Link Between the One-Dimensional Dirac Equations and Two Velocities in Nonrelativistic Quantum Mechanics?

Francesco R. Ruggeri Hanwell, N.B. Aug. 22, 2019

The Dirac equation is often obtained through linearization of the Klein-Gordon equation. This leads to a 2x2 matrix system for one dimension, thus two functions u(x) and v(x) from the two-vector appear naturally. The Schrodinger equation may then be obtained through the nonrelativistic limit, as has shown in the literature. In this note, we consider the idea of two velocities in nonrelativistic quantum mechanics, namely 1/m d/dx W / W and (1/m) sqrt[- d/dx d/dx W(x)/W(x)] where W(x) is the nonrelativistic wavefunction. We note the Einstein's energy momentum relation with a scalar potential is: [E-mo-V(x)][E+mo+V(x)] = p*p. The 2m in the 1/2m factor (in the nonrelativistic equation) may be associated with E+mo+V(x) and E-mo-V(x) with the time-independent Schrodinger equation. In a series of notes, we argued there seem to be two velocities in nonrelativistic quantum mechanics: 1/m d/dx W/W and (1/m)sqrt(- d/dx d/dx W/W). We suggest these two velocities might be the starting point for introducing two functions u(x) and v(x). These might then be linked to the relativistic quantities E+mo+V(x) and E-mo-V(x). From this point, we try to obtain the two Dirac equations in one dimension.

Dirac Equations in One Dimension

We list the one dimensional Dirac equations as:

-d/dx v(x) = [E-mo-V(x)] u(x)   ((1a)) and

d/dx u(x) = [E+mo+V(x)] v(x)   ((1b))

In the nonrelativistic limit, ((1b)) yields d/dx u = 2m v. Inserting into ((1a)) yields the time-independent Schrodinger equation. This has been shown in the literature many times.

Two Velocities in Nonrelativistic Quantum Mechanics

In this note, we suggest the one dimensional Dirac equations may follow from ideas already present in the time-independent Schrodinger case. First, the time-independent case is related to the two factors [E+mo+V(x)] and [E-mo-V(x)] which when multiplied together and set equal to momentum squared give Einstein's energy-momentum relationship (from which the Klein-Gordon equation with a scalar potential follows). The time-independent Schrodinger equation itself is linked to [E-mo-V(x)]. Secondly, one may argue that the 1/2m coefficient of the kinetic energy term -1/2m d/dx d/dx W(x) is linked to [E+mo+V(x)].

Next, one may consider the idea of two velocities present in nonrelativistic quantum mechanics namely:

1/m d/dx W / W ((2a)) and (1/m)sqrt(- d/dx d/dx W / W) = v/sqrt(2m) ((2b))

These, we argue, may justify the designation of two functions u(x) and v(x) in the following manner. (First, however, note that d/dx of each velocity should be related to force as the first velocity is an average momentum and the second (squared) kinetic energy. We also note the d/dx derivatives of the factors [E+mo+V(x)] and [E-mo-V(x)] yield force or negative force (in the nonrelativistic sense).

Thus, we start with d/dx u(x) = [E+mo+V(x)] v(x) ((3)).

This defines v(x) in terms of u(x) i.e. v(x)=d/dx u(x) / [E+mo+V(x)]. v(x)/u(x) is related to velocity ((2a)). We include the factor 1/ [E+mo+V(x)] otherwise v(x) would be completely expressible in terms of u(x) and there would be no need to define two functions u(x) and v(x), u(x) would suffice.

Then d/dx v(x) should be related to the kinetic energy as this type of relationship exists for ((2a)) and ((2b)) i.e.

-d/dx v(x) = [E-mo-V(x)] u(x). ((4))

One may see that both ((3)) and ((4)) are now linked to force=-d/dx V(x) if one takes d/dx. For equation ((4)) this is expected as it is the time-independent Schrodinger equation in the nonrelativistic limit. Thus:

d/dx (-1/2m d/dx d/dx W(x) / W(x)) = -d/dx V(x).

Equation ((3)) is little more surprising as force appears as second order correction. Nevertheless, one might expect the d/dx derivative of the average momentum to be linked to force:

d/dx v(x) = d/dx { d/dx u(x) / (2m+V(x))] = [d/dx d/dx u(x)] (1/2m) (1-V(x)/2m) - d/dx u(x) d/dx V(x) /(2m*2m) + lower order terms ((5))

The term d/dx d/dx u(x) in ((5)) is related to kinetic energy, thus taking d/dx v(x) yields a term related to kinetic energy as well as a lower order term related to force. This does not appear directly in nonrelativistic quantum mechanics, but is a consequence of relating the spatial derivatives of the two functions to the factors [E+mo+V(x)] and [E-mo-V(x)].

Thus, the same force seems to be active in two places, first in linking v(x) to kinetic energy by taking d/dx and secondly, in computing d/dx of d/dx d/dx v(x).

Conclusion

In conclusion, we suggest in this brief note, that the one dimensional Dirac equations may follow from considerations of the time-independent Schrodinger equation with the idea of two velocities being present. These two velocities are linked by d/dx. Introducing [E+mo+V(x)] and [E-mo-V(x)] may justify introducing two functions u(x) and v(x). These two equations may then have significance in relativistic quantum mechanics, even though they use ideas from nonrelativistic quantum mechanics.