PRISMS-PF
Mechanics (Infinitesimal Strain)

Consider a elastic free energy expression of the form:

$$\Pi(\varepsilon) = \int_{\Omega} \frac{1}{2}\varepsilon : C : \varepsilon \ dV - \int_{\partial\Omega} u \cdot t \ dS \tag{1}$$

where ε is the infinitesimal strain tensor, given by $\varepsilon = \frac{1}{2}(\nabla u + \nabla u^T)$, $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ is the fourth order elasticity tensor, (λ, mu) are the Lame parameters, u is the displacement, and t is the surface traction.

1 Governing equation

Considering variations on the displacement u of the from $u + \alpha w$, we have

$$\begin{aligned}
\delta\Pi &= \frac{d}{d\alpha}\left(\int_{\Omega} \frac{1}{2}\epsilon(u+\alpha w) : C : \epsilon(u+\alpha w) \ dV - \int_{\partial\Omega} u \cdot t \ dS\right)\Bigg|_{\alpha=0} && (2)\\
&= \int_{\Omega} \nabla w : C : \epsilon \ dV - \int_{\partial\Omega} w \cdot t \ dS && (3)\\
&= \int_{\Omega} \nabla w : \sigma \ dV - \int_{\partial\Omega} w \cdot t \ dS && (4)
\end{aligned}$$

where $\sigma = C : \varepsilon$ is the stress tensor and $t = \sigma \cdot n$ is the surface traction.

The minimization of the variation, $\delta\Pi = 0$, gives the weak formulation of the governing equation of mechanics:

$$\int_{\Omega} \nabla w : \sigma \ dV - \int_{\partial\Omega} w \cdot t \ dS = 0 \tag{5}$$

If surface tractions are zero:

$$R = \int_{\Omega} \nabla w : \sigma \ dV = 0 \tag{6}$$

2 Terms for Input into PRISMS-PF

In PRISMS-PF, two sets of terms are required for elliptic PDEs (such as this one), one for the left-hand side of the equation (LHS) and one for the right-hand side of the equation (RHS). We solve $R = 0$ by casting this in a form that can be solved as a matrix inversion problem. This will involve a brief detour into the discretized form of the equation. First we derive an expression for the solution, given an initial guess, u_0:

$$0 = R(u) = R(u_0 + \Delta u) \tag{7}$$

where $\Delta u = u - u_0$. Then, applying the discretization that $u = \sum_i w^i U^i$, we can write the following linearization:

$$\frac{\delta R(u)}{\delta u} \Delta U = -R(u_0) \tag{8}$$

The discretized form of this equation can be written as a matrix inversion problem. However, in PRISMS-PF, we only care about the product $\frac{\delta R(u)}{\delta u} \Delta U$. Taking the variational derivative of $R(u)$ yields:

$$\frac{\delta R(u)}{\delta u} = \frac{d}{d\alpha} \int_\Omega \nabla w : C : \epsilon(u + \alpha w) \ dV \Big|_{\alpha=0} \tag{9}$$

$$= \int_\Omega \nabla w : C : \frac{1}{2} \frac{d}{d\alpha} \left[\nabla(u + \alpha w) + \nabla(u + \alpha w)^T \right] \ dV \Big|_{\alpha=0} \tag{10}$$

$$= \int_\Omega \nabla w : C : \frac{d}{d\alpha} \nabla(u + \alpha w) \ dV \Big|_{\alpha=0} \quad (\textit{due to the symmetry of } C) \tag{11}$$

$$= \int_\Omega \nabla w : C : \nabla w \ dV \tag{12}$$

In its discretized form $\frac{\delta R(u)}{\delta u} \Delta U$ is:

$$\frac{\delta R(u)}{\delta u} \Delta U = \sum_i \sum_j \int_\Omega \nabla N^i : C : \nabla N^j dV \ \Delta U^j \tag{13}$$

Moving back to the non-discretized form yields:

$$\frac{\delta R(u)}{\delta u} \Delta U = \int_\Omega \nabla w : C : \nabla(\Delta u) dV \tag{14}$$

Thus, the full equation relating u_0 and Δu is:

$$\int_\Omega \nabla w : \underbrace{C : \nabla(\Delta u)}_{r_{ux}^{LHS}} dV = -\int_\Omega \boldsymbol{\nabla} w : \underbrace{\sigma}_{r_{ux}} \ dV \tag{15}$$

The above values of r_{ux}^{LHS} and r_{ux} are used to define the equation terms in the following input file:
applications/mechanics/equations.cc