Acoustic scattering simulations in coupled fluid-porous media problems

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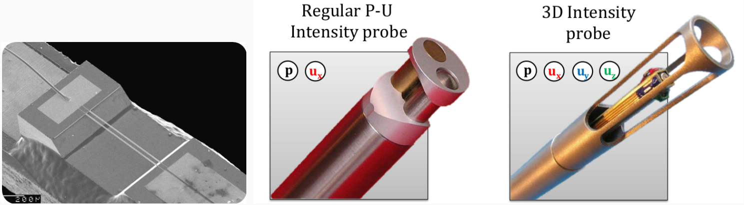
Introduction

Introduction : The Microflown

- Microflown PU probe is an acoustic sensor measuring particle velocity.
- Transducer designed on thermal principle
- Sensitive to fluid flow conditions
- Requires windscreens for wind velocities > 15 m/s



A microscopic view of the Microflown transducer (*left*), a regular PU probe (*middle*) and a 3D intensity probe (*right*)

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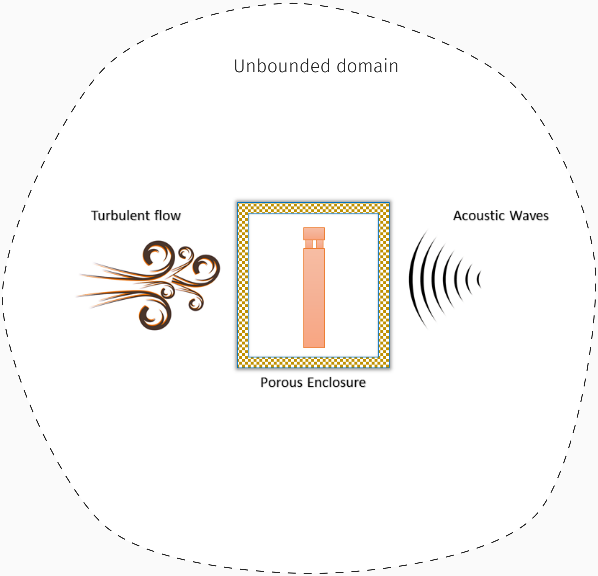
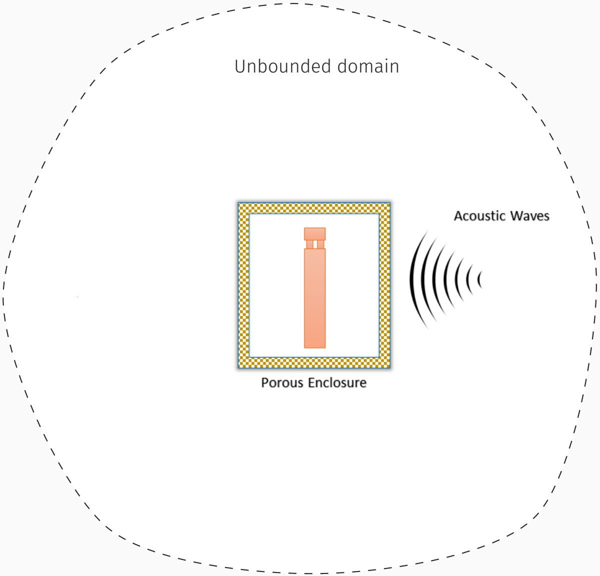
Main goal of the project



A schematic of the objective coupled problem

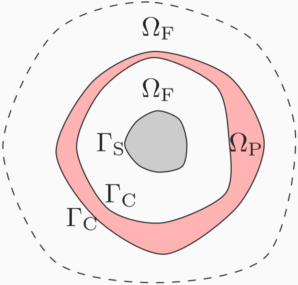
Simplified problem setting



A schematic of the simplified coupled problem

Mathematical model

Mathematical model



General problem setting

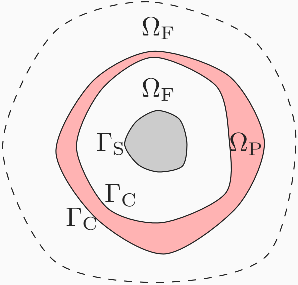
For a given angular frequency $\omega > 0$:

$$
\begin{aligned}
-\nabla(\rho_F c_F^2 \operatorname{div} \boldsymbol{u}_F) - \rho_F \omega^2 \boldsymbol{u}_F &= \boldsymbol{f}_F \quad \text{in } \Omega_F, \\
-\nabla(K_P(\omega) \operatorname{div} \boldsymbol{u}_P) - \rho_P(\omega)\omega^2 \boldsymbol{u}_P &= \boldsymbol{f}_P \quad \text{in } \Omega_P, \\
\boldsymbol{u}_F \cdot \boldsymbol{n} &= g \quad \text{on } \Gamma_S, \\
\boldsymbol{u}_F \cdot \boldsymbol{n} - \boldsymbol{u}_P \cdot \boldsymbol{n} &= 0 \quad \text{on } \Gamma_C, \\
\rho_F c_F^2 \operatorname{div} \boldsymbol{u}_F - K_P(\omega) \operatorname{div} \boldsymbol{u}_P &= 0 \quad \text{on } \Gamma_C, \\
\lim_{|\boldsymbol{x}| \to \infty} |\boldsymbol{x}| \left(\operatorname{div} \boldsymbol{u}_F - \mathrm{i} k_F \boldsymbol{u}_F \cdot \frac{\boldsymbol{x}}{|\boldsymbol{x}|} \right) &= \boldsymbol{0},
\end{aligned}
$$

Assumptions:

- Fluid: Homogeneous, non-viscous, compressible, isotropic, isentropic acoustic fluid
- Porous medium: Homogeneous, isotropic, isothermal porous material

Mathematical model



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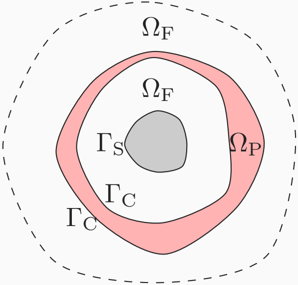
$$
\begin{aligned}
-\nabla(\rho_\mathrm{F}\, c_\mathrm{F}^2 \operatorname{div} \boldsymbol{u}_\mathrm{F}) - \rho_\mathrm{F}\omega^2 \boldsymbol{u}_\mathrm{F} &= \boldsymbol{f}_\mathrm{F} \quad \text{in } \Omega_\mathrm{F},\\
-\nabla(K_\mathrm{P}(\omega) \operatorname{div} \boldsymbol{u}_\mathrm{P}) - \rho_\mathrm{P}(\omega)\omega^2 \boldsymbol{u}_\mathrm{P} &= \boldsymbol{f}_\mathrm{P} \quad \text{in } \Omega_\mathrm{P},\\
\boldsymbol{u}_\mathrm{F} \cdot \boldsymbol{n} &= g \quad \text{on } \Gamma_\mathrm{S},\\
\boldsymbol{u}_\mathrm{F} \cdot \boldsymbol{n} - \boldsymbol{u}_\mathrm{P} \cdot \boldsymbol{n} &= 0 \quad \text{on } \Gamma_\mathrm{C},\\
\rho_\mathrm{F}\, c_\mathrm{F}^2 \operatorname{div} \boldsymbol{u}_\mathrm{F} - K_\mathrm{P}(\omega) \operatorname{div} \boldsymbol{u}_\mathrm{P} &= 0 \quad \text{on } \Gamma_\mathrm{C},\\
\lim_{|\boldsymbol{x}|\to\infty} |\boldsymbol{x}| \left(\operatorname{div} \boldsymbol{u}_\mathrm{F} - \mathrm{i} k_\mathrm{F}\, \boldsymbol{u}_\mathrm{F} \cdot \frac{\boldsymbol{x}}{|\boldsymbol{x}|} \right) &= \boldsymbol{0},
\end{aligned}
$$

where

- Ω_F: fluid subdomain, $\boldsymbol{f}_\mathrm{F}$: fluid source term
- Ω_P: porous subdomain, $\boldsymbol{f}_\mathrm{P}$: porous source term
- Γ_S: rigid solid boundary, g: structural normal displacement
- Γ_C: coupled fluid-porous boundary

Mathematical model



General problem setting

For a given angular frequency $\omega > 0$:

$$
\begin{aligned}
\boxed{-\nabla(\rho_\mathrm{F} c_\mathrm{F}^2 \operatorname{div} \boldsymbol{u}_\mathrm{F}) - \rho_\mathrm{F}\omega^2 \boldsymbol{u}_\mathrm{F} = \boldsymbol{f}_\mathrm{F} \quad \text{in } \Omega_\mathrm{F},} \\
-\nabla(K_\mathrm{P}(\omega) \operatorname{div} \boldsymbol{u}_\mathrm{P}) - \rho_\mathrm{P}(\omega)\omega^2 \boldsymbol{u}_\mathrm{P} = \boldsymbol{f}_\mathrm{P} \quad \text{in } \Omega_\mathrm{P}, \\
\boxed{\boldsymbol{u}_\mathrm{F} \cdot \boldsymbol{n} = g \quad \text{on } \Gamma_\mathrm{S},} \\
\boldsymbol{u}_\mathrm{F} \cdot \boldsymbol{n} - \boldsymbol{u}_\mathrm{P} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma_\mathrm{C}, \\
\rho_\mathrm{F} c_\mathrm{F}^2 \operatorname{div} \boldsymbol{u}_\mathrm{F} - K_\mathrm{P}(\omega) \operatorname{div} \boldsymbol{u}_\mathrm{P} = 0 \quad \text{on } \Gamma_\mathrm{C}, \\
\lim_{|\boldsymbol{x}|\to\infty} |\boldsymbol{x}| \left(\operatorname{div} \boldsymbol{u}_\mathrm{F} - \mathrm{i} k_\mathrm{F} \boldsymbol{u}_\mathrm{F} \cdot \frac{\boldsymbol{x}}{|\boldsymbol{x}|} \right) = \boldsymbol{0},
\end{aligned}
$$

where

- $\boldsymbol{u}_\mathrm{F}$, $\boldsymbol{u}_\mathrm{P}$: fluid and porous displacement field
- ρ_F, $\rho_\mathrm{P}(\omega)$: fluid and dynamic porous mass density
- c_F: fluid sound speed
- $K_\mathrm{P}(\omega)$: dynamic porous bulk modulus

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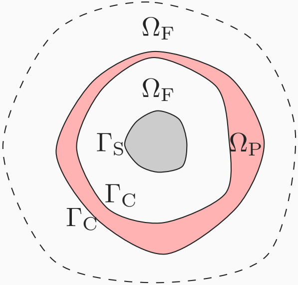
Mathematical model



General problem setting

For a given angular frequency $\omega > 0$:

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\begin{aligned}
-\nabla(\rho_{\mathrm{F}} c_{\mathrm{F}}^2 \operatorname{div} \boldsymbol{u}_{\mathrm{F}}) - \rho_{\mathrm{F}}\omega^2 \boldsymbol{u}_{\mathrm{F}} = \boldsymbol{f}_{\mathrm{F}} \quad & \text{in } \Omega_{\mathrm{F}}, \\
\boxed{-\nabla(K_{\mathrm{P}}(\omega) \operatorname{div} \boldsymbol{u}_{\mathrm{P}}) - \rho_{\mathrm{P}}(\omega)\omega^2 \boldsymbol{u}_{\mathrm{P}} = \boldsymbol{f}_{\mathrm{P}} \quad \text{in } \Omega_{\mathrm{P}},} & \\
\boldsymbol{u}_{\mathrm{F}} \cdot \boldsymbol{n} = g \quad & \text{on } \Gamma_{\mathrm{S}}, \\
\boxed{\boldsymbol{u}_{\mathrm{F}} \cdot \boldsymbol{n} - \boldsymbol{u}_{\mathrm{P}} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma_{\mathrm{C}},} & \\
\boxed{\rho_{\mathrm{F}} c_{\mathrm{F}}^2 \operatorname{div} \boldsymbol{u}_{\mathrm{F}} - K_{\mathrm{P}}(\omega) \operatorname{div} \boldsymbol{u}_{\mathrm{P}} = 0 \quad \text{on } \Gamma_{\mathrm{C}},} & \\
\lim_{|\boldsymbol{x}|\to\infty} |\boldsymbol{x}| \left(\operatorname{div} \boldsymbol{u}_{\mathrm{F}} - \mathrm{i} k_{\mathrm{F}} \boldsymbol{u}_{\mathrm{F}} \cdot \frac{\boldsymbol{x}}{|\boldsymbol{x}|} \right) = \boldsymbol{0}, &
\end{aligned}
$$

where

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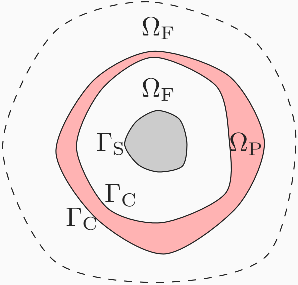
Mathematical model



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-\nabla(K_{\mathrm{P}}(\omega) \operatorname{div} \boldsymbol{u}_{\mathrm{P}}) - \rho_{\mathrm{P}}(\omega) \omega^2 \boldsymbol{u}_{\mathrm{P}} &= \boldsymbol{f}_{\mathrm{P}} \quad \text{in } \Omega_{\mathrm{P}}, \\
\boldsymbol{u}_{\mathrm{F}} \cdot \boldsymbol{n} &= g \quad \text{on } \Gamma_{\mathrm{S}}, \\
\boldsymbol{u}_{\mathrm{F}} \cdot \boldsymbol{n} - \boldsymbol{u}_{\mathrm{P}} \cdot \boldsymbol{n} &= 0 \quad \text{on } \Gamma_{\mathrm{C}}, \\
\rho_{\mathrm{F}} c_{\mathrm{F}}^2 \operatorname{div} \boldsymbol{u}_{\mathrm{F}} - K_{\mathrm{P}}(\omega) \operatorname{div} \boldsymbol{u}_{\mathrm{P}} &= 0 \quad \text{on } \Gamma_{\mathrm{C}},
\end{aligned}
$$

$$
\lim_{|\boldsymbol{x}| \to \infty} |\boldsymbol{x}| \left(\operatorname{div} \boldsymbol{u}_{\mathrm{F}} - \mathrm{i} k_{\mathrm{F}} \boldsymbol{u}_{\mathrm{F}} \cdot \frac{\boldsymbol{x}}{|\boldsymbol{x}|} \right) = \boldsymbol{0},
$$

where

- $\boldsymbol{u}_{\mathrm{F}}$, $\boldsymbol{u}_{\mathrm{P}}$: fluid and porous displacement field
- ρ_{F}, $\rho_{\mathrm{P}}(\omega)$: fluid and dynamic porous mass density
- c_{F}: fluid sound speed
- $K_{\mathrm{P}}(\omega)$: dynamic porous bulk modulus

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Fluid-equivalent porous model

Johnson-Champoux-Allard-Lafarge (JCAL) model

- Valid for porous materials with arbitrarily shaped pores
- Six-parameter model to describe rigid-frame porous media properties

Parameter	#	Value
Porosity	ϕ	0.94
Flow Resistivity	σ	4e4
Tourtuosity	α_∞	1.06
Viscous Characteristic Length	Λ	56e-6
Thermal Characteristic Length	Λ'	110e-6
Static Thermal Permeability	${k_0}'$	2.5e-10

The model aso involves the fluid mass density ρ_{F}, specific heat ratio γ, Prandtl Number Pr, and equillibrium fluid pressure P_{F}

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Johnson-Champoux-Allard-Lafarge (JCAL) model

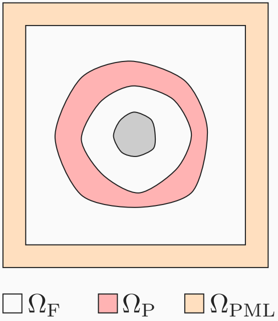
- Valid for porous materials with arbitrarily shaped pores
- Six-parameter model to describe rigid-frame porous media properties

$$\rho_{\mathrm{P}}(\omega) = \frac{\rho_{\mathrm{F}}}{\phi}\alpha_\infty \left(1 - \mathrm{i}\frac{\sigma\phi}{\omega\rho_{\mathrm{F}}\alpha_\infty}\sqrt{1 + \mathrm{i}\frac{4\alpha_\infty^2\eta\rho_{\mathrm{F}}\omega}{\sigma^2\Lambda^2\phi^2}}\right),$$

$$K_{\mathrm{P}}(\omega) = \frac{\gamma P_{\mathrm{F}}/\phi}{\gamma - (\gamma-1)\left(1 - \mathrm{i}\frac{\eta\phi}{\rho_{\mathrm{F}}k_0'\omega\mathrm{Pr}}\sqrt{1 + \mathrm{i}\frac{4{k_0'}^2\rho_{\mathrm{F}}\omega\mathrm{Pr}}{\eta\Lambda'^2\phi^2}}\right)^{-1}},$$

The model aso involves the fluid mass density ρ_{F}, specific heat ratio γ, Prandtl Number Pr, and equillibrium fluid pressure P_{F}

Perfectly Matched Layers (PML) technique



Schematic of PML Model

- Truncate the domain at some finite distance
- Wrap by an absorption layer Ω_{PML},
  - Involve complex-valued stretching of the spatial coordinates:

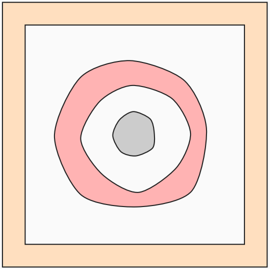
$$\widetilde{\nabla}\phi = \sum_{j=1}^{3} \frac{1}{\gamma_j} \frac{\partial \phi}{\partial x_j} \boldsymbol{e}_j,$$

$$\widetilde{\operatorname{div}}\, \boldsymbol{w} = \sum_{j=1}^{3} \frac{1}{\gamma_j} \frac{\partial w_j}{\partial x_j},$$

with $\quad \gamma_j \in \mathbb{C},\ j = 1, 2, 3.$

Bermúdez A., Hervella-Nieto L. M., Prieto A., Rodríguez R., “An optimal perfectly matched layer with unbounded absorbing function for time-harmonic acoustic scattering problems”, J. Comput. Phys., 223, 469-48, 2007

Perfectly Matched Layers (PML) technique



Ω_{F} Ω_{P} Ω_{PML}

Schematic of PML Model

$$\Omega_{\mathrm{PML}} = \prod_{j=1}^{3}[-L_j^\infty, L_j^\infty] \setminus \prod_{j=1}^{3}[-L_j, L_j]$$

$\gamma_j \in \mathbb{C},\ j = 1, 2, 3$ are modeled by a piecewise smooth function,

$$\gamma_j(x_j) = \begin{cases} 1 & |x_j| \leq L_j, \\ 1 + \mathrm{i}\sigma_j(x_j) & L_j \leq |x_j| \leq L_j^\infty, \end{cases}$$

with $\sigma_j(x_j) = c_{\mathrm{F}}/(\omega(L_j^\infty - |x_j|))$

- Singular on boundary
- Optimally tuned to absorb waves of any frequency

Bermúdez A., Hervella-Nieto L. M., Prieto A., Rodríguez R., “An optimal perfectly matched layer with unbounded absorbing function for time-harmonic acoustic scattering problems”, J. Comput. Phys., 223, 469-48, 2007

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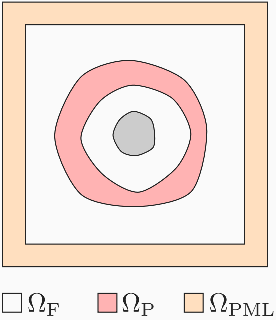
Perfectly Matched Layers (PML) technique



Schematic of PML Model

The governing (PML) equation in Ω_{PML}:

$$-\widetilde{\nabla}(\rho_{\mathrm{PML}}\, c_{\mathrm{F}}^2\, \widetilde{\mathrm{div}}\, \boldsymbol{u}_{\mathrm{PML}}) - \rho_{\mathrm{F}}\omega^2 \boldsymbol{u}_{\mathrm{PML}} = \mathbf{0},$$

or, in an equivalent form,

$$-\,\mathrm{div}(\rho_{\mathrm{F}}\, c_{\mathrm{F}}^2 \widetilde{\mathrm{C}}(\nabla \boldsymbol{u}_{\mathrm{PML}})) - \rho_{\mathrm{F}}\omega^2 \widetilde{\boldsymbol{M}} \boldsymbol{u}_{\mathrm{PML}} = \mathbf{0},$$

where

4th order tensor: $\widetilde{\mathrm{C}}(\nabla \boldsymbol{w}) = (\widetilde{\mathrm{div}}\, \boldsymbol{w})\boldsymbol{I}$,

2nd order tensor: $\widetilde{\boldsymbol{M}} = \sum_{j=1}^{3} \gamma_j \boldsymbol{e}_j \otimes \boldsymbol{e}_j$.

Bermúdez A., Hervella-Nieto L. M., Prieto A., Rodríguez R., “An optimal perfectly matched layer with unbounded absorbing function for time-harmonic acoustic scattering problems”, J. Comput. Phys., 223, 469-48, 2007

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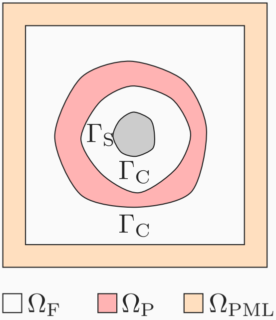
Planewave scattering problem

Total fields are decomposed in incident and scattering fields: $\boldsymbol{u}_{\mathrm{F}} = \boldsymbol{u}_{\mathrm{F}}^{\mathrm{inc}} + \boldsymbol{u}_{\mathrm{F}}^{\mathrm{sc}}$ with $\boldsymbol{u}_{\mathrm{F}}^{\mathrm{inc}} = \dfrac{1}{\rho_{\mathrm{F}}\omega^2}\nabla\big(A_{\mathrm{F}}\exp(\boldsymbol{k}_{\mathrm{F}}\cdot\boldsymbol{x})\big)$, $|\boldsymbol{k}_{\mathrm{F}}| = \omega/c_{\mathrm{F}}$.

- $\boldsymbol{u}_{\mathrm{F}}^{\mathrm{inc}}$ is a solution of Helmholtz-like equation in Ω_{F}:
  $$-\nabla(\rho_{\mathrm{F}}\, c^2 \operatorname{div} \boldsymbol{u}_{\mathrm{F}}^{\mathrm{inc}}) - \rho_{\mathrm{F}}\omega^2 \boldsymbol{u}_{\mathrm{F}}^{\mathrm{inc}} = \boldsymbol{0},$$
- but *not* in the porous subdomain Ω_{P}, where $\boldsymbol{u}_{\mathrm{P}} = \boldsymbol{u}_{\mathrm{P}}^{\mathrm{sc}} + \boldsymbol{u}_{\mathrm{F}}^{\mathrm{inc}}$, so
  $-\nabla(K_{\mathrm{P}}(\omega) \operatorname{div} \boldsymbol{u}_{\mathrm{P}}^{\mathrm{inc}}) - \rho_{\mathrm{P}}(\omega)\omega^2 \boldsymbol{u}_{\mathrm{P}}^{\mathrm{inc}} = \boldsymbol{f}_{\mathrm{P}} \neq \boldsymbol{0}$.



Schematic of Final Model

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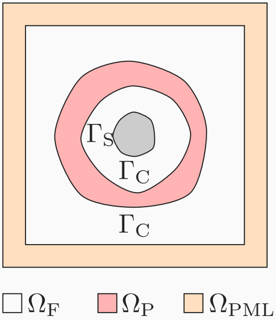
Planewave scattering problem

Since the normal components of displacement fields are continuous across interfaces, the scattering displacement field is defined as,

$$\boldsymbol{u} = \begin{cases} \boldsymbol{u}_{\mathrm{F}}^{\mathrm{sc}} & \text{in } \Omega_{\mathrm{F}}, \\ \boldsymbol{u}_{\mathrm{P}}^{\mathrm{sc}} & \text{in } \Omega_{\mathrm{P}}, \\ \boldsymbol{u}_{\mathrm{PML}} & \text{in } \Omega_{\mathrm{PML}}. \end{cases}$$



Schematic of Final Model

Variational Formulation

Introduce $\Omega = \Omega_{\mathrm{F}} \cup \Omega_{\mathrm{P}} \cup \Omega_{\mathrm{PML}}$ and the functional space

$$\mathbf{V} = \left\{ \boldsymbol{v} \in [\mathrm{L}^2(\Omega)]^3 : \begin{array}{l} \boldsymbol{v}|_{\Omega_{\mathrm{F}}} \in \mathbf{H}(\mathrm{div}, \Omega_{\mathrm{F}}),\ \boldsymbol{v}|_{\Omega_{\mathrm{P}}} \in \mathbf{H}(\mathrm{div}, \Omega_{\mathrm{P}}), \\ \widetilde{\boldsymbol{M}}\boldsymbol{v}|_{\Omega_{\mathrm{PML}}} \in [\mathrm{L}^2(\Omega_{\mathrm{PML}})]^3,\ \widetilde{\mathrm{div}}(\boldsymbol{v}|_{\Omega_{\mathrm{PML}}}) \in \mathrm{L}^2(\Omega_{\mathrm{PML}}), \\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_\infty \end{array} \right\}$$

Variational problem

Given $\omega > 0$, find $\boldsymbol{u} \in \mathbf{V}$ such that $\boldsymbol{u} \cdot \boldsymbol{n} = -\boldsymbol{u}_{\mathrm{F}}^{\mathrm{inc}} \cdot \boldsymbol{n}$ on Γ_{S}, and

$$\left.\begin{array}{l} \displaystyle\int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} c^2 (\mathrm{div}\, \boldsymbol{u})(\mathrm{div}\, \boldsymbol{v}) \,\mathrm{d}V - \int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} \omega^2 \boldsymbol{u} \cdot \boldsymbol{v} \,\mathrm{d}V \\ \displaystyle + \int_{\Omega_{\mathrm{P}}} \mathrm{K}_{\mathrm{P}}(\omega)(\mathrm{div}\, \boldsymbol{u})(\mathrm{div}\, \boldsymbol{v}) \,\mathrm{d}V - \int_{\Omega_{\mathrm{P}}} \rho_{\mathrm{P}}(\omega) \omega^2 \boldsymbol{u} \cdot \boldsymbol{v} \,\mathrm{d}V \\ \displaystyle + \int_{\Omega_{\mathrm{PML}}} \rho_{\mathrm{F}} c^2 \widetilde{\mathsf{C}}(\nabla \boldsymbol{u}) : \nabla \boldsymbol{v} \,\mathrm{d}V - \int_{\Omega_{\mathrm{PML}}} \rho_{\mathrm{F}} \omega^2 \widetilde{\boldsymbol{M}} \boldsymbol{u} \cdot \boldsymbol{v} \,\mathrm{d}V \end{array}\right\} = \int_{\Omega_{\mathrm{P}}} \boldsymbol{f}_{\mathrm{P}} \cdot \boldsymbol{v} \,\mathrm{d}V,$$

for all $\boldsymbol{v} \in \mathbf{V}$ with $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ_{S}.

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Finite Element Method

We discretize using the first-order Raviart-Thomas Finite Elements ($\mathbf{RT}_{\mathrm{h}}^{1}$) over a tetrahedral mesh $\mathcal{T}_h$ of Ω,

Raviart-Thomas Finite Element Space

$$\mathbf{RT}_{\mathrm{h}}^{1}(\Omega) = \left\{ \boldsymbol{v} \in \mathbf{H}(\mathrm{div}, \Omega) : \quad \boldsymbol{v}|_T = \boldsymbol{a} + b\boldsymbol{x}, \boldsymbol{a} \in \mathbb{C}^3, b \in \mathbb{C}, T \in \mathcal{T}_h \right\}$$

- Constant divergence in each element
- Constant normal component on all faces of tetrahedra
- Completely determined if their normal components are known on each face.

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Post-Processing

Pressure fields are computed from the fluid and porous displacement field:

$$p_{\mathrm{F}} = -\rho_{\mathrm{F}}\, c^2 \,\mathrm{div}\, \boldsymbol{u}_{\mathrm{F}} \quad \text{in } \Omega_{\mathrm{F}},$$

$$p_{\mathrm{P}} = -\mathrm{K}_{\mathrm{P}}(\omega)\,\mathrm{div}\, \boldsymbol{u}_{\mathrm{P}} \quad \text{in } \Omega_{\mathrm{P}}.$$

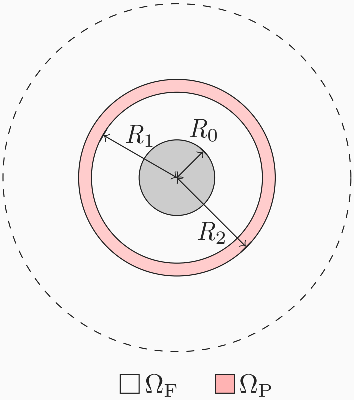
Since $\boldsymbol{u}_{\mathrm{F}}^h \in \mathbf{RT}_h^1(\Omega_{\mathrm{F}})$ and $\boldsymbol{u}_{\mathrm{P}}^h \in \mathbf{RT}_h^1(\Omega_{\mathrm{P}})$,
the numerical approximation for induced pressure fields:
$p_{\mathrm{F}}^h \in \mathbf{DG}_h^0(\Omega_{\mathrm{F}})$ and $p_{\mathrm{P}}^h \in \mathbf{DG}_h^0(\Omega_{\mathrm{P}})$.

3D Directivity patterns are also computed by evaluating pressure fields at uniformly distributed points on a sphere, $\boldsymbol{m}_j$, $j = 1, \ldots, M$:

$$\mathrm{Directivity}^h(\boldsymbol{m}_j) = \frac{|p_{\mathrm{F}}^h(m_j)| - \min_{\boldsymbol{m}_j} |p_{\mathrm{F}}^h(\boldsymbol{m}_j)|}{||p_{\mathrm{F}}^h||_{\mathrm{L}^\infty(\Omega_{\mathrm{F}})} - \min_{\boldsymbol{m}_j} |p_{\mathrm{F}}^h(\boldsymbol{m}_j)|}.$$

Benchmark case

Benchmark case : Monopole



Schematic of the benchmark monopole case

For a given $\omega > 0$, writing the problems in term of pressure fields we obtain the exact solution:

$$p_1(r) = A_1 \frac{e^{-\mathrm{i}k_F r}}{r} + B_1 \frac{e^{\mathrm{i}k_F r}}{r}, \;\; r \in [R_0, R_1],$$

$$p_2(r) = A_2 \frac{e^{-\mathrm{i}k_P r}}{r} + B_2 \frac{e^{\mathrm{i}k_P r}}{r}, \;\; r \in [R_1, R_2],$$

$$p_3(r) = B_3 \frac{e^{\mathrm{i}k_F r}}{r}, \;\; r \in [R_2, \infty),$$

with boundary conditions

$$p'(R_0) = \rho_F \omega^2 g_0, \;\; g_0 = \text{constant},$$

$$p_1(R_1) = p_2(R_1), \quad \rho_P p_1'(R_1) = \rho_F p_2'(R_1),$$

$$p_2(R_2) = p_3(R_2), \quad \rho_F p_2'(R_2) = \rho_P p_3'(R_2).$$

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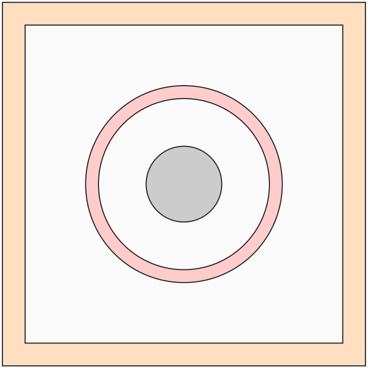
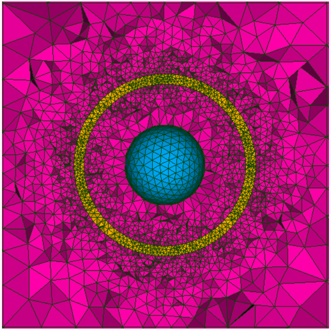
Benchmark case: Implementation



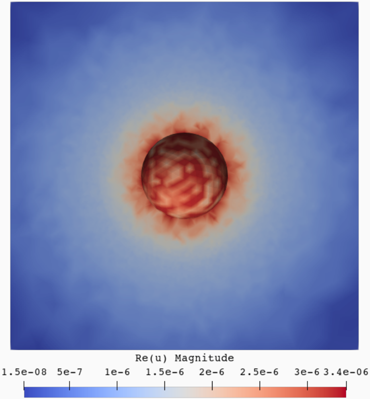


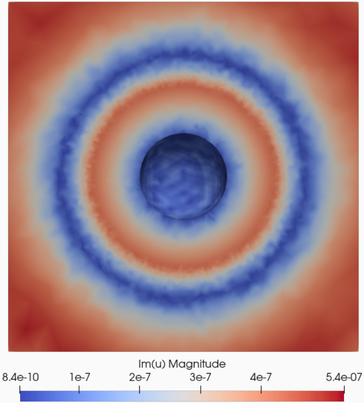
$\square\,\Omega_{\mathrm{F}}$ $\square\,\Omega_{\mathrm{P}}$ $\square\,\Omega_{\mathrm{PML}}$

Schematic of Benchmark case (left) and snapshot of mesh(right).
The PML layer is not shown in mesh.

- Conformal mesh in subdomains and coupling boundaries
- Local refinements around the porous layer and the rigid solid

Benchmark case: Displacement Field





The computed real (*left*) and imaginary (*right*) parts of displacement field.

The field is qualitatively uniform the radial directions with aberrations due to interpolations in a non-uniform mesh.

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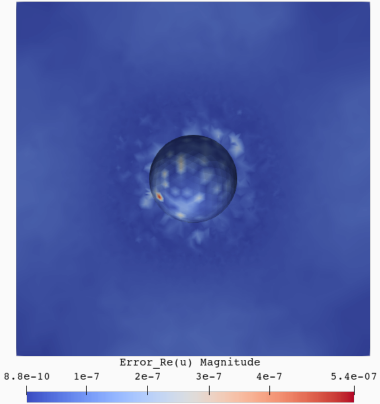
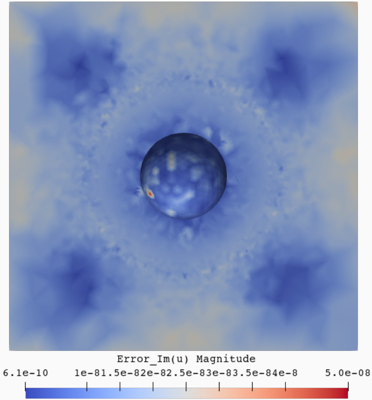
Benchmark case: Displacement Field

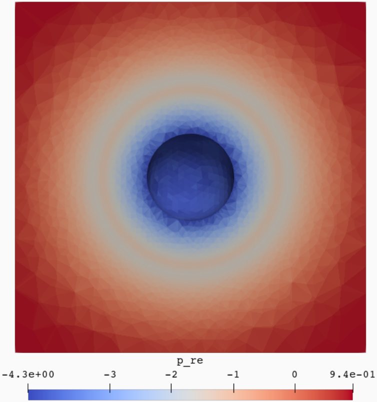


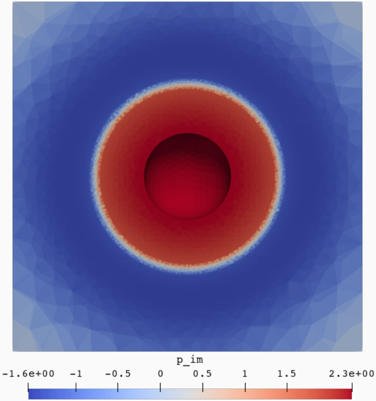


The computed real (*left*) and imaginary (*right*) parts of errors in displacement field.

The errors are mostly minimal in orders of magnitude of the displacement field.

Benchmark case: Pressure Field

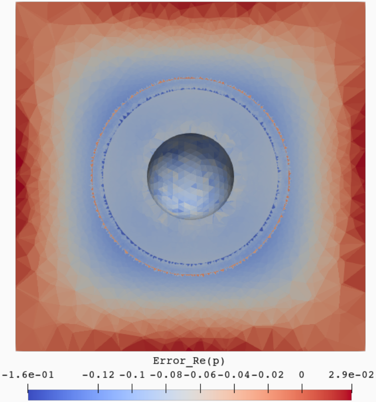


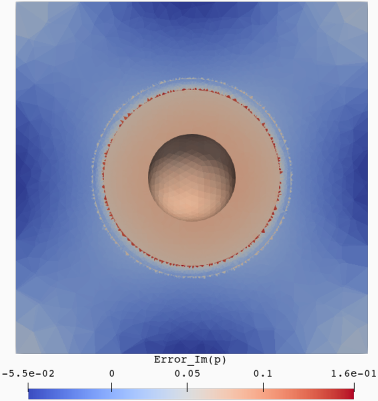


The computed real (*left*) and imaginary (*right*) parts of the pressure field.

The field is qualitatively uniform along the radial directions.

Benchmark case: Pressure field

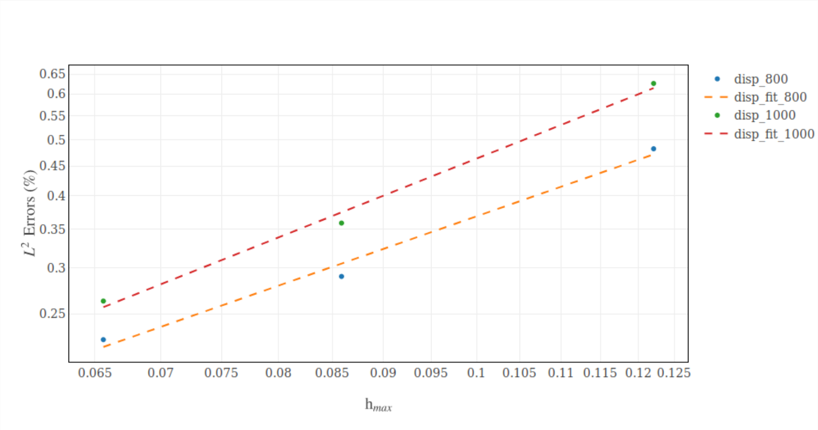




The computed real (*left*) and imaginary (*right*) parts of the errors in the pressure field.

The field is qualitatively agreeable with the errors smaller by an order of magnitude.

Benchmark case: Error Convergence



Slopes of fitted lines: 1.2370 (800Hz) and 1.41 (1000Hz)

$$\mathrm{L}^2\text{-rel. error (\%)} = 100\frac{||\boldsymbol{u} - \boldsymbol{u}_{\mathrm{ex}}||_{\mathrm{L}^2(\Omega_\mathrm{F})}}{||\boldsymbol{u}_{\mathrm{ex}}||_{\mathrm{L}^2(\Omega_\mathrm{F})}}$$

Numerical results

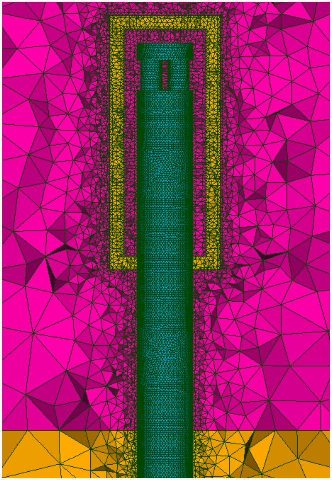
Scenario

Mesh Information

- # Elements: 902971
- # Vertices: 169633
- Conformal with layers
- Local refinements

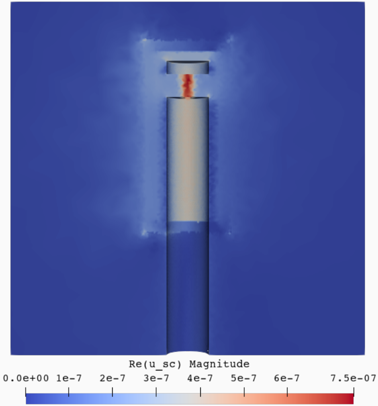
Frequency = 800Hz

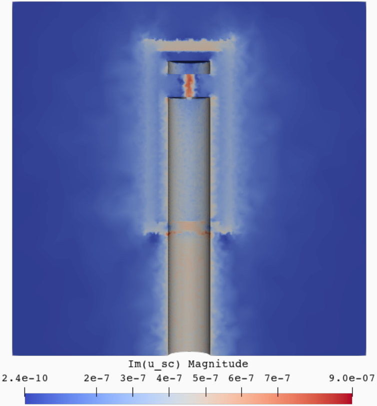
Incident Planewave in the $+x$ direction.



Snapshot of the PU Probe Mesh cross-section. Only the bottom PML layer is partly visible.

Results: Scattered Displacement Field



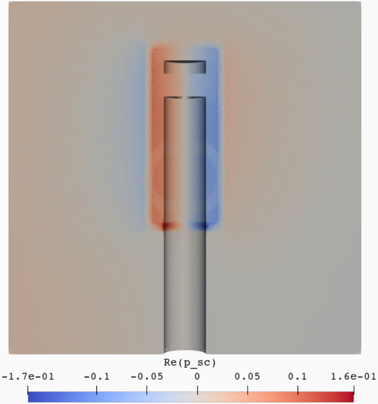


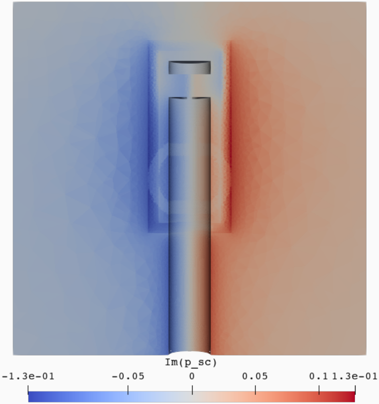
The real (*left*) and imaginary (*right*) parts of the scattered displacement field.

The field clearly shows higher displacement amplitudes between the probe pillars.

Also visible are acoustic dispersions at corners of the porous layers.

Results: Scattered Pressure Field

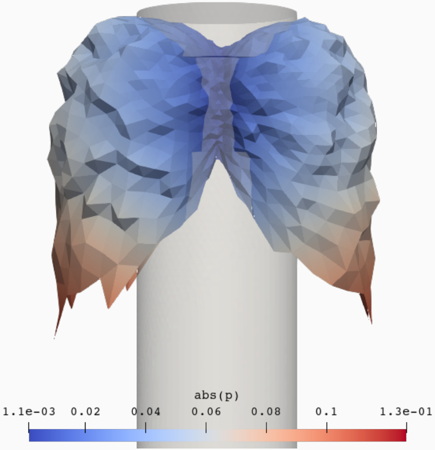




The computed real (*left*) and imaginary (*right*) parts of the scattered pressure field.

The field clearly shows scatter behaviour of the plane wave along direction of propagation.

Directivity



The computed directivity of the scattered pressure field.

The plot allows for highlighting directions of influence by the scattering object.

Conclusions

Conclusions

- A mathematical model has been developed for coupling porous layers and deal with radiating boundary conditions
- The model has been validated
- A customizable tool has been developed which lets deeper insights in sensor design
    - Prototyping different porous-layer configurations
- Frequency response studies and other analysis is made available.

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Thank You!