\pi r' \tag{144}$$

Since that

$$r' = \frac{r}{\gamma} \tag{145}$$

$$\frac{1}{\gamma} \frac{2v'}{(c^2 - v'^2)} = \gamma \frac{2v'}{c^2} \tag{146}$$

and

$$v' = v \tag{147}$$

we obtain

$$\Delta t' = \frac{4v}{c^2 - v^2} \pi r = \gamma \frac{4\omega(\pi r^2)}{c^2} = \gamma \frac{4\omega A}{c^2} \tag{148}$$

where A is the area limited by the circumference. If

$$v << c \tag{149}$$

then

$$\Delta t' \cong \frac{4\omega A}{c^2} \tag{150}$$

The equation (150) has the same form that the equation (30) found in Post [3].

10 Comparisons with the special relativity and a consideration on space properties

10.1 The energy that capacitors can store in S' after a rotation

Consider capacitance of the capacitors shown in Figure 8. Each capacitor consists of two parallel square plates both of area A separated by a distance d such that $d << \sqrt{A}$. Both capacitors are presented in system S. The two plates of one of capacitors (denoted by α) are parallel to the plane *x-z* and the two plate of the other capacitor (denoted by β) are parallel to the plane *y-z*. Both capacitors are moving in direction *axis-*x^+ with velocity u. The system S' is moving with respect to the system S with the same speed u.

The capacitance of the capacitors α and β are obtained with good approximation by

$$C = \epsilon_0 \frac{A}{d} \tag{151}$$

and the stored energy in each capacitor is

$$U = \frac{1}{2} \frac{Q^2}{C} \tag{152}$$

Using the proposed transformation the values of C' and U' becomes in the system S'

$$C' = \epsilon_0 \frac{A/\gamma^2}{d/\gamma} = \frac{1}{\gamma} \epsilon_0 \frac{A}{d} \tag{153}$$

and the stored energy in each capacitor is

$$U' = \frac{1}{2} \frac{Q^2}{C'} = \frac{\gamma}{2} \frac{Q^2}{C} \tag{154}$$

The mass correspondent to the energy given in (152) is

$$m_{\alpha or \beta} = \frac{1}{c^2} \frac{Q^2}{C} \tag{155}$$

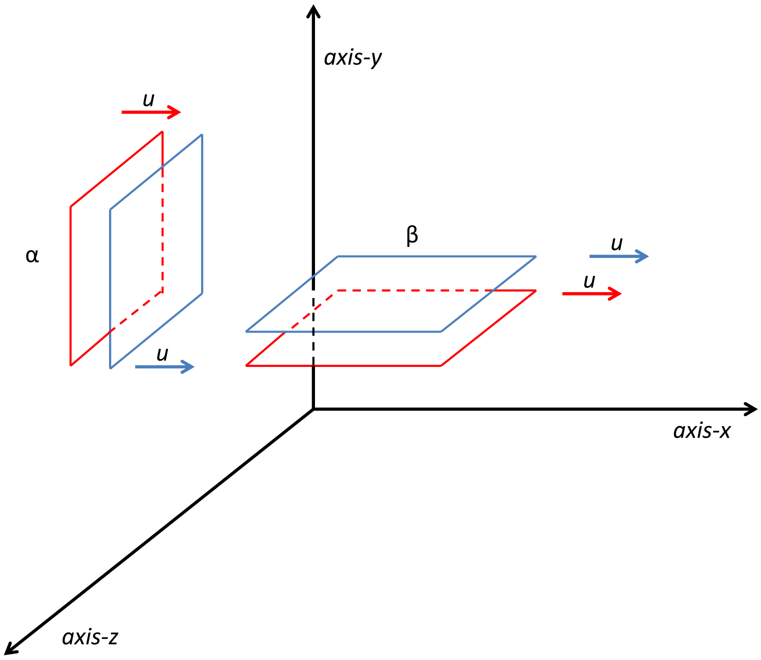

Figure 8: Two capacitors in the system S

In the system S' the mass $m'_{\alpha or \beta}$ must be

$$m'_{\alpha or \beta} = \gamma m_{\alpha or \beta} \tag{156}$$

and this is the case when we consider the equation (154).

Using the Lorentz transformation the energy stored in the capacitors α and β are not equal in system S'. In this case we obtain two values

$$\frac{\gamma}{2}\frac{Q^2}{C} \tag{157}$$

and

$$\frac{1}{2\gamma}\frac{Q^2}{C} \tag{158}$$

Using the Lorentz transformation, a simple 90-degree rotation in a capacitor (in the system S') considerably modifies the energy that the capacitor can store.

10.2 Helical movement of a charge in a magnetic field with component of velocity in the direction of the field

The Figure 9 shows a particle with positive electric charge q traversing a helical path in the system S. The speed of the particle is v, the component of v parallel to the *axis-x* is denoted by v_x and the component of v parallel to the *plane-yz* is denoted by v_{yz}. The magnetic field $\boldsymbol{B}$ is uniform and parallel to the *axis-x*. The mass of the particle is denoted by m. The electric charge path is a composition of uniform circular motion with uniform motion. The following equation relates electromagnetic and centrifugal forces

$$q v_{yz} B = m \frac{v_{yz}^2}{r} \tag{159}$$

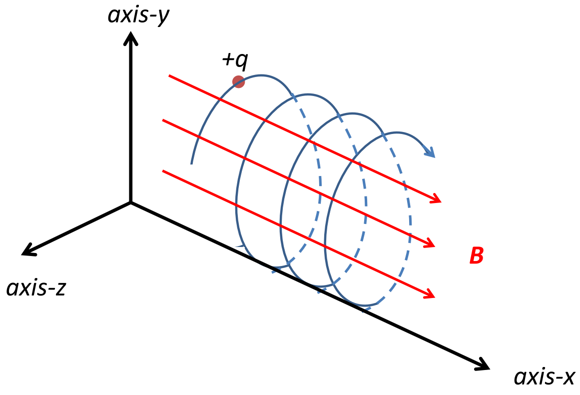

Figure 9: Helical movement of a charge in a magnetic field

or

$$m = \frac{q\ B\ r}{v_{xy}} \tag{160}$$

where r is the radius of the uniform circular motion. Let $v_x = u$ be where u is the velocity the system S' with respect to S.

The velocities transformations in the especial relativity are

$$v'_x = \frac{v_x - u}{1 - \frac{u}{c^2} v_x} = 0 \tag{161}$$

$$v'_y = \frac{v_y}{\gamma\left(1 - \frac{u}{c^2} v_x\right)} = \gamma v_y \tag{162}$$

and

$$v'_z = \frac{v_z}{\gamma\left(1 - \frac{u}{c^2} v_x\right)} = \gamma v_z \tag{163}$$

Considering (2) and (3) we have

$$r' = r \tag{164}$$

Since $\boldsymbol{B}$ is uniform and parallel to the *axis-x* we have

$$\boldsymbol{B}' = \boldsymbol{B} \tag{165}$$

For the system S' we have

$$m' = \frac{q r' B'}{v'_{yz}} \tag{166}$$

From (162) and (163) we can write

$$v'_{yz} = \gamma v_{yz} \tag{167}$$

Therefore

$$m' = \frac{q r B}{\gamma v_{yz}} = \frac{m}{\gamma} \tag{168}$$

Now let us use the proposed transformation. Substituting the values of r' B' and v'_{yz} we obtain

$$m' = \frac{qr'B'}{v'_{yz}} = \frac{q\left(\frac{r}{\gamma}\right)\left(\gamma^2 B\right)}{v_{yz}} = \gamma\frac{qrB}{v_{yz}} = \gamma\ m \tag{169}$$

Using the proposed transformation the period of the uniform circular motion in S' is

$$T' = \frac{2\pi r'}{v'_{yz}} = \frac{2\pi}{v_{yz}}\frac{r}{\gamma} = \frac{T}{\gamma} \tag{170}$$

where T is the period of the uniform circular motion in S.

10.3 On the space properties

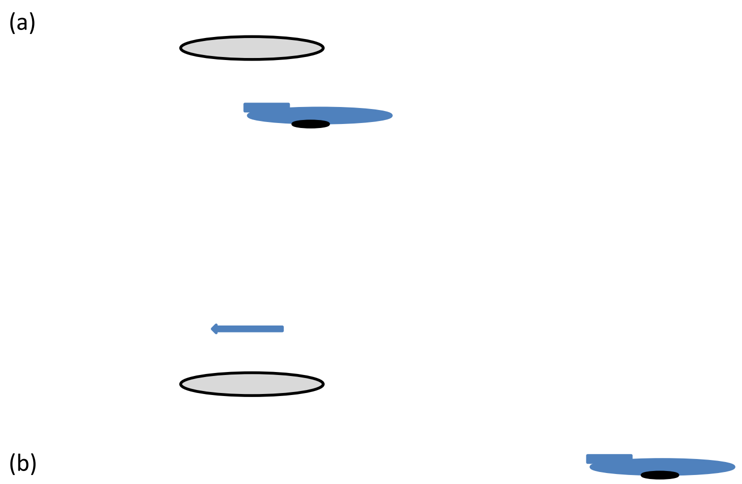

Figure 10: Base and rockets

Let be the figures 10 and 11. The Figure 10 (a) shows a base (in gray) and a blue rocket. The rocket is stationary with respect to the base. The Figure 10 (a) corresponds to the instant t_0 in the Figure 11. The clocks of the base and the blue rocket are synchronized at t_0. The Figure 10 (b) corresponds to the instant t_2 in the Figure 11. The Figure 10 (b) shows the base moving away from the blue rocket. In the Figure 11 the speed variation corresponds to the blue rocket, for $t > t_2$ the speed v is near to c and

$$t_2 - t_1 >> t_1 - t_0 \tag{171}$$

The blue rocket has a small nave (in black) which has a clock synchronized with the clock of the blue rocket at t_2. The Figure 13 shows how the blue rocket observes the black nave and the base moving away. For this figure we have

$$t_4 - t_3 >> t_3 - t_2 \tag{172}$$

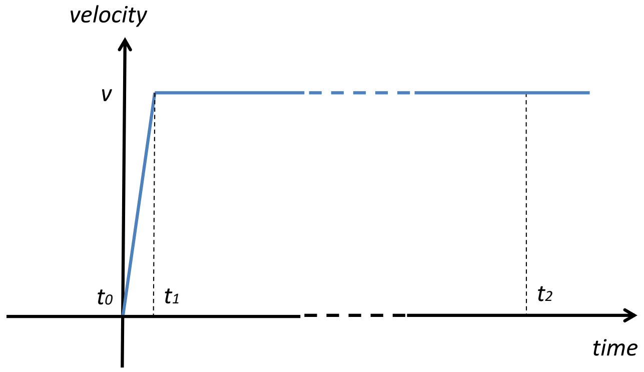

Figure 11: Rocket velocity with the time

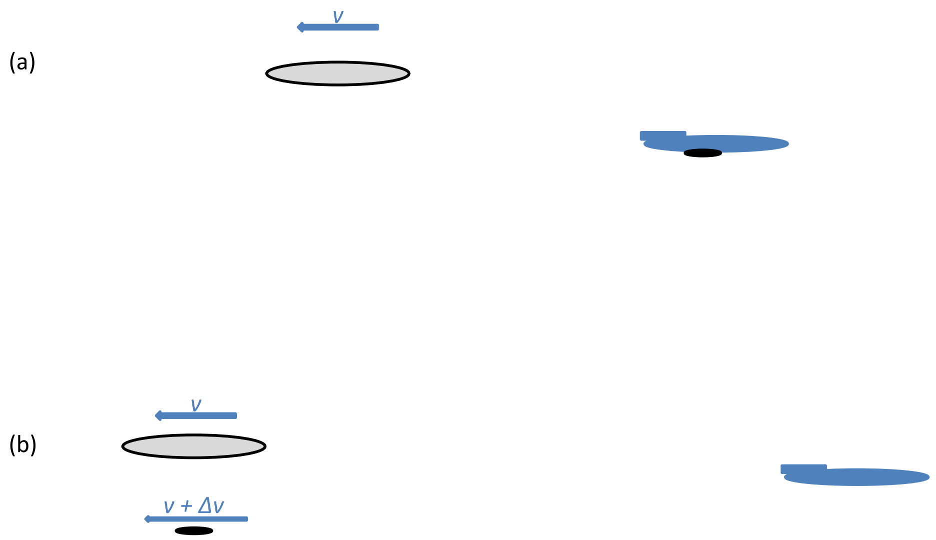

Figure 12: The base, the blue rocket and the black nave

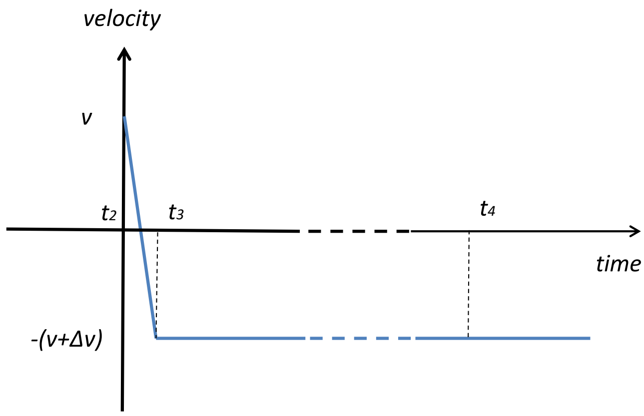

Figure 13: Black nave velocity with the time

and

$$0 < \Delta v <<< v \tag{173}$$

For an observer X inside the blue rocket that do not know what happened before t_2 the time in the black nave at t_4 is expected flows slower than the time measured with the clock of the blue rocket. At the time t_4 the black nave is very near of the gray base and an observer Y inside the black nave, who also does not know what happened before t_2, compares the clock of the black nave with the clock of the base. For Y which measure of time is expected to be almost equal with the measure of time of the clock of the black nave at t_4? The time measured with the clock of the gray base? Or the time expected by X?

Let us notice that for the Figure 10 (a) there is no information about what happened before t_0. The system formed by the base, the blue rocket and the black nave in the Figure 10 (a) is analogous to the system formed by blue rocket and black nave (Figure 12 (a) without the base) for the observers X and Y at t_2.

References

[1] A. Einstein, *Zur Elektrodynamik bewegter Körper*, Annalen der Physik, Band 17, 891-921, 1905.

[2] John von Neumann, *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, 1955.

[3] E. J. Post, *Sagnac Effect*, Reviews of Modern Physics, 39(2):475-493, 1967.