Espil short proof of generalized Cauchy's residue theorem

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Abstract: shortly we can derive the Cauchy's residue theorem (its general form) just by direct integration of a Taylor series "without" making any radius go to zero, even without the limit circumference idea take place.

H) Let D be a simply connected open subset of the complex plane, where $z = a \epsilon D$, enclosed by a rectificable positively oriented simple curve (C^+) in D, and f a function defined and holomorphic on D

T) $\oint_{C^+} \frac{f(z)}{(\mathrm{z}-\mathrm{a})^n}\mathrm{dz} = \lim_{z\to a} \frac{2\pi \mathrm{i}}{(n-1)!}\frac{d^{n-1}f(z)}{\mathrm{dz}^{n-1}}$.

D) Being f holomorphic on D ,its infinitely differentiable and equal to its own Taylor series at z=a and in the neighborhood.

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{d^k f(a)}{\mathrm{dz}^k}(z-a)^k$$

$$\oint_{C^+} \frac{f(z)}{(\mathrm{z}-\mathrm{a})^n}\mathrm{dz} = \oint_{C^+} \sum_{k=0}^{\infty} \frac{1}{k!}\frac{d^k f(a)}{\mathrm{dz}^k}(z-a)^k \frac{1}{(\mathrm{z}-\mathrm{a})^n}\mathrm{dz} = \oint_{C^+} \sum_{k=0}^{\infty} \frac{1}{k!}\frac{d^k f(a)}{\mathrm{dz}^k}(z-a)^{k-n}\mathrm{dz}$$

$$\oint_{C^+} \{\sum_{k=0}^{n-2} \frac{1}{k!}\frac{d^k f(a)}{\mathrm{dz}^k}(z-a)^{k-n} + \frac{1}{(n-1)!}\frac{d^{n-1}f(a)}{\mathrm{dz}^{n-1}}\frac{1}{(z-a)} + \sum_{k=n}^{\infty} \frac{1}{k!}\frac{d^k f(a)}{\mathrm{dz}^k}(z-a)^{k-n}\}\mathrm{dz}$$

Let $z_0 = a + \rho_0 e^{\mathrm{i}\theta_0} \epsilon\, \partial D$,being $\theta_0 = \arg\,(z-a)$ when travelling counterclockwise over $\partial D\ around\ z = a$, being the start point: z_0 and the end point: $z_1 = a + \rho_0 e^{i(\theta_0 + 2\pi)} \epsilon\, \partial D$, then integrating ...

$$\{\sum_{k=0}^{n-2}\frac{1}{k!}\frac{d^k f(a)}{\mathrm{dz}^k}\frac{(z-a)^{k-n+1}}{k-n+1}+\frac{1}{(n-1)!}\frac{d^{n-1}f(a)}{\mathrm{dz}^{n-1}}\mathrm{Ln}(z-a)+\sum_{k=n}^{\infty}\frac{1}{k!}\frac{d^k f(a)}{\mathrm{dz}^k}\frac{(z-a)^{k-n+1}}{k-n+1}\}]_{z_0}^{z_1}$$

Both lateral sums are canceled, remaining the middle term

$$\frac{1}{(n-1)!}\frac{d^{n-1}f(a)}{\mathrm{dz}^{n-1}}\mathrm{Ln}\left(\frac{z_1-a}{z_0-a}\right)=\frac{1}{(n-1)!}\frac{d^{n-1}f(a)}{\mathrm{dz}^{n-1}}\mathrm{Ln}\left(\frac{\rho_0 e^{i(\theta_0+2\pi)}}{\rho_0 e^{\mathrm{i}\theta_0}}\right)=\frac{1}{(n-1)!}\frac{d^{n-1}f(a)}{\mathrm{dz}^{n-1}}\mathrm{Ln}\left(e^{2\pi\mathrm{i}}\right)$$

$$=\frac{2\pi\mathrm{i}}{(n-1)!}\frac{d^{n-1}f(a)}{\mathrm{dz}^{n-1}}$$

thus

$$\oint_{C^+}\frac{f(z)}{(\mathrm{z}-\mathrm{a})^n}\mathrm{dz}=\lim_{z\to a}\frac{2\pi\mathrm{i}}{(n-1)!}\frac{d^{n-1}f(z)}{\mathrm{dz}^{n-1}}$$