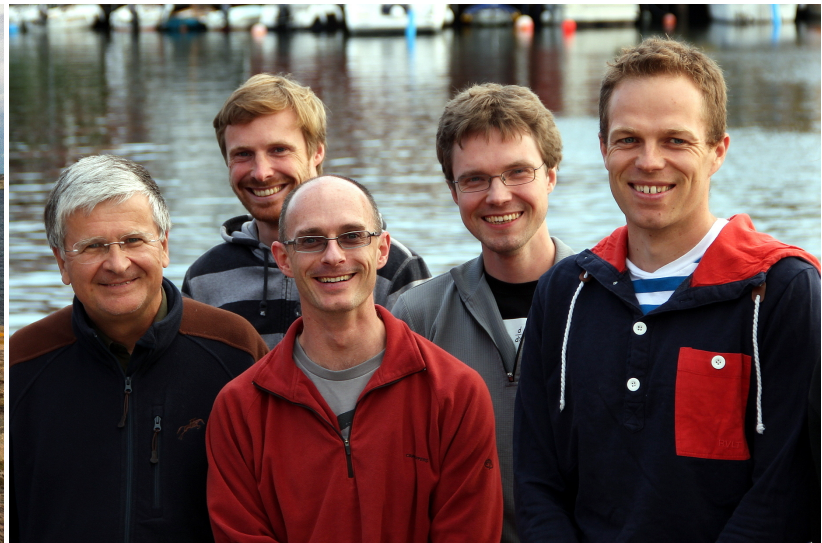*Proceedings of*

MATH ON THE ROCKS

Shape Analysis Workshop in Grundsund

Grundsund, Sweden, July 27 – August 1, 2015

Editors:

Klas Modin & Stefan Sommer

Preface

The workshop "Math on the Rocks" took place in Grundsund in the archipelago of western Sweden during the period July 27th - August 1st, 2015. The workshop was the the third edition of the Shape Analysis workshops, and it continued with the succesful format used in the earlier workshops in New Zealand and Austria:

- Math on the Beach - Shape Analysis Workshop in Foxton Beach, New Zealand, 2013
- Math in the Cabin - Shape Analysis Workshop in Bad Gastein, Austria, 2014

The aim of the week was to bring together a group of researchers with diverse backgrounds - ranging from differential geometry to applied medical image analysis - to discuss questions of common interest, that can be vaguely summarized under the heading "shape analysis". Topics discussed at the workshop included infinite-dimensional Riemannian geometry, shape analysis, image matching, computational anatomy, and topological hydrodynamics. These Proceedings contain a summary of selected discussions held during the meeting.

Participants

1. Martin Bauer, University of Vienna, Austria
2. Geir Bogfjellmo, NTNU, Norway
3. Martins Bruveris, Brunel University London, UK
4. Philipp Harms, ETHZ, Switzerland
5. Boris Khesin, University of Toronto, Canada
6. Stig Larsson, Chalmers University of Technology and the University of Gothenburg, Sweden
7. Stephen Marsland, Massey University, New Zealand
8. Peter Michor, University of Vienna, Austria
9. Gerard Misiolek, University of Notre Dame, USA
10. Klas Modin, Chalmers University of Technology and the University of Gothenburg, Sweden
11. Jakob Møller-Andersen, DTU, Denmark
12. Stephen Preston, University of Colorado Boulder, USA
13. Stefan Sommer, University of Copenhagen, Denmark
14. Olivier Verdier, Bergen University College, Norway
15. François-Xavier Vialard, Université Paris-Dauphine, France

Schedule

We kept the formal schedule of the week to a minimum, so that the participants would have time to talk and work with each other. Every day we had two or three talks or discussion sessions led by a participant. These talks or the resulting discussions are summarized by each participant in the second half of this report.

Tuesday July 28th
    Olivier Verdier: What is equivariance of numerical methods?
    Stephen Preston: N/A
    Martins Bruveris: How to define Sobolev metrics?

Wednesday July 29th
    Gerard Misiolek: Continuity properties of the solution map of the Euler equations in Hölder spaces
    Boris Khesin: Invariants of functions on symplectic surfaces and ideal hydrodynamics
    Stefan Sommer: Anisotropic distributions on manifolds and most probable paths

Thurday July 30th
    François-Xavier Vialard: Generalized optimal transport
    Geir Bogfjellmo: Character groups of Hopf algebras are Lie groups

Friday July 31th
    Klas Modin: Information geometry and matrix factorizations
    Stephen Marsland: Image registration for landmarks with uncertainty
    Stig Larsson: N/A
    Jakob Møller-Andersen: Geodesics of constant coefficient Sobolev metric on curves

Saturday August 1st
    Martin Bauer and Philipp Harms: Metrics with prescribed horizontal bundle on spaces of curves
    Peter Michor: Olaf Müller's "k-safe"-theory of elliptic differential operators with Sobolev coefficients, and its uses for Laplacians of Sobolev Riemannian metrics

Gothenburg, Copenhagen
November 2015

Klas Modin
Stefan Sommer

Acknowledgements

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Contents

Chapter 1
Metrics with prescribed horizontal bundle on spaces of curves

Martin Bauer and Philipp Harms

Abstract We study metrics on the shape space of curves that induce a prescribed splitting of the tangent bundle. More specifically, we consider reparametrization invariant metrics G on the space $\mathrm{Imm}(S^1,\mathbb{R}^2)$ of parametrized regular curves. For many metrics the tangent space $T_c\,\mathrm{Imm}(S^1,\mathbb{R}^2)$ at each curve c splits into vertical and horizontal components (with respect to the projection onto the shape space $B_i(S^1,\mathbb{R}^2) = \mathrm{Imm}(S^1,\mathbb{R}^2)/\mathrm{Diff}(S^1)$ of unparametrized curves and with respect to the metric G). In a previous article we characterized all metrics G such that the induced splitting coincides with the natural splitting into normal and tangential parts. In these notes we extend this analysis to characterize all metrics that induce any prescribed splitting of the tangent bundle.

1.1 Introduction

Let $\mathrm{Imm}(S^1,\mathbb{R}^2)$ be the space of regular planar curves. Our center of attention lies on the shape space of unparametrized curves, which can be identified with the quotient space

$$B_i(S^1,\mathbb{R}^2) = \mathrm{Imm}(S^1,\mathbb{R}^2)/\mathrm{Diff}(S^1)\,.$$

Here, $\mathrm{Diff}(S^1)$ denotes the Lie group of all smooth diffeomorphisms on the circle, which acts smoothly on $\mathrm{Imm}(S^1,\mathbb{R}^2)$ via composition from the right:

$$\mathrm{Imm}(S^1,\mathbb{R}^2)\times\mathrm{Diff}(S^1) \to \mathrm{Imm}(S^1,\mathbb{R}^2), \qquad (c,\varphi)\mapsto c\circ\varphi\,.$$

The quotient space $B_i(S^1,\mathbb{R}^2)$ is not a manifold, but only an orbifold with isolated singularities (see [6] for more information). A strong motivation for considering this space – and in particular Riemannian metrics thereon – comes from the field of shape analysis [8, 9, 13, 14, 15, 2]. See [3] for a recent overview on various metrics on these spaces.

Given a reparametrization invariant metric G on $\mathrm{Imm}(S^1,\mathbb{R}^2)$, we can (under certain conditions) induce a unique Riemannian metric on the quotient space $B_i(S^1,\mathbb{R}^2)$ such that the projection

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$$\pi : \mathrm{Imm}(S^1, \mathbb{R}^2) \to B_i(S^1, \mathbb{R}^2) := \mathrm{Imm}(S^1, \mathbb{R}^2)/\mathrm{Diff}(S^1) \tag{1.1}$$

is a Riemannian submersion. A detailed description of this construction is given in [5, Section 4]. For many metrics, $T\pi$ induces a splitting of the tangent bundle $T\mathrm{Imm}(S^1, \mathbb{R}^2)$ into a vertical bundle, which is defined as the kernel of $T\pi$, and a horizontal bundle, defined as the G-orthogonal complement of the vertical bundle:

$$T\mathrm{Imm}(S^1, \mathbb{R}^2) = \ker T\pi \oplus (\ker T\pi)^{\perp, G} =: \mathrm{Ver} \oplus \mathrm{Hor}. \tag{1.2}$$

If one can lift any curve in $B_i(S^1, \mathbb{R}^2)$ to a horizontal curve in $\mathrm{Imm}(S^1, \mathbb{R}^2)$, then there is a one-to-one correspondence between geodesics on shape space $B_i(S^1, \mathbb{R}^2)$ and horizontal geodesics on $\mathrm{Imm}(S^1, \mathbb{R}^2)$ [5, Section 4.8].

In [4] we described all metrics on $\mathrm{Imm}(S^1, \mathbb{R}^2)$ such that the splitting (1.2) coincides with the natural splitting into components that are tangential and normal to the immersed surface. In this article we characterize all metrics that induce an arbitrary given splitting of $T\mathrm{Imm}(S^1, \mathbb{R}^2)$, generalizing our previous result.

As an application of our result we investigate a splitting that could be used to develop efficient numerics for the horizontal geodesic equation. The splitting is the decomposition of $T_c\mathrm{Imm}(S^1, \mathbb{R}^2)$ into deformations preserving the speed $\|\dot{c}\|$ and a suitable complement.

1.2 The decomposition theorem

1.2.1 Assumptions

Following [5], we now describe the class of metrics that we study in this article. We define all metrics via a so-called inertia operator L by the formula

$$G_c^L(h, k) = \int_{S^1} \langle L_c h, k \rangle ds, \tag{1.3}$$

where $ds = |c'| d\theta$ denotes integration by arc-length. We assume that L is a smooth bundle automorphism of $T\mathrm{Imm}(S^1, \mathbb{R}^2)$ such that at every $c \in \mathrm{Imm}(S^1, \mathbb{R}^2)$, the operator

$$L_c : T_c\mathrm{Imm}(S^1, \mathbb{R}^2) \to T_c\mathrm{Imm}(S^1, \mathbb{R}^2)$$

is a pseudo-differential operator of order $2l$ which is symmetric and positive with respect to the L^2-metric on $\mathrm{Imm}(S^1, \mathbb{R}^2)$. Moreover, we assume that L is invariant under the action of the reparametrization group $\mathrm{Diff}(S^1)$ acting on $\mathrm{Imm}(S^1, \mathbb{R}^2)$, i.e.,

$$L_{c \circ \varphi}(h \circ \varphi) = L_c(h) \circ \varphi \qquad \text{for all } \varphi \in \mathrm{Diff}(S^1).$$

These assumptions remain in place throughout this work. Their immediate use is as follows: being symmetric and positive, L induces a Sobolev-type metric on the manifold of immersions through equation (1.3). The $\mathrm{Diff}(S^1)$-invariance of L implies the $\mathrm{Diff}(S^1)$-invariance of the metric G^L. Assuming that the decomposition in horizontal and vertical bundles exists, there is a unique metric on $B_i(S^1, \mathbb{R}^2)$ such that the projection (1.1) is a Riemannian submersion (see [5, Thm. 4.7]). Then the resulting geometry of shape space is mirrored by the "horizontal geometry" on the manifold of immersions.

1.2.2 Splitting into horizontal and vertical subbundles

By definition, the horizontal and vertical bundles are given by

$$\mathrm{Ver}_c := \ker(T\pi) = \mathrm{Tan}_c\,,$$
$$\mathrm{Hor}_c := (\mathrm{Ver}_c)^{\perp,G} = \{h \in T_c\mathrm{Imm}(S^1,\mathbb{R}^2) : L_c h \in \mathrm{Nor}_c\}\,.$$

Note, that $\mathrm{Hor} \oplus \mathrm{Ver}$ might not span all of $T\mathrm{Imm}$ in this infinite-dimensional setting.

1.2.3 Constructing metrics that induce a prescribed splitting.

We now state our main result.

Theorem 1.1 (Decomposition theorem). *Let $c \in \mathrm{Imm}(S^1,\mathbb{R}^2)$ and let $H(c)$ be any complement of $\mathrm{Tan}(c)$, i.e.,*

$$T_c\mathrm{Imm}(S^1,\mathbb{R}^2) = \mathrm{Tan}(c) \oplus H(c)\,. \tag{1.4}$$

Then the following conditions on a metric G^L are equivalent:

(a)The tangent bundle splits into a vertical and horizontal bundle (1.2) *and this splitting coincides with* (1.4).
(b)The inertia operator L_c admits a decomposition

$$L_c = (P^{\mathrm{tan}})^* \circ \tilde{L}_c \circ P^{\mathrm{tan}} + (P^H)^* \circ \tilde{L}_c \circ P^H,$$

where $\tilde{L}_c : T_c\mathrm{Imm}(S^1,\mathbb{R}^2) \to T_c\mathrm{Imm}(S^1,\mathbb{R}^2)$ is an invertible pseudo-differential operator and where * denotes the adjoint with respect to the reparametrization-invariant L^2-metric.

The theorem follows directly as a special case of the following lemma by setting $K(c) = \mathrm{Tan}(c)$.

Lemma 1.1. *Let $c \in \mathrm{Imm}(S^1,\mathbb{R}^2)$ and let*

$$T_c\mathrm{Imm}(S^1,\mathbb{R}^2) = H(c) \oplus K(c) \tag{1.5}$$

be a given splitting with corresponding projections P^H, P^K. Then the following conditions on a metric G^L are equivalent:

(a)The subspace $H(c)$ is G^L-orthogonal to $K(c)$.
(b)The operator L_c has a decomposition

$$L_c = (P^H)^* \circ \tilde{L}_c \circ P^H + (P^K)^* \circ \tilde{L}_c \circ P^K,$$

where $\tilde{L}_c : T_c\mathrm{Imm}(S^1,\mathbb{R}^2) \to T_c\mathrm{Imm}(S^1,\mathbb{R}^2)$ is an invertible pseudo-differential operator and where * denotes the adjoint with respect to the reparametrization-invariant L^2-metric.

Proof. Assume (a) and let $h = P^H(h) + P^K(h) := h^H + h^K$. We have

$$\begin{aligned}
G_c(h^H &+ h^K, k^H + k^K) \\
&= G_c(h^H, k^H) + G_c(h^H, k^K) + G_c(h^K, k^H) + G_c(h^K, k^K) \\
&= G_c(h^H, k^H) + 0 + G_c(h^K, k^K)\,,
\end{aligned}$$

where the last equality follows from the orthogonality of the splitting with respect to the metric G. Now the formula for the operator L follows directly.

Conversely, assume (b). To see the orthogonality we calculate

$$G_c(h^H, k^K) = \int_{S^1} \langle Lh^H, k^K \rangle ds = \int_{S^1} \langle 0 + (P^H)^*(\tilde{L}_c(h^H)), k^K \rangle ds$$
$$= \int_{S^1} \langle (\tilde{L}_c(h^H), P^H(k^K)) \rangle ds = 0 \,.$$

1.3 Applications

1.3.1 The splitting into tangential and normal vector fields

In this section, we want to recover the results from [4]. Letting n denote the unit length normal vector field to the curve c, we define

$$\text{Nor}(c) := \left\{ h = a.n : a \in C^\infty(S^1) \right\} \,. \tag{1.6}$$

This yields a splitting

$$T_c \text{Imm}(S^1, \mathbb{R}^2) = \text{Tan}(c) \oplus \text{Nor}(c) \tag{1.7}$$

with corresponding projections

$$(P^{\text{tan}})^*(h) = P^{\text{tan}}(h) = \langle h, v \rangle v,$$
$$(P^{\text{nor}})^*(h) = P^{\text{nor}}(h) = \langle h, n \rangle n \,.$$

By Theorem 1.1 the splitting into horizontal and vertical bundles coincides with the above splitting (1.7) if and only if L can be written as

$$L_c = (P^{\text{tan}})^* \circ \tilde{L}_c \circ P^{\text{tan}} + (P^{\text{tan}})^* \circ \tilde{L}_c \circ P^{\text{tan}}. \tag{1.8}$$

This is the content of the main theorem of [4].

A particular class of metrics inducing this splitting are almost local metrics [7, 1]. Further examples of higher order metrics are given in [4].

1.3.2 The splitting into tangential and constant speed preserving vector fields

In this section we consider a different splitting, which is motivated by investigations of Riemannian metrics on the space of arc length parametrized curves [10, 11, 12]. In the following lemma we characterize all tangent vectors that preserve constant speed parametrization.

Lemma 1.2. *Let $c \in \text{Imm}(S^1, \mathbb{R}^2)$ be parametrized by constant speed. Then a tangent vector $h \in T_c \text{Imm}$ preserves the parametrization of c if and only if*

$$\langle D_s^2 h, v \rangle + \kappa \langle D_s h, n \rangle = 0 \tag{1.9}$$

Here $v = \frac{c'}{|c'|}$ the unit length tangent vector field, $n = iv$ denotes the unit length normal vector field and κ the curvature of the curve.

Proof. This follows immediately from the infinitesimal action of h on the volume form ds:

$$D_{c,h}(ds) = D_{c,h}(|c'|d\theta) = \frac{\langle c',h'\rangle}{|c'|}d\theta = \langle v, D_s h\rangle ds . \tag{1.10}$$

Equation (1.9) is obtained by setting $D_s\langle v, D_s h\rangle = 0$.

We now describe a decomposition of $T\mathrm{Imm}(S^1,\mathbb{R}^2)$ in a subspace that preserves constant speed and a complement with values in the tangential bundle.

Lemma 1.3. *For each regular curve c the tangent bundle $T_c\mathrm{Imm}(S^1,\mathbb{R}^2)$ can be decomposed as*

$$T_c\mathrm{Imm}(S^1,\mathbb{R}^2) = \mathrm{Tan}(c) \oplus \mathrm{Arc}^0(c) , \tag{1.11}$$

where

$$\mathrm{Tan}(c) := \left\{h = f.v : a \in C^\infty(S^1)\right\} , \tag{1.12}$$

$$\mathrm{Arc}^0(c) := \left\{h = a.n + b.v : D_s^2 b = D_s(a\kappa) \text{ and } b(0) = 0\right\} . \tag{1.13}$$

The corresponding projections onto these subspaces are given by

$$P^{\mathrm{Arc}^0}(k) := k^{\mathrm{arc}} = \langle k,n\rangle n + bv , \tag{1.14}$$

$$P^{\mathrm{Tan}}(k) := k^{\mathrm{tan}} = \langle k,v\rangle v - bv , \tag{1.15}$$

where b solves

$$D_s^2 b = D_s(\langle k,n\rangle\kappa) \text{ with } b(0) = 0 . \tag{1.16}$$

Proof. We start by showing that the projections take values in the correct spaces. Let $a = \langle k,n\rangle$. We calculate:

$$\langle D_s P^{\mathrm{Arc}^0}(k),n\rangle = \langle D_s(an+bv),n\rangle = \langle (D_s a + b\kappa)n + (D_s b - a\kappa)v,n\rangle$$
$$= D_s a + b\kappa$$
$$\langle D_s^2 P^{\mathrm{Arc}^0}(k),v\rangle = \langle D_s^2(an+bv),v\rangle = \langle D_s((D_s a + b\kappa)n + (D_s b - a\kappa)v),v\rangle$$
$$= -\kappa(D_s a + b\kappa) + D_s^2 b - D_s(a\kappa)$$

Thus we have

$$\langle D_s^2 h,v\rangle + \kappa\langle D_s h,n\rangle = -\kappa(D_s a + b\kappa) + D_s^2 b - D_s(a\kappa) + \kappa(D_s a + b\kappa)$$
$$= D_s^2 b - D_s(a\kappa) .$$

This shows that $P^{\mathrm{Arc}^0}(k)$ preserves constant speed parametrization. We choose $b(0) = 0$ to enforce uniqueness of the solutions. The mapping P^{Tan} takes values in Tan by definition. The projection property for P^{Arc^0} is clear. For P^{Tan} we have

$$(P^{\mathrm{Tan}})^2(k) = (\mathrm{Id} - P^{\mathrm{Arc}^0})^2(k) = (\mathrm{Id} - P^{\mathrm{Arc}^0})(k) = P^{\mathrm{Tan}}(k) .$$

Note, that this proves also that P^{Tan} (resp. P^{Arc^0}) are surjective mappings onto Tan (resp. Arc^0). As P^{Tan} and P^{Arc^0} are continuous mappings, their kernels Arc and Tan are closed subspaces. This shows that $T\text{Imm}(S^1,\mathbb{R}^2)$ splits as in (1.11).

Using Theorem 1.1 we can now construct metrics on $\text{Imm}(S^1,\mathbb{R}^2)$ such that the horizontal bundle coincides with $\text{Arc}^0(c)$. The general formula for these metrics is given by:

$$G_c(h,k) = \int_{S^1} \langle L_c(h^{\text{tan}}), k^{\text{tan}} \rangle ds + \int_{S^1} \langle L_c(h^{\text{arc}}), k^{\text{arc}} \rangle ds$$

The simplest example, which corresponds to the identity operator L, is

$$\begin{aligned}
G_c(h_1,h_2) &= \int_{S^1} \langle h_1^{\text{tan}}, h_2^{\text{tan}} \rangle ds + \int_{S^1} \langle h_1^{\text{arc}}, h_2^{\text{arc}} \rangle ds \\
&= \int_{S^1} \langle h_1 - b_1.v, h_2 - b_2.v \rangle ds + \int_{S^1} b_1 b_2 + a_1 a_2 ds \\
&= \int_{S^1} 2a_1 a_2 + \tilde{b}_1 \tilde{b}_2 - b_1 \tilde{b}_2 - \tilde{b}_1 b_2 + b_1 b_2 \, ds
\end{aligned}$$

with

$$a_i = \langle h_i, n \rangle, \quad \tilde{b}_i = \langle h_i, v \rangle, \quad D_s^2 b_i = D_s(a_i \kappa), \quad b_i(0) = 0 .$$

For $h_1 = h_2 = h$ this reads as

$$G_c(h,h) = \int_{S^1} 2a^2 + (\tilde{b} - b)^2 \, ds .$$

On the space of constant-speed parametrized immersions this induces the L^2-metric studied in [10, 11].

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Chapter 2
Hörmander's condition for normal bundles on spaces of immersions

Martin Bauer and Philipp Harms

Abstract Several representations of geometric shapes involve quotients of mapping spaces. The projection onto the quotient space defines two sub-bundles of the tangent bundle, called the horizontal and vertical bundle. We investigate in these notes the sub-Riemannian geometries of these bundles. In particular, we show for a selection of bundles which naturally occur in applications that they are either bracket generating or integrable.

2.1 Introduction

Several representations of geometric shapes involve quotients of mapping spaces. Three examples are presented in the diagram below:

$$\begin{array}{ccc}
\mathrm{Imm}(M,N) & \mathrm{Diff}(N) & \mathrm{Diff}(M) \\
\downarrow & \downarrow & \downarrow \\
\mathrm{Imm}(M,N)/\mathrm{Diff}(M) & \mathrm{Emb}(M,N) & \mathrm{Dens}(M) = \mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)
\end{array}$$

The first example is the quotient of embeddings modulo reparametrizations. In the second example, $\mathrm{Diff}(N)$ acts on some fixed "template" element of $\mathrm{Emb}(M,N)$. The third example is Moser's representation of densities as diffeomorphisms modulo volume preserving diffeomorphisms [8, 4].

Let us abstract from these examples and consider a submersion $\pi\colon P \to Q$ between possibly infinite-dimensional manifolds. If P is endowed with a (weak) Riemannian metric G, two natural sub-bundles of TP appear: the vertical bundle Ver is defined as the kernel of $T\pi$ and the horizontal bundle Hor as the set of tangent vectors in TP which are G-orthogonal to Ver. Note that Ver + Hor might or might not span all of TP. However, any closed complement of the vertical bundle is the horizontal bundle of some Riemannian metric, as was recently shown in [1] for the special case of planar curves.

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While the vertical bundle is always integrable (the integral manifolds are the fibers of the projection), it is in general not clear whether the horizontal bundles are integrable or, at the other extreme, bracket generating. This question is interesting for several reasons.

- Integrability of the horizontal bundle is necessary for lifting loops in Q to horizontal loops in P. This is a natural task in, for example, the analysis of cardiac cycles, which can be represented as loops in shape space.
- If the horizontal bundle is integrable, then the horizontal geodesic equation can be solved in the lower-dimensional coordinate system of the integral manifold instead of the higher-dimensional coordinate system of P.
- If on the other hand the horizontal bundle is bracket generating, then its integral manifold is a dense subset of P, and any two points in the integral manifold can be connected by a horizontal curve [6, 5].

2.2 Results

Definition 2.1. Let M be a compact manifold and $(N,\overline{g})$ a Riemannian manifold. Then the sub-bundles Tan and Nor of $T\mathrm{Imm}(M,N)$ are given at each $f \in \mathrm{Imm}(M,N)$ by

$$\mathrm{Tan}_f = \{Tf \circ X : X \in \mathfrak{X}(M)\},$$
$$\mathrm{Nor}_f = \{h \in T_f\mathrm{Imm}(M,N) : \forall x \in M, \forall X \in T_xM, \overline{g}(h(x), Tf(X)) = 0\}.$$

Remark 2.1. The bundle Tan is the vertical bundle of the projection onto the space of unparametrized immersions and is integrable. Indeed, the group $\mathrm{Diff}(M)$ acts on $\mathrm{Imm}(M,N)$ by composition from the right and the $\mathrm{Diff}(M)$-orbits are integral manifolds for Tan.

The following theorem shows that Nor is bracket generating and that the first bracket is enough to generate all of the tangent space.

Theorem 2.1. *Let M be compact and* $\dim(N) = \dim(M) + 1$. *Then*

$$\mathrm{Nor} + [\mathrm{Nor}, \mathrm{Nor}] = T\mathrm{Imm}(M,N).$$

Proof. Assume that a normal vector field n to $f \in \mathrm{Imm}(M,N)$ is defined on all of M and locally around f. Then any functions a,b on M define local vector fields an, bn on $\mathrm{Imm}(M,N)$. Let ∇ denote the covariant derivative on $\mathrm{Imm}(M,N)$ which is associated to $(N,\overline{g})$; see Section 2.3. Then the Lie bracket $[an,bn]$ can be expressed using covariant derivatives because ∇ is torsion-free by Lemma 2.1. By the variational formula for the normal vector in Lemma 2.2,

$$[an,bn] = \nabla_{an}(bn) - \nabla_{bn}(an) = b\nabla_{an}n - a\nabla_{bn}n = Tf\left(a\,\mathrm{grad}^g\,b - b\,\mathrm{grad}^g\,a\right)$$
$$= Tf\left(g^{-1}(a\mathrm{d}b - b\mathrm{d}a)\right). \tag{2.1}$$

By Lemma 2.3, all one-forms on M are linear combinations of one-forms $a\mathrm{d}b - b\mathrm{d}a$. This shows $\mathrm{Tan} \supseteq [\mathrm{Nor} + \mathrm{Nor}]$.

The assumption that n is defined globally on M can be eliminated by localization. Indeed, as M is compact, any vector field $X \in \mathfrak{X}(M)$ is a finite sum of vector fields supported in domains U such that n is defined in a neighborhood of $\overline{U}$. By Remark 2.3, the functions a,b can be chosen with support in $\overline{U}$. It follows that an, bn are well-defined.

Remark 2.2. If N is a linear space, the covariant derivatives in the proof of Theorem 2.1 can be replaced by directional derivatives. For example, in the case of planar curves $M = S^1, N = \mathbb{R}^2$, Equation (2.1) evaluated at some $c \in \text{Imm}(S^1, \mathbb{R}^2)$ takes the form

$$[an, bn]_c = D_{c,an}(bn) - D_{c,bn}(an) = (aD_s b - bD_s a)v.$$

Example 2.1. A further example of an integrable bundle are the arc-length preserving deformations of planar curves. Let $M = S^1, N = \mathbb{R}^2$, and define the vector bundle Arc $\subset T\text{Imm}(S^1, \mathbb{R}^2)$ at each $c \in \text{Imm}(S^1, \mathbb{R}^2)$ by

$$\text{Arc}_c = \{h \in T_c \text{Imm}(S^1, \mathbb{R}^2) : D_s(D_{c,h} ds/ds) = 0\}.$$

Then the bundle Arc is integrable and the collections of curves c whose velocities $|\partial_\theta c| \in C^\infty(S^1)$ are multiples of each other are integral manifolds for Arc.

2.3 Covariant derivative on $\text{Imm}(M, N)$

We recall some definitions and results of [2, Section 4.2]. Let ∇ be the Levi-Civita covariant derivative of the Riemannian manifold $(N, \overline{g})$. Then $\nabla_X h \colon Q \to TN$ is well-defined for any manifold Q, vector field $X \in \mathfrak{X}(Q)$, and mapping $h \colon Q \to TN$. This covariant derivative can be extended to $\text{Imm}(M, N)$ using the isomorphism $\wedge \colon C^\infty(Q, C^\infty(M, TN)) \to C^\infty(Q \times M, TN)$ and its inverse $\vee$. Let $h \colon Q \to T\text{Imm}(M, N)$ and $X \in \mathfrak{X}(Q)$. Then $\nabla_X h$ is defined as $(\nabla_{X \times 0} h^\wedge)^\vee$.

Lemma 2.1. *The covariant derivative ∇ on $\text{Imm}(M, N)$ is torsion-free, i.e., $\nabla_X Y - \nabla_Y X = [X, Y]$ holds for any $X, Y \in \mathfrak{X}(\text{Imm}(M, N))$.*

Proof. Let X, Y be vector fields on a manifold Q and $f \colon Q \times M \to TN$. Then $X \times 0$ and $Y \times 0$ are vector fields on $Q \times M$ and

$$\nabla_{X \times 0} Tf(Y \times 0) - \nabla_{Y \times 0} Tf(X \times 0) = Tf([X \times 0, Y \times 0]) = Tf([X, Y] \times 0)$$

because the Levi-Civita covariant derivative on $(N, \overline{g})$ is torsion-free [7, Section 22.10]. The statement of the Lemma follows by setting $Q = \text{Imm}(M, N)$, $f(g, x) = g(x)$ for all $g \in \text{Imm}(M, N)$ and $x \in M$, and noting that $Tf(X \times 0) = X^\wedge$. $\square$

2.4 Variational formula for the normal vector field

Lemma 2.2. *[3, Section 4.11] Let X be a vector field on $\text{Imm}(M, N)$. Then the variation of the normal vector field n in the direction of X is*

$$\nabla_X n = -Tf\left(LX^\top\right) + \text{grad}^g\, \overline{g}(X, n)\Big),$$

where $X = Tf \circ X^\top + \overline{g}(X, n)n$ is the decomposition in tangential and normal components, $g = f^*\overline{g}$ is the pull-back of $\overline{g}$ to M, and L is the Weingarten map.

2.5 Auxiliary result about one-forms

Lemma 2.3. *Let M be compact. Then $\Omega^1(M) = \mathrm{span}_{\mathbb{R}}\{a\,\mathrm{d}b - b\,\mathrm{d}a : a, b \in C^\infty(M)\}$.*

Proof. All one-forms $f\,\mathrm{d}g$ with positive f can be generated by elements of the form $a\,\mathrm{d}b - b\,\mathrm{d}a$. Indeed, set $a = (fe^{-g})^{1/2}, b = (fe^g)^{1/2}$ and check that $a\,\mathrm{d}b - b\,\mathrm{d}a = f\,\mathrm{d}g$. Moreover, the closed one-form $\mathrm{d}g$ can be generated by setting $a = 1, b = g$. This allows one to generate all one-forms $f\,\mathrm{d}g$ with f bounded from below. As every function can be decomposed in a function bounded from below, and one from above, this allows one to generate all one-forms $f\,\mathrm{d}g$.

Let U be an open set in M such that there exist functions $x^1, \dots, x^d$ defined on all of M providing a coordinate system on U. Then any one-from α with support in U can be written as $\alpha = \sum_{i=1}^d \alpha(\partial x^i)\mathrm{d}x^i$, showing that α is a linear combination of expressions of the form $f\,\mathrm{d}g$.

Finally, any one-form on M is a sum of finitely many one-forms supported in open sets U as above. To see this, note that any point x in M has an open neighborhood U with the above properties. As M is compact, finitely many such neighborhoods $U_1, \dots, U_n$ cover M. Let $\phi_1, \dots, \phi_n$ be a partition of unity subordinate to $U_1, \dots, U_n$. Then any one-form α can be written as $\alpha = \sum_{i=1}^n \phi_i \alpha$ and $\phi_i \alpha$ is supported in U_i.

Remark 2.3. If α is a one-form with support in an open set $U \subseteq M$, then it can be represented as a linear combination of forms $a\,\mathrm{d}b - b\,\mathrm{d}a$ with functions a, b supported in $\overline{U}$. Indeed, at each step of the proof of Lemma 2.3, the functions a, b may be multiplied by a bump function which vanishes outside of U and equals 1 on the support of α.

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Chapter 3
How to define Sobolev metrics?

Martins Bruveris

3.1 Variety of metrics

Take the simple sentence

> The geodesic equation of the right-invariant H^1-metric on $\mathrm{Diff}(S^1)$ is the Camassa–Holm equation.

For this sentence to have a mathematical meaning, we have to specify, which H^1-metric we mean. In this case we mean the metric induced by the inner product

$$\langle X, Y \rangle_{H^1} = \int_{S^1} XY + \alpha^2 X'Y' \, \mathrm{d}x \,.$$

But there are other possible inner products, for example

$$\langle X, Y \rangle_{\mu \dot{H}^1} = \int_{S^1} X \, \mathrm{d}x \int_{S^1} Y \, \mathrm{d}x + \int_{S^1} X'Y' \, \mathrm{d}x \,.$$

However the geodesic equation for this metric is the μ-Hunter–Saxton equation (the name comes from the averaging operator $\mu(X) = \int_{S^1} X \, \mathrm{d}x$).

3.2 What is the aim?

We want to have a general definition, such that the statement

> Let G be a Sobolev metric of order q on $\mathcal{X}$.

has a rigorous mathematical meaning; furthermore this definition should encompass all or most used examples of Sobolev metrics. At the moment we have a pool of examples; metrics that have been studied, given names, used in applications; what we are lacking is a classification.

What is $\mathcal{X}$? The two most-studied cases are the diffeomorphism group and the space of curves. Regarding the diffeomorphism group, both $\mathrm{Diff}(M)$, where M is a compact manifold, as well as $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^d)$, with $\mathcal{A}$ denoting decay conditions, have been considered. The space of curves is $\mathrm{Imm}(S^1, \mathbb{R}^d)$, although

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open curves as well as manifold-valued curves are possible as well. An answer should include the space $\operatorname{Imm}(M,N)$ as the unifying case. The following diagram shows the relations between these spaces.

$$
\begin{array}{ccc}
\operatorname{Imm}(M,N) & \longleftarrow & \operatorname{Diff}(M) \\
\nearrow \quad \nwarrow & & \\
\operatorname{Imm}(S^1,N) \qquad \operatorname{Imm}(M,\mathbb{R}^d) & & \\
\nwarrow \qquad \nearrow & & \\
\operatorname{Imm}(S^1,\mathbb{R}^2) \longrightarrow \operatorname{Imm}(S^1,\mathbb{R}^d) & &
\end{array}
$$
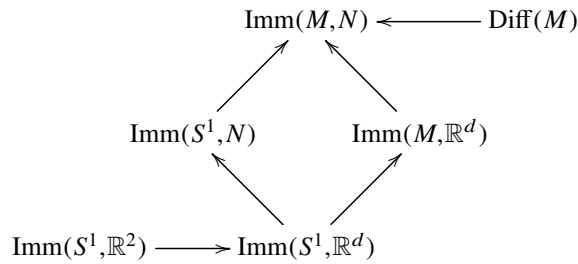
Of course there are other spaces, that we haven't even touched: the space of Riemannian metrics, $\operatorname{Met}(M)$, the space of densities $\operatorname{Dens}(M)$ as well as others.

3.3 Metrics on the space of curves

Let us mention some examples, that we would like to include in a comprehensive definition. The simplest Sobolev metrics are those with constant coefficients.

$$
G_c(h,k) = \int_{S^1} a_0 \langle h,k \rangle + \cdots + a_n \langle D_s^n h, D_s^n k \rangle \, \mathrm{d}s \, .
$$

Sometimes we do not want a_j to be constants. This is the case for scale-invariant metrics, which have the form

$$
G_c(h,k) = \int_{S^1} a_0 \ell_c^{-3} \langle h,k \rangle + \cdots + a_n \ell_c^{2n-3} \langle D_s^n h, D_s^n k \rangle \, \mathrm{d}s \, .
$$

The familiy of elastic metrics, given by

$$
G_c(h,k) = \int_{S^1} a \langle D_s h, v \rangle \langle D_s k, v \rangle + b \langle D_s h, n \rangle \langle D_s k, n \rangle \, \mathrm{d}s \, ,
$$

require even more general expressions for the coefficients. Then there are metrics, which are constant on constant-speed curves. They are given by

$$
G_c(h,k) = \langle\!\langle h \circ \psi_c^{-1}, k \circ \psi_c^{-1} \rangle\!\rangle, \text{ with } \psi_c' = \frac{2\pi}{\ell_c} |c'| \, ,
$$

where $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is a Sobolev inner product, possibly of fractional order. Fractional order metrics can also be written directly using powers of the Laplace operator

$$
G_c(h,k) = \int_{S^1} \left\langle \left(\operatorname{Id} - D_s^2 \right)^q h, k \right\rangle \, \mathrm{d}s \, .
$$

3.4 The diffeomorphism group

For right-invariant metrics on the diffeomorphism group the situation is a bit simpler, although by no means trivial. A right-invariant metric is determined by the inner product on the tangent space at the identity, which can be written using an operator A as

$$G_\varphi(X,Y) = \langle X \circ \varphi^{-1}, Y \circ \varphi^{-1} \rangle_A = \int_M g\left(A\left(X \circ \varphi^{-1}\right), Y \circ \varphi^{-1}\right) \mathrm{dvol},$$

when (M,g) is a Riemannian manifold. What shall A be? A first attempt could be

> Let A be a symmetric (w.r.t. L^2), positive, elliptic pseudo-differential operator (belonging to some symbol class) of order $2q$.

A slightly different approach would be to start with an operator B and apply it on both sides to ensure symmetry,

$$G_\varphi(X,Y) = \int_M g(B(X \circ \varphi^{-1}), B(Y \circ \varphi^{-1})) \, \mathrm{dvol} .$$

In this case $A = B^*B$ and the question becomes, for which classes of operators do both definitions lead to the same class of metrics. In other words, given A, can we always find B and given B, does B^*B always lie in the correct class of operators?

Then again, ellipticity might be too restrictive. For example, the following family of metrics,

$$G_{\mathrm{Id}}(X,Y) = \int_{\mathbb{R}^d} \left\langle \left(\mathrm{Id} - \frac{\eta^2}{p}\Delta\right)^p \circ \left(\mathrm{Id} - \frac{1}{\epsilon^2}\nabla \circ \mathrm{div}\right) X, Y \right\rangle \mathrm{d}x .$$

has been used to approximate Euler equations for incompressible fluids by compressible flow, where deviation from incompressibility is increasingly penalized. The corresponding operator is not elliptic.

3.5 Why do we bother?

There seems to exist a collection of results in the literature, that seem to depend only on the order of the metric and to a much lesser extent on its particular form. Examples include

- Smoothness of the geodesic spray on suitable Sobolev completions
- Vanishing and non-vanishing behaviour of the geodesic distance
- Fredholmness properties of the exponential map
- Completeness properties of the metric.

It would be helpful to future researchers to identify or at least conjecture about the "correct" class of metrics, where these results hold.

There are features for Sobolev metrics, that seem to be quite general. The most striking example is the fact that the geodesic equation of a Sobolev metric preserves the smoothness of the initial conditions exactly (within the family of H^q-spaces); this property seems to require only that the Sobolev metric in question is invariant under reparametrizations (this equals right-invariance in the case of the diffeomorphism group).

3.6 An attempt at a definition

We conclude with an attempt at a definition.

A Riemannian metric G on $\mathrm{Imm}(M,\mathbb{R}^d)$ is called a *Sobolev metric of order q*, if G can be extended to a smooth map

$$G : \mathrm{Imm}(M,\mathbb{R}^d) \times H^q(M,\mathbb{R}^d) \times H^q(M,\mathbb{R}^d) \to \mathbb{R}.$$

We call G a *strong Sobolev metric*, if G can be extended to

$$G : \mathcal{I}^q(M,\mathbb{R}^d) \times H^q(M,\mathbb{R}^d) \times H^q(M,\mathbb{R}^d) \to \mathbb{R}.$$

Whether this is the right definition or not, and if not, how it is to be improved, is left open for debate. Note however, that somewhere analysis has to come into the picture. One can make the definition easy to verify, in which case analysis will be needed to prove interesting properties. Or the definition can be written, such that analysis is necessary to show that a given metrics satisfies the definition. This is the case with the above definition. For example, showing that a metric of fractional order extends to a smooth strong metric on the Sobolev completions is usually hard work.

Chapter 4
On the Wasserstein-Fisher-Rao metric

François-Xavier Vialard

Abstract This note gives a summary of the presentation that I gave at the workshop on shape analysis[1]. Based on [CSPV15, CPSV15], we present a generalization of optimal transport to measures that have different total masses. This generalization enjoys most of the properties of standard optimal transport but we will focus on the geometric formulation of the model. We expect this new metric to have interesting applications in imaging.

4.1 Motivation and a Dynamical Model

In several contexts of applications including imaging, it is natural to consider data that can be represented by densities and these densities might have different masses. Often, optimal transport has been used in these applications (for instance, [HZTA04, AKS15]) since it provides an "easily computable" (at least, an efficient approximation [Cut13]) distance between probability measures that reflects a geometric displacement between them. Therefore, the mass constraint on the densities has to be taken into account and this problem seems to bring renewed interest in the applied mathematics literature [PR13, PR14, FG10, LM13, MRSS15] although this issue has been addressed since Kantorovich [Gui02].

In the following, we describe a dynamical approach to define optimal transport between general non-negative Radon measures. We will present the model only in a smooth setting although it is well defined on the space of Radon measures.

The Benamou-Brenier formulation: In [BB00], the authors formulated the Wasserstein L^2 distance as a convex variational problem, inspired by a fluid dynamic approach. In what follows, M will be a compact manifold without boundary. Let $\rho \in C^\infty(M, \mathbb{R}_+)$ be a positive function, note that all the quantities will be implicitly time dependent. The dynamic formulation of the Wasserstein distance consists in minimizing

$$\mathcal{E}(v) = \frac{1}{2} \int_0^1 \int_M |v(t,x)|^2 \rho(t,x) \, \mathrm{d}x \, \mathrm{d}t \ , \tag{4.1}$$

subject to the constraints $\dot{\rho} + \nabla \cdot (v\rho) = 0$ and initial condition $\rho(0) = \rho_0$ and final condition $\rho(1) = \rho_1$. Equivalently, following [BB00], a convex reformulation using the momentum $m = \rho v$ reads

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$$\mathcal{E}(m) = \frac{1}{2} \int_0^1 \int_M \frac{|m(t,x)|^2}{\rho(t,x)} \, \mathrm{d}x \, \mathrm{d}t, \tag{4.2}$$

subject to the constraints $\dot{\rho} + \nabla \cdot m = 0$ and initial condition $\rho(0) = \rho_0$ and final condition $\rho(1) = \rho_1$. Let us underline that the functional (4.2) is convex in ρ, m and the constraint is linear.

The Wasserstein-Fisher-Rao metric: The continuity equation enforces the mass conservation property. In view of the optimal transport generalization, this constraint needs to be relaxed, for instance by introducing a source term $\mu \in C^\infty(M, \mathbb{R})$,

$$\dot{\rho} = -\nabla \cdot (v\rho) + \mu. \tag{4.3}$$

For a given variation of the density $\dot{\rho}$, there exist a priori many couples (v, μ) that reproduce this variation. Following [TY05], it can be determined via the minimization of a norm of (v, μ) for an arbitrary choice of the norm. The penalization of μ was chosen in [MRSS15] as the L^2 norm but a natural choice is rather the Fisher-Rao metric

$$FR(\mu) = \frac{1}{2} \int_M \frac{\mu(t,x)^2}{\rho(t,x)} \, \mathrm{d}x \, \mathrm{d}t$$

since (1) it is 1-homogeneous with respect to the couple (μ, ρ) and (2) it is parametrization invariant [MBM14]. The first point is important for convex analysis properties in order to define the model on singular measures and the second point is natural from a modeling point of view if one thinks that μ represents a growth term. Thus, the action functional becomes:

$$WF(m, \mu) = \frac{1}{2} \int_0^1 \int_M \frac{|m(t,x)|^2}{\rho(t,x)} \, dx + \frac{1}{2} \int_0^1 \int_M \frac{\mu(t,x)^2}{\rho(t,x)} \, \mathrm{d}x \, \mathrm{d}t, \tag{4.4}$$

subject to the constraints $\dot{\rho} + \nabla \cdot m = \mu$ and initial condition $\rho(0) = \rho_0$ and final condition $\rho(1) = \rho_1$. This dynamical formulation enjoys most of the analytical properties of the initial Benamou-Brenier formulation (4.1) and especially convexity. An important consequence is the existence of optimal paths in the space of time-dependent measures [CSPV15].

4.2 A Geometric Point of View

Not only analytical properties are conserved but also some interesting geometrical properties of standard optimal transport such as the Riemannian submersion of Otto [Ott01]. Namely, for a fixed reference measure ρ_0, the map $\varphi \mapsto \varphi_*(\rho_0)$ from the group of diffeomorphisms of M with the $L^2(\rho_0)$ metric into the space of densities with the Wasserstein L^2 metric. See the appendix of [KW08] for more details. This property is simply proved by passing from the Eulerian point of view of the formulation (4.1) to a Lagrangian formulation. In this section, we extend this property to the generalized model.

A cone metric: Let us first discuss informally what happens for a particle of mass $m(t)$ at a spatial position $x(t)$ in a Riemannian manifold (M, g) under the generalized continuity constraint (4.3). The system reads

$$\begin{cases} \dot{x}(t) = v(x(t)) \\ \dot{m}(t) = \alpha(x(t))m(t) \end{cases} \tag{4.5}$$

where $\alpha \overset{\text{def.}}{=} \frac{\mu}{\rho}$ is the growth rate. The action associated with the functional defined in (4.4) is $\int_0^1 |v(x(t))|^2 m(t) + \frac{\dot{m}(t)^2}{m(t)} \, dt$. Thus, considering the particle as a point in $M \times \mathbb{R}_+^*$, the Riemannian metric seen by the particle is $mg + \frac{\mathrm{d}m^2}{m}$. Using the change of variable $r = \sqrt{m}$, we get $r^2 g + 4 \, \mathrm{d}r^2$ which is known under the name

of cone metric in Riemannian geometry. Note that if $M = \mathbb{R}$, a local isometry with the Euclidean space is given by $(x,m) \mapsto \sqrt{m}e^{ix/2} \in \mathbb{C}$. The distance on $M \times \mathbb{R}_+^*$ is explicit in terms of the distance on M with a Riemannian metric g,

$$\frac{1}{4}d((x_1,m_1),(x_2,m_2))^2 = m_2 + m_1 - 2\sqrt{m_1 m_2}\cos\left(\frac{1}{2}d_M(x_1,x_2) \wedge \pi\right). \tag{4.6}$$

This implies that mass can appear and disappear at a finite cost. In other words, the cone metric is not complete but adding the vertex of the cone, which represents $M \times \{0\}$, to $M \times \mathbb{R}_+^*$ turns it into a complete metric space.

Note that this distance squared is 1-homogeneous in (m_1,m_2).

A semi-direct product of groups: Going from Eulerian to Lagrangian coordinates in this new model is properly done by introducing a semi-direct product of group that extends the group of diffeomorphisms by introducing an action on mass that can be described as pointwise multiplication with a positive function on M. Working in a smooth context, we define $\Lambda(M) \stackrel{\text{def.}}{=} \{\lambda \in C^\infty(M,\mathbb{R}) : \lambda > 0\}$. It is a group under pointwise multiplication. We will also denote the same space as $\text{Dens}(M)$ to represent densities, that are smooth and positive L^1 function w.r.t. a reference measure ν. We define the semi-direct product of group between $\text{Diff}(M)$ and $\Lambda(M)$ in order to turn the map π defined by

$$\pi : (\text{Diff}(M) \ltimes_\Psi \Lambda(M)) \times \text{Dens}(M) \mapsto \text{Dens}(M)$$

$$\pi((\varphi,\lambda),\rho) \stackrel{\text{def.}}{=} \varphi \cdot \lambda \varphi_* \rho = \varphi_*(\lambda\rho)$$

into a left-action of the group $\text{Diff}(M) \ltimes_\Psi \Lambda(M)$ on the space of (generalized) densities. The group composition law is defined by:

$$(\varphi_1,\lambda_1) \cdot (\varphi_2,\lambda_2) = (\varphi_1 \circ \varphi_2, (\lambda_1 \circ \varphi_2)\lambda_2) \tag{4.7}$$

The important result is the following:

Proposition 1 *Let $\rho_0 \in \text{Dens}(M)$ and $\pi_0 : \text{Diff}(M) \ltimes_\Psi \Lambda(M) \mapsto \text{Dens}(M)$ be the map defined by $\pi_0(\varphi,\lambda) \stackrel{\text{def.}}{=} \varphi_*(\lambda\rho_0)$.*

Then, the map π_0 is a Riemannian submersion of the metric $L^2(M, M \times \mathbb{R}_+^*)$ (where $M \times \mathbb{R}_+^*$ is endowed with the cone metric (4.6)) on the group $\text{Diff}(M) \ltimes_\Psi \Lambda(M)$ to the Wasserstein-Fisher-Rao on the space of generalized densities $\text{Dens}(M)$.

A direct application of this result is the formal computation of the sectional curvature of the Wasserstein-Fisher-Rao in this smooth setting by applying O'Neill's formula, see [CPSV15].

The corresponding Monge formulation: Another important consequence of the L^2 metric on the group is that one can define a Monge formulation of the Wasserstein-Fisher-Rao metric as follows:

$$WF(\rho_0,\rho_1) = \inf_{(\varphi,\lambda)}\left\{\|(\varphi,\lambda) - (Id,1)\|_{L^2(\rho_0)} : \varphi_*(\lambda\rho_0) = \rho_1\right\} \tag{4.8}$$

4.3 The Kantorovich Formulation

From a variational point of view, it is important to derive a relaxation of the Monge formulation. It is of interest to understand first the simple situation when the source and target measures are single Dirac masses and when M is a convex and compact domain in the Euclidean space [CSPV15].

Proposition 2 *Let M be a convex and compact domain in $\mathbb{R}^d$ with the Euclidean metric. Let $m_1 \delta_{x_1}$ and $m_2 \delta_{x_2}$ be two Dirac masses.*

If $\frac{1}{2}d(x_1, x_2) < \pi/2$, there exists a unique geodesic which is $m(t)\delta_{x(t)}$ where $(x(t), m(t))$ is the geodesic in $M \times \mathbb{R}_+^*$ with the cone metric between (x_1, m_1) and (x_2, m_2).

If $\frac{1}{2}d(x_1, x_2) > \pi/2$, there exists a unique geodesic which is $m_1(t)\delta_{x_1} + m_2(t)\delta_{x_2}$ where $m_1(t) = m_1(1 - t)^2$ and $m_2(t) = m_2 t^2$ which describe the geodesics between (x_i, m_i) and the vertex of the cone denoted by $\mathcal{S}$ for $i = 1, 2$.

If $\frac{1}{2}d(x_1, x_2) = \pi/2$, there exists an infinite number of geodesics which are interpolations of the two first types defined above.

The important point is that passing to the case of measures the angle of the cone has been (surprisingly) divided by 2. This is because we are not looking for geodesics on $M \times \mathbb{R}_+^*$ but on the space of measures on M. This new distance is in fact the Fenchel-Legendre biconjugate of the initial distance with respect to the mass variable. The generalization to any positive Radon measures gives a Kantorovich relaxation: For two given positive Radon measures ρ_1, ρ_2, we define, for $\mathcal{M}(M^2)$ the space of positive Radon measures on M^2,

$$\Gamma(\rho_1, \rho_2) \stackrel{\text{def.}}{=} \left\{ (\gamma_1, \gamma_2) \in (\mathcal{M}_+(M^2))^2 : (\text{Proj}_1)_* \gamma_1 = \rho_1, (\text{Proj}_2)_* \gamma_2 = \rho_2 \right\}, \tag{4.9}$$

where Proj_1 and Proj_2 denote the projection on the first and second factors of M^2. The variational problem associated with the Wasserstein-Fisher-Rao distance is

$$WF(\rho_1, \rho_2)^2 = \inf_{(\gamma_1, \gamma_2) \in \Gamma(\rho_1, \rho_2)} \int_{M^2} \check{d}^2 \left((x, \frac{d\gamma_1}{d\gamma}), (y, \frac{d\gamma_2}{d\gamma}) \right) d\gamma(x, y), \tag{4.10}$$

where $4\check{d}^2$ is $m_2 + m_1 - 2\sqrt{m_1 m_2} \cos\left(\frac{1}{2}d_M(x_1, x_2) \wedge \pi/2\right)$ and γ is any measure that dominates ρ_1 and ρ_2. The fact that the integration does not depend on this choice is because of the 1-homogeneity of d^2 in function of the mass. We also state the dual formulation:

Proposition 3 *It holds*

$$WF^2(\rho_0, \rho_1) = \sup_{(\phi, \psi) \in C(M)^2} \int_M \phi(x) \, d\rho_0 + \int_M \psi(y) \, d\rho_1 \tag{4.11}$$

subject to $\forall (x, y) \in M^2$,

$$\begin{cases} \phi(x) \leq 1, \quad \psi(y) \leq 1, \\ (1 - \phi(x))(1 - \psi(y)) \geq \cos^2\left(\check{d}(x, y)\right) \end{cases} \tag{4.12}$$

For numerical computation, this formulation can be further reduced with a change of variable given by taking the logarithm of the multiplicative constraint (4.12).

4.4 Conclusion

We generalized the Wasserstein L^2 distance to a Riemannian-like metric on the space of densities whose total masses are different. Of important interest for application is that a static formulation is equivalent to the original dynamic one, which reduces the computational time. This Wasserstein-Fisher-Rao distance might be a useful tool in applications: On one hand, it can be seen as a modification of the Fisher-Rao metric that is stable under small spatial deformations and on the other hand as a modification of the

Wasserstein metric which does not allow for mass transfer if masses are too far apart (note once again that mass creation and destruction is enabled due to the cone metric).

This natural generalization introduces a cone metric on the product between space and mass. In a smooth setting, it is possible to formally apply O'Neill's formula to obtain the sectional curvature of the space of generalized densities. However, we did not study the global geometry of the space: one expects that, as for the Euclidean cone, the curvature is concentrated at its singularity. We refer to [CSPV15, CPSV15] for more details and generalizations.

After the presentation at the workshop, two important papers [LMS15b, LMS15a] also appeared on the same model motivated by different applications.

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Chapter 5
Image registration for landmarks with uncertainty

Stephen Marsland

Together with Tony Shardlow at the University of Bath in the UK I've been looking into landmark-based image registration with noise.

Consider the problem of finding a diffeomorphism $\mathbf{\Phi}\colon B \to B$ of minimum bending energy so that $u_r \circ \Phi$ is as close as possible to u_t, where $u_r, u_t\colon B \to \mathbb{R}$ are target and reference images in a domain $B \subset \mathbb{R}^d$.

If we parameterise the diffeomorphisms by time-dependent vector fields $v\colon [0,1] \times B \to \mathbb{R}^d$ and define $\mathbf{\Phi}(\mathbf{Q}) = \mathbf{q}(1)$ for $\mathbf{Q} \in B$, where $\mathbf{q}(t)$ satisfies the initial-value problem

$$\frac{d\mathbf{q}}{dt} = \mathbf{v}(t, \mathbf{q}(t)), \qquad \mathbf{q}(0) = \mathbf{Q}, \tag{5.1}$$

we can define the bending energy to be the norm

$$\|\mathbf{\Phi}\| = \left(\int_0^1 \|\mathcal{L}\mathbf{v}(t, \cdot)\|^2 \, dt \right)^{1/2},$$

for some differential operator $\mathcal{L}$.

Specifically, we consider a set of noisy observations of landmarks $\mathbf{q}_i^r$ and $\mathbf{q}_i^t$ for $i = 1, \ldots, N$ in B and we demand that $\Phi(\mathbf{q}_i^r) = \mathbf{q}_i^t$. The case where landmarks are fully observed is well studied, and our focus is uncertainty around landmark positions and sensitivity of the diffeomorphism to noise. To study this problem, we introduce a Bayesian formulation and define a prior distribution on the set of diffeomorphisms. We then condition the distribution on the noisy observation of landmarks to define a posterior distribution. Our approach is motivated by the Gibbs canonical distribution and we consider Langevin-type perturbations of the Hamiltonian equations, which has the Gibbs distribution $\exp(-\beta H)$, for inverse temperature β, as an invariant measure (modulo choices of boundary conditions and regularity of H). The advantage is that, with suitable initial data, the solutions of the Langevin equation $(\mathbf{q}_i(t), \mathbf{p}_i(t))$ all follow the same distribution $\exp(-\beta H)$ for $t \in [0,1]$. Therefore, when we condition on the data for $\mathbf{q}_i(0)$ and $\mathbf{q}_i(1)$, each end is equally treated.

The dynamics in a Hamiltonian model are constant-energy (constant-H). Instead, we can connect the system to a heat bath and look at constant-temperature dynamics. The heat bath perturbs the Hamiltonian system by an amount determined by the temperature of the heat bath, which we parameterise by the inverse temperature β. One method of constant-temperature particle dynamics is the Langevin equation. That is, we consider the system of stochastic ODEs given by

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$$dp_i = \left[-\lambda \nabla_{\mathbf{p}_i} H - \nabla_{\mathbf{q}_i} H \right] dt + \sigma d\mathbf{W}_i(t), \qquad \frac{d\mathbf{q}_i}{dt} = \nabla_{\mathbf{p}_i} H \tag{5.2}$$

for a dissipation $\lambda > 0$ and diffusion $\sigma > 0$. Here $\mathbf{W}_i(t)$ are i.i.d.$\mathbb{R}^d$ Brownian motions. In the case where $H = \frac{1}{2}p^2 + V(q)$ for a potential V, (5.2) is the classical Langevin equation where the marginal invariant distribution for $\mathbf{p}$ is $\mathcal{N}(0, \beta^{-1} I)$ and hence the average temperature per degree of freedom is the constant β^{-1}. Let $(\mathbf{q}_i(t), \mathbf{p}_i(t))$ for $t \in [0,1]$ satisfy (5.2) and define $\mathbf{\Phi}(\mathbf{Q})$ as in (5.1). Notice that $\mathbf{\Phi}(\mathbf{q}_i(0)) = \mathbf{q}_i(1)$. In perturbing (5.2), only the momentum equation is changed, so the equations for $\mathbf{q}$ are untouched and are consistent with definition of $\mathbf{v}(t, \mathbf{q})$ and hence $\mathbf{\Phi}$.

To define a distribution on the family of diffeomorphisms, it remains to choose initial data. If we specify a distribution on $[\mathbf{q}_1, \ldots, \mathbf{q}_N, \mathbf{p}_1, \ldots, \mathbf{p}_N]$ at $t = 0$, (5.2) implies a distribution on the paths and hence on $\mathbf{\Phi}$ via (5.1). The obvious choice is the Gibbs distribution. The Gibbs distribution is $\exp(-\beta H)$ (excluding a normalising constant) and, if $\sigma^2 \beta = 2\lambda$ (the fluctuation–dissipation relation), it is an invariant measure of (5.2). To see this, the generator of (5.2) is

$$L = \nabla_{\mathbf{p}} H \cdot \nabla_{\mathbf{q}} + (-\lambda \nabla_{\mathbf{p}} H - \nabla_{\mathbf{q}} H) \cdot \nabla_{\mathbf{p}} + \frac{1}{2}\sigma^2 \nabla_{\mathbf{p}}^2$$

and its adjoint

$$L^* \rho = -\nabla_{\mathbf{q}} \cdot ((\nabla_{\mathbf{p}} H)\rho) - \nabla_{\mathbf{p}} \cdot ((-\lambda \nabla_{\mathbf{p}} H - \nabla_{\mathbf{q}} H)\rho) + \frac{1}{2}\sigma^2 \nabla_{\mathbf{p}}^2 \rho.$$

The Fokker–Planck equation for pdf $\rho(\mathbf{q}, \mathbf{p}, t)$ is

$$\frac{\partial \rho}{\partial t} = -\nabla_{\mathbf{q}} \rho \cdot \nabla_{\mathbf{p}} H + \left(\lambda \nabla_{\mathbf{p}} H \cdot \nabla_{\mathbf{p}} + \nabla_{\mathbf{p}} \cdot \lambda \nabla_{\mathbf{p}} H \right) \rho + \nabla_{\mathbf{p}} \rho \cdot \nabla_{\mathbf{q}} H + \frac{1}{2}\nabla_{\mathbf{p}}^2(\sigma^2 \rho).$$

Put $\rho = e^{-H\beta}$, to see

$$\frac{\partial \rho}{\partial t} = (-\beta \nabla_{\mathbf{q}} H) \cdot \nabla_{\mathbf{p}} H \rho + (-\lambda \nabla_{\mathbf{p}} H \cdot \beta \nabla_{\mathbf{p}} H + \nabla_{\mathbf{p}} \cdot \lambda \nabla_{\mathbf{p}} H)\rho - \beta \nabla_{\mathbf{p}} H \cdot \nabla_{\mathbf{q}} H \rho$$
$$+ \frac{1}{2}\sigma^2 (-\beta \nabla_{\mathbf{p}}^2 H + \beta^2 \nabla_{\mathbf{p}} H \nabla_{\mathbf{p}} H^\mathsf{T})\rho.$$

Then, $\partial \rho / \partial t = 0$ if $\sigma^2 \beta = 2\lambda$ and ρ is an invariant measure.

In many cases G is translation invariant (e.g., $G(\mathbf{q}_1, \mathbf{q}_2) = \exp(-\|\mathbf{q}_1 - \mathbf{q}_2\|/\ell)$ for a length scale ℓ) and this means $\exp(-\beta H)$ cannot be a probability measure on $\mathbb{R}^{2dN}$. While the desire to have an invariant measure is appealing, we view the trajectories as convenient parameterisations of the diffeomorphism and therefore not themselves of interest. It is simpler to ask for a distribution on the diffeomorphism that is invariant under taking the inverse: that is, $\mathbf{\Phi}$ and $\mathbf{\Phi}^{-1}$ have the same distribution. To achieve this, $[\mathbf{q}_1(t), \ldots, \mathbf{q}_N(t), \mathbf{p}_1(t), \ldots, \mathbf{p}_N(t)]$ should have the same distribution under the time reversal $t \mapsto 1 - t$. This can be achieved very simply by setting initial data at $t = 1/2$ and flowing forward and backward using the same dynamics. Precisely, choose an initial probability distribution $\mu_{1/2}$ on $\mathbb{R}^{2dN}$. Given $[\mathbf{p}_i(1/2), \mathbf{q}_i(1/2)] \sim \mu_{1/2}$, compute $\mathbf{p}_i(t)$ and $\mathbf{q}_i(t)$ for $t > 1/2$ by solving (5.2). For $t < 1/2$, solve

$$d\mathbf{p}_i = \left[\lambda \nabla_{\mathbf{p}_i} H - \nabla_{\mathbf{q}_i} H \right] dt + \sigma d\mathbf{W}_i(t), \qquad \frac{d\mathbf{q}_i}{dt} = \nabla_{\mathbf{p}_i} H. \tag{5.3}$$

Here the sign of the dissipation is changed as we evolve the system forward by decreasing t. Note that the distribution of $(\mathbf{q}_1, \ldots, \mathbf{q}_N, \mathbf{p}_1, \ldots, \mathbf{p}_N)$ is unchanged by $t \mapsto 1 - t$, as can be verified using the Fokker–Planck equation.

In the case that $\mu_{1/2}$ is the Gibbs distribution, this method is identical to the originally proposed method. However, we achieve a time-symmetric distribution with any choice of $\mu_{1/2}$ and the Gibbs distribution does not need to be known or sampled.

Sampling of Bayesian inverse problems such as this are often approached using MCMC. We take first a simpler approach that allows quick approximation of, for example, the joint distribution of the landmark positions. That is, we choose to linearise (5.2) about some distinguished paths and replace (5.2) by a linear system of SDEs

$$d\boldsymbol{\delta} = -\lambda \begin{pmatrix} \nabla_{\mathbf{p}} H(\mathbf{p}^*(t), \mathbf{q}^*(t)) \\ \mathbf{0} \end{pmatrix} + B^+(t)\boldsymbol{\delta}\, dt + \begin{pmatrix} \sigma I_{dN} \\ 0 \end{pmatrix} d\mathbf{W}(t), \tag{5.4}$$

where $\mathbf{W}(t)$ is a $\mathbb{R}^{dN}$ Brownian motion. Here, $B(t)$ is a $2dN \times 2dN$ matrix that describes the linearisation: to linearise around $\mathbf{p}^*(t), \mathbf{q}^*(t)$, take

$$B^+(t) = \begin{pmatrix} -\lambda\nabla_{\mathbf{PP}}H - \nabla_{\mathbf{QP}}H & -\lambda\nabla_{\mathbf{PQ}}H - \nabla_{\mathbf{QQ}}H \\ \nabla_{\mathbf{PP}}H & \nabla_{\mathbf{PQ}}H \end{pmatrix}.$$

all evaluated at $\mathbf{p}^*(t), \mathbf{q}^*(t)$. This system is linear and exact expressions are available for this system in terms of deterministic integrals. We prefer to use a time-stepping method, and have worked out forward and backward versions of the Euler-Maruyama method with temporal covariance. The method seems to work very well, and experiments are under way.

We are also planning to develop a method for full bridge diffusion in this case, and to consider how to incorporate learning the Green's function from data.

Chapter 6
Invariants of functions on symplectic surfaces and ideal hydrodynamics

Boris Khesin

6.1 Abstract

This talk is based on the paper [1], to which we refer for more detail. We classify generic coadjoint orbits of several diffeomorphism groups of surfaces. In particular, we answer a question about a complete set of invariants for generic isovorticed fields in 2D ideal hydrodynamics posed by V.Arnold in [2], Section I.5. Recall that the corresponding classification problem for diffeomorphisms of the circle was solved by A. Kirillov in [3]. He showed that it is equivalent to classification of periodic quadratic differentials and described Casimirs for generic orbits. Orbits of the Virasoro-Bott group, a nontrivial extension of the circle diffeomorphism group, were classified independently in different terms by G.Segal, A.Kirillov, and other authors. The latter problem is also equivalent to the classification of Hill's operators or projective structures on the circle. All those results deal with diffeomorphisms of one-dimensional manifolds.

In paper [1] we give an answer to this question by describing the orbit classification for symplectic and Hamiltonian diffeomorphisms of an arbitrary 2D oriented surface. To obtain these classifications we first solve an auxiliary problem, which is of interest by itself: classify (and describe invariants of) generic Morse functions on closed surfaces with respect to the action of area-preserving diffeomorphisms (possibly isotoped to the identity). It turns out that invariants of those actions on functions are given by the Reeb graphs of functions equipped with various collections of structures, such as a measure on the graph, homomorphisms of (local) homology groups of surfaces to that graph, a choice of a pants de- composition, and the flux across certain cycles as we describe in the corresponding sections. Also the corresponding measures on Reeb graphs are not arbitrary but satisfy certain con- straints in terms of asymptotic expansions at all three-valent vertices of the graph. To pass from classification of functions to classification of coadjoint orbits one needs to supplement the above data by the equality of appropriately defined circulation functions.

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Chapter 7
Geodesics of constant coefficient Sobolev metric on curves

Jakob Møller-Andersen

When matching curves using a Riemannian approach, the choice of metric on the tangent space often comes with a choice of parameters describing the metric. On the space of immersions,

$$\mathrm{Imm}(S^1, \mathbb{R}^d) = \{c \in C^\infty(S^1, \mathbb{R}^d) : c'(\theta) \neq 0\} \subset C^\infty(S^1, \mathbb{R}^d).$$

we consider the family of constant coefficient Sobolev type metrics given by

$$G_c(h,k) = \int_{S^1} a_0 \langle h,k \rangle + a_1 \langle D_s h, D_s k + a_2 \langle D_s^2 h, D_s^2 k \rangle \, \mathrm{d}s,$$

where $a_i \in \mathbb{R}$, $D_s = \frac{1}{|c'|}\partial_\theta$, $\mathrm{d}s = |c'(\theta)| \, \mathrm{d}\theta$. Given two curves c_0, c_1, we can consider the space of all paths, c connecting them. Geodesics in the space is are minimizers of the energy functional

$$E(c) = \int_0^1 G_c(\dot{c}, \dot{c}) \, dt.$$

Here $\cdot$ means differentiation with respect to t. To approximate these minimizers, we consider the space of tensor product B-spline paths c connecting c_0 and c_1, explicitly we have the paths

$$c(t,\theta) = \sum_{i=1}^{M} \sum_{j=1}^{N} c_{i,j} B_i(t) C_j(\theta),$$

where B_i and C_j are B-splines chosen such that $c(0,\theta) = c_0(\theta)$ and $c(1,\theta) = c_1(\theta)$. Figure 1 shows three examples of geodesics connecting two fixed curves, for increasing values of a_2. For observing the influence of the constant on the geodesics, we note that we can always scale the metric by a constant value, so we can assume $a_0 = 1$. We can observe that a_2 has a "blow-up" effect for increasing values, see figure 1: certain features are smudged out and become bigger during the geodesic. This effect is also apparent in the simpler situation in figure 2, where the effect is very pronounced between the two simple curves. A heuristic explanation is that the H^2 term in the metric measures the cost of changing the curvature of the curve, so if this term dominates it is cheap in energy to scale the curve up, hence decrease the curvature, then do a simple linear transformation to a similarly scaled version of the other curve and then rescaling to the original since. The influence of the H^1 term is not so easy to describe, especially not its interplay with a_2. Figure 3 shows how increasing values of a_1 decreases the blow-up effect. Since the choice of
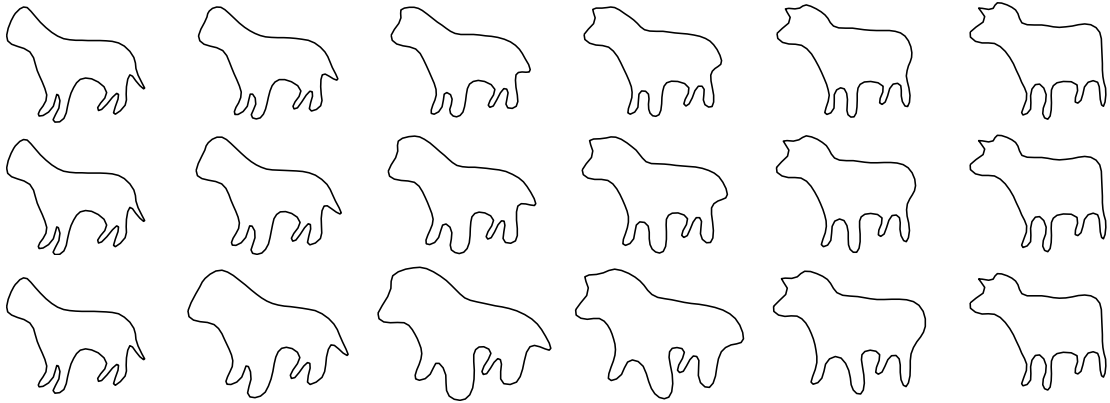
DTU, Denmark

Fig. 7.1 The effects of a_2 on geodesics, here $a_0 = a_1 = 1$ and $a_2 = 1, 10, 100$ in the first, second and third row respectively.
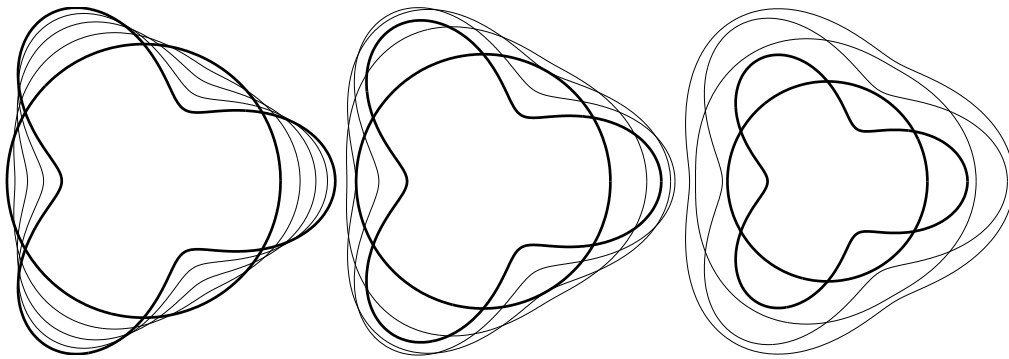

Fig. 7.2 Here $a_0 = a_1 = 1$ and $a_2 = 0.1, 10, 10$ in the first, second and third row respectively.

parameters has a great influences on the geodesics, and their corresponding energies, the question is: for a given application, how do you choose these parameters optimally to get the matching you need?
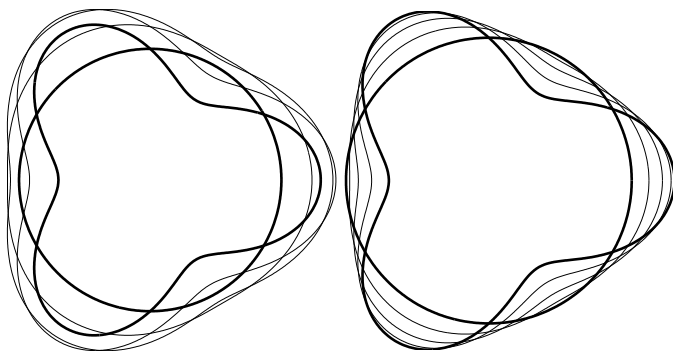

Fig. 7.3 Here $a_0 = 1, a_2 = 10$ and $a_1 = 10, 100$ in the first and second row respectively.

Chapter 8
Holonomy, curvature, and anisotropic diffusions

Stefan Sommer[1] and Anne Marie Svane[2]

8.1 Introduction

Let M be a smooth, connected, compact manifold of finite dimension with connection and a fixed volume form. In [7], a class of distributions on M is introduced that generalizes Euclidean normal distributions with anisotropic covariance to the non-linear geometry on M. The distributions arise as transition distributions of Euclidean diffusion processes that through horizontal development in the frame bundle FM of M are mapped to M. This process is denoted *stochastic development*, see e.g. [2].

In connection with this, it becomes relevant to study horizontal paths on FM and the naturual sub-Riemannian structure on FM [9, 8]. The aim of this abstract is to review and outline the relation between the curvature of M, the holonomy group and its Lie algebra, and the Hörmander condition of the horizontal distribution on FM.

8.2 Brownian Motion in the Frame Bundle

The frame bundle FM of a differentiable manifold M is the smooth vector bundle consisting of points $x \in M$ and corresponding frames (ordered bases) in the tangent spaces $T_x M$. FM is a principal bundle over M with fiber $\mathrm{GL}(n)$. A fundamental property of FM is the existence of $n = \dim(M)$ globally defined *horizontal* vector fields $H_1, \ldots, H_n$. These vector fields correspond to infinitesimal displacements δx on M and parallel transport of frames along δx. Let $\pi : FM \to M$ be the bundle projection, and let $\mathcal{H}$ denote the distribution in TFM spanned by the horizontal vector fields. We denote by π_* the pushforward $TFM \to TM$ of π and π_u^* the horizontal lift of $x \in TM$ to $\mathcal{H}_u$.

Given a stochastic processes X_t in $\mathbb{R}^n$ starting at 0, a *stochastic development* of X_t is a stochastic process U_t on FM satisfying the Stratonovich stochastic differential equation $dU_t = H_i \circ dX_t^i$ with initial condition $U_0 = u_0 \in FM$. If X_t is a Brownian motion, the projection $Y_t = \pi U_t$ of U_t onto M may be considered an anisotropic diffusion on M starting at πu_0 with covariance $\Sigma = S^T S$ where S denotes the frame part of u_0.

If Hörmander's condition is satisfied, the distribution of U_t will have a smooth density [9]. As we shall see below, Hörmander's condition is not satisfied on FM but in some situations on a subbundle of FM.

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8.2.1 Sub-Riemannian Structure

There exists a natural sub-Riemannian metric $g_{FM} : TFM^* \to HFM \subset TFM$ on FM defined by

$$\langle w, g_{FM}(\xi) \rangle = (\xi|w) \, , \, \forall w \in H_u FM \, , \tag{8.1}$$

where the inner product at $u = (x, X_\alpha) \in FM$ is

$$\langle v, w \rangle = \left\langle X_\alpha^{-1} \pi(v), X_\alpha^{-1} \pi(w) \right\rangle_{\mathbb{R}^n} \, ,$$

see [9, 8]. The sub-Riemannian length of an absolutely continuous path $\gamma : [0,1] \to FM$ whose derivative is a.e. horizontal is defined by $l(\gamma) = \int_0^1 |\dot\gamma(t)| dt$. If $\dot\gamma$ is not a.e. horizontal, we set $l(\gamma) = \infty$. The corresponding sub-Riemannian distance between u_1 and u_2 in FM is then

$$d(u_1, u_2) = \inf\{l(\gamma) \mid \gamma(0) = u_1, \gamma(1) = u_2\}.$$

As in the Riemannian case, one may now ask for length minimizing curves (geodesics), exponential maps, etc. To answer such questions, it is convenient to assume that the horizontal vector fields satisfy the Hörmander condition.

8.3 The Reachable Set and Holonomy

The sub-Riemannian distance between two points may be infinite, even if M is connected, because not all points can be reached by a horizontal path. For this reason, we consider the set of reachable points. Write $u \sim p$ if $u, p \in FM$ and there exists a horizontal curve in FM joining u and p. Then

$$Q(u) = \{p \in FM \mid p \sim u\}$$

is the set of points in FM reachable by horizontal curves from u. $Q(u)$ is a smooth immersed submanifold [10].

The holonomy group $\mathrm{Hol}_u(FM, \mathcal{H})$ of $\mathcal{H}$ at $u \in FM$ is

$$\mathrm{Hol}_u(FM, \mathcal{H}) = \{a \in \mathrm{GL}(n) \mid u \cdot a \sim u\}$$

where $u \cdot a$ denotes the natural action on each fibre. The holonomy group corresponds to the set of frames reachable by parallel transport around loops of $\pi(u)$. We denote by Hol_u^0 the connected component in Hol_u containing the identity.

Proposition 8.1. *Let M be Riemannian and fix $u \in FM$. Then $Q(u)$ is a principal subbundle of FM with fibre $\mathrm{Hol}_u(FM, \mathcal{H})$.*

Proof. Theorem 3.2.8 of [3] asserts that the holonomy subgroup is closed because M is Riemannian. The result then follows from Theorem 2.3.6 of [3].

If M is Riemannian, Hol_u is isomorphic to a subgroup of $\mathrm{O}(n)$ and if M is orientable, it is a subgroup of $\mathrm{SO}(n)$. In this case, if u is orthonormal, $Q(u)$ is a subbundle of the orthonormal frame bundle OM.

8.3.1 The Hörmander condition

The horizontal distribution is said to satisfy the Hörmander condition if $\mathcal{H}$ is bracket generating, i.e. if $\mathrm{Lie}(\mathcal{H}) = TFM$ where $\mathrm{Lie}(\mathcal{H})$ denotes the Lie saturate of $\mathcal{H}$, i.e. the linear span of $\mathcal{H}$ and all finite brackets. The discussion below is based on [10, 1, 3, 5].

The Hörmander condition is not satisfied on FM, as one may realize as follows: The tangent space TFM naturally splits into a horizontal part $\mathcal{H}$ and a vertical part V tangent to each fiber, that is $T_u FM \cong \mathcal{H}_u \oplus V_u$. The vertical part of $\mathrm{Lie}(\mathcal{H})_u$ is contained in the Lie algebra $\mathfrak{h}_u$ of Hol_u and $\mathfrak{h}_u \subseteq V_u$. This rules out that $\mathcal{H}$ is bracket-generating since $\mathrm{Hol}_u \neq \mathrm{GL}(n)$. In general, it is not even bracket generating on OM since we may have $\dim \mathrm{Hol}_u < \dim O(n)$.

We can give conditions under which the Hörmander condition is satisfied on $Q(u)$. Let M be Riemannian. Injectivity of the curvature tensor $R_x : \Lambda^2(T_x M) \to \mathfrak{so}(T_x M)$ implies surjectivity because of dimensions of $\Lambda^2(T_x M)$ and $\mathrm{SO}(T_x M)$. Such injective curvature metrics are generic, i.e. they form an open and dense subset of all metrics on M [1]. In this situation, the Hörmander condition is satisfied on the subbundle $Q(u)$:

Theorem 8.1. *If M is Riemannian and the curvature map is surjective then the horizontal distribution is bracket generating on $Q(u)$ and $\mathrm{Hol}_u^0 = \mathrm{SO}(n)$.*

Proof. Since $\mathrm{Lie}(\mathcal{H})_u \subseteq T_u Q(u) \subseteq \mathcal{H} + \mathfrak{so}(n)$, it suffices to show that $\mathrm{Lie}(\mathcal{H})_u = \mathcal{H} + \mathfrak{so}(n)$. For this, it is enough that the span of $\mathcal{H}$ and its first bracket equals $\mathcal{H} + \mathfrak{so}(n)$, i.e. $\mathcal{H} + [\mathcal{H}, \mathcal{H}] = \mathcal{H} + \mathfrak{so}(n)$. Thus, let $z = z_v + z_h \in \mathfrak{so}(n) \oplus \mathcal{H}_u$. By assumption, R is surjective onto $\mathfrak{so}(\mathfrak{n})$ so we can find horizontal vector fields V, W s.t. $R(V, W) = z_v$. Since $R(V, W) = [V, W] - \pi^*([\pi_*(V), \pi_*(V)])$, we have $z = z_v + z_h = [V, W] - \pi^*([\pi_*(V), \pi_*(V)]) + z_h$. The first term is in $[\mathcal{H}, \mathcal{H}]$ and two last terms are in $\mathcal{H}$ giving the result. ∎

When R is not injective, it is still possible that $Q(u)$ satisfies Hörmander's condition in some non-degenerate situations:

Theorem 8.2. *If $\mathrm{Lie}(\mathcal{H})_u$ has constant rank for all $u \in FM$, then $Q(u)$ satisfies the Hörmander condition.*

The constant rank condition is for instance satisfied for analytic manifolds [5, Appendix C] and homogeneous spaces.

Proof. The distribution $\mathrm{Lie}(\mathcal{H})$ is involutive by definition. The constant rank ensures that the Frobenius theorem (see [4] Theorem 3.20) applies. Thus for any $u \in FM$ there exists a maximal connected immersed submanifold $Q_{\mathrm{Lie}(\mathcal{H})}(u)$ containing u of dimension $\dim \mathrm{Lie}(\mathcal{H})$ with tangent space $\mathrm{Lie}(\mathcal{H})$. By construction, the Hörmander condition is satisfied on $Q_{\mathrm{Lie}(\mathcal{H})}(u)$.

Chow's theorem [5, Theorem 2.2] yields that $Q_{\mathrm{Lie}(\mathcal{H})}(u) \subseteq Q(u)$. On the other hand, any two points in $Q(u)$ can be joined by a horizontal curve. By the construction of $Q_{\mathrm{Lie}(\mathcal{H})}(u)$, this curve must lie in $Q_{\mathrm{Lie}(\mathcal{H})}(u)$. We deduce that $Q(u)$ and $Q_{\mathrm{Lie}(\mathcal{H})}(u)$ are equal as sets. Moreover, it follows from [5, Exercise C.4], see also [10], that $\dim \mathrm{Lie}(\mathcal{H})_u = \dim \mathrm{Hol}_u$, so $Q(u) = Q_{\mathrm{Lie}(\mathcal{H})}(u)$ as differentiable manifolds. ∎

In general, however, $\mathrm{Lie}(\mathcal{H})_u$ may not have constant dimension. In this case, it is not possible to find a submanifold of FM where $\mathcal{H}$ is bracket generating. For instance, if M is flat in a neighborhood of $\pi(u)$ then $\dim \mathrm{Lie}(\mathcal{H})_u = n$ while the dimension of $\mathrm{Lie}(\mathcal{H})$ may be larger in curved parts of M. While $Q(u)$ and Hol_u are global constructions, $\mathrm{Lie}(\mathcal{H})_u$ is local and the corresponding Lie group is known as the infinitesimal holonomy group [6].

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