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# Second strain gradient theory in orthogonal curvilinear coordinates: prediction of the relaxation of a solid nanosphere and embedded spherical nanocavity

Farzaneh Ojaghnezhad[a,*], Hossein M. Shodja[b,c]

[a] *School of Engineering, Alzahra University, Vanak St., 1993891176, Tehran, Iran*
[b] *Department of Civil Engineering, Sharif University of Technology, P.O. Box 11155-4313, Tehran, Iran*
[c] *Institute for Nanoscience and Nanotechnology, Sharif University of Technology, P.O. Box 11155-9161, Tehran, Iran*

**Abstract**

In this paper, Mindlin's second strain gradient theory is formulated and presented in an arbitrary orthogonal curvilinear coordinate system. Equilibrium equations, generalized stress-strain constitutive relations, components of the strain tensor and their first and second gradients, and the expressions for three different types of traction boundary conditions are derived in any orthogonal curvilinear coordinate system. Subsequently, for demonstration, Mindlin's second strain gradient theory is represented in the spherical coordinate system as a highly-practical coordinate system in nanomechanics. Second strain gradient elasticity have been developed mainly for its ability to capture the surface effects in the presence of micro-/nano- structures. As a numeric illustration of the theory, the surface relaxation of spherical domains in Mindlin's second strain gradient theory is considered and compared with that in the framework of Gurtin-Murdoch surface elasticity. It is observed that Mindlin's second strain

*Corresponding author
*Email address:* f.ojaghnezhad@alzahra.ac.ir (Farzaneh Ojaghnezhad)

gradient theory predicts much larger value for the radial displacement just near the surface in comparison to Gurtin-Murdoch surface elasticity.

*Keywords:* second strain gradient theory, orthogonal curvilinear coordinates, surface effect, nanosphere, nanocavity, relaxation

## 1. Introduction

Recently, there has been a flurry of interest in such unusual forms as nanowires, nanotubes, and nano-particles of commonplace materials like metals, semiconductors, insulators, and organic compounds. In view of their vast possible applications in electronics, energy conversion, optics, chemical sensing, cancer therapy, and drug delivery, among other fields, consideration of the mechanics and physics of these forms of nano-structures is inevitable. The most important key features in these structures are their nanometer scale in two/three dimensions as well as their special symmetrical shapes.

It is well-known that traditional continuum theories are inadequate in treating mechanical aspects of nano-scale structures and resorting to augmented continuum theories seems to be remedial. For example, Lazar [1] deals with strain gradient elasticity of defects to give a non-singular dislocation continuum theory. Lazar and Agiasofitu [2] provide fundamental quantities in generalized elasticity and dislocation theory of crystals. The development of higher order continuum theories such as strain gradient elasticity has been brought into focus, mainly in the period of about 1960-1975. In first strain gradient theory, Toupin [3] assumed that the potential energy density function of the material depends on both the second order strain tensor and its first gradient. The correspondence

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between first strain gradient theory and the atomic structure of the material is exhibited by Toupin and Gazis [4] through consideration of the nearest and next nearest interatomic interactions; they realized that the drawing in or pushing out the surface layer happens only in non-centrosymmetric materials. Later, Toupin, in a private communication with Mindlin [5], suggested that one can remove this restriction with the inclusion of the components of the second gradient of the strain tensor in the potential energy density function. Subsequently, Mindlin [5] proposed second strain gradient theory in which the strain energy density function depends on, not only the strain field and its first derivative, but also the second derivative of the strain field. Formulation within Mindlin's second strain gradient theory gives rise to two surface parameters, namely, surface characteristic length and modulus of cohesion, enabling this theory to capture the surface effect of nano-structures on their mechanical properties [6, 7, 8]. Factually, consideration of the surface effect in nano-scale structures has been one of the most important stimuli in the development of higher order continuum theories. Recently, Shodja et al. [8] have proposed an atomistic model for the calculation of the additional constants for fcc materials in second strain gradient elasticity. Subsequently, they studied the surface effects on the behavior of nano-size Bernoulli-Euler beams. Moreover, Ojaghnezhad and Shodja [6] employed a combined first principles and analytical approach for determination of the modulus of cohesion, surface energy, and the additional constants in second strain gradient elasticity. Later, Ojaghnezhad and Shodja [7] reformulated Gurtin and Murdoch [9] surface elasticity theory in the context of second strain gradient theory. In surface elasticity theory which has been formulated to capture the surface effects, the bulk material and surface layer are treated as two separate entities. Formulation in this framework entails the introduction of the notion of two surface parameters as surface residual stresses and surface elastic moduli tensor. The work of Ojaghnezhad and Shodja [7] lead to a linkage between the surface elastic parameters such as surface stress and surface elastic constants of surface elasticity theory and the elastic parameters stemming from second strain gradient elasticity.

However, utilization of Mindlin's second strain gradient theory to treat a vast variety of nano-scale problems involving various geometries in a mathematically rigorous manner requires the employment of suitable coordinate systems. Eringen [10] provides a simple but effective mathematical tool to transform any formulation written in the Cartesian coordinates to any curvilinear coordinate system. As an effort, Ji et al. [11] derived the general formulations of the simplified first strain gradient theory proposed by Zhou et al. [12] in the framework of orthogonal curvilinear coordinates. However, to date, in spite of the vast application of Mindlin's second strain gradient theory in prediction of the mechanical behavior of nano-structures, its general formulation in orthogonal curvilinear coordinates is absent in the literature. In this paper, the methodology proposed by Eringen [10] is utilized to provide the formulation of Mindlin's second strain gradient theory in any arbitrary orthogonal curvilinear coordinates. As it was alluded to, such formulation would be of great value for the treatment of nanostructures of various shapes where the surface effect is important. In continue,

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in order to illustrate the usefulness of the presented formulation in applications, relaxation of a spherical domain and a spherical cavity is examined based on Mindlin's second strain gradient theory. Recently, solid and hollow nanospheres made of dielectric and precious metals such as gold and silver have absorbed great attention of the researchers due to their effective application in nanotechnology. Dielectric nanospheres are promising structures for light trapping in plannar thin-film solar cells [13]. Moreover, metal nanospheres due to their optical properties, have various technological applications such as surface plasmon resonance detection and imaging, surface-enhanced Raman scattering, and biomedical imaging and therapy. In the case of hollow gold nanospheres, the unique combination of small size, spherical shape, and strong tunable surface plasmon resonance is ideal for biomedical applications [14]. Hollow Pd spheres have been fabricated for usage as heterogeneous catalyst for suzuki coupling reactions [15]. Tunability of surface plasmon resonance by interior cavity size in Au hollow nanospheres has been examined by Liang et al. [16]. Au hollow nanosphere has also been used for drug delivery [17]. Metal nanoshell has found application in tumor therapy [18]. The present work focuses on the phenomenon of relaxation as an application of the current theoretical developments. In particular, we examine the relaxation of spherical domain as well as spherical cavity made of Ag, Au, and Pt based on Mindlin's second strain gradient theory. The results are compared with the corresponding ones obtained from Gurtin-Murdoch surface elasticity as well as molecular dynamics simulation.

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## 2. Second strain gradient theory in Cartesian coordinates

Based on strain gradient theory formulated by Mindlin [5], strain energy density of a homogeneous and centrosymmetric material depends not only on the traditional infinitesimal strain, $\epsilon_{ij}$, but also on its first and second spatial gradients, $\epsilon_{ijk}$ and $\epsilon_{ijkl}$, respectively, as below

$$\begin{aligned} W = &\frac{1}{2}C_{ijkl}\epsilon_{ij}\epsilon_{kl} + F_{ijklmn}\epsilon_{ij}\epsilon_{klmn} + \frac{1}{2}G_{ijklmn}\epsilon_{ijk}\epsilon_{lmn} \\ &+ \frac{1}{2}I_{ijklmnpq}\epsilon_{ijkl}\epsilon_{mnpq} + B^{\circ}_{ijkl}\epsilon_{ijkl}. \end{aligned} \tag{2.1}$$

where the summation convention for repeated indices is employed and

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.2a}$$

$$\epsilon_{ijk} = u_{k,ij}, \tag{2.2b}$$

$$\epsilon_{ijkl} = u_{l,ijk}. \tag{2.2c}$$

In the above relations $u_i$ is the displacement component and "," in subscript denotes the usual partial differentiation with respect to the Cartesian coordinates $x_i$, $i = 1, 2, 3$. Based on the considered strain energy density, second-, third-, and forth-order stress tensors of any hyperelastic material are defined as $\tau_{ij} = \dfrac{\partial W}{\partial \epsilon_{ij}}$, $\tau_{ijk} = \dfrac{\partial W}{\partial \epsilon_{ijk}}$, and $\tau_{ijkl} = \dfrac{\partial W}{\partial \epsilon_{ijkl}}$. Thus,

$$\tau_{ij} = C_{ijkl}\epsilon_{kl} + F_{ijklmn}\epsilon_{klmn}, \tag{2.3a}$$

$$\tau_{ijk} = G_{ijklmn}\epsilon_{lmn}, \tag{2.3b}$$

$$\tau_{ijkl} = F_{pqijkl}\epsilon_{pq} + I_{ijklmntu}\epsilon_{mntu} + B^{\circ}_{ijkl}. \tag{2.3c}$$

In the last relation, $B^{\circ}_{ijkl} = \dfrac{b_0}{3}\delta_{ijkl}$, where $b_0$ is Mindlin's modulus of cohesion and $\delta_{ijkl} = \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}$ in which $\delta_{ij}$ is the Kronecker delta. Utilizing

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the above relations together with the symmetry considerations of the strain and stress tensors, it is inferred that the fourth, sixth, and eighth order tensors, respectively, $C_{ijkl}$, $F_{ijklmn}$, $G_{ijklmn}$, and $I_{ijklmntu}$ have the following symmetry properties

$$C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}, \tag{2.4a}$$

$$F_{ijklmn} = F_{jiklmn} = F_{ijlkmn} = F_{ijmlkn} = F_{ijkmln}, \tag{2.4b}$$

$$G_{ijklmn} = G_{lmnijk} = G_{jiklmn} = G_{ijkmln}, \tag{2.4c}$$

$$\begin{aligned} I_{ijklmnrs} &= I_{mnrsijkl} = I_{jiklmnrs} = I_{kjilmnrs} = I_{ijlkmnrs} \\ &= I_{ijklrnms} = I_{ijklnmrs} = I_{ijklmrns}. \end{aligned} \tag{2.4d}$$

Based on the above-mentioned symmetries and Eqs. (2.2), the strain energy density function can be represented in the following form

$$\begin{aligned} W = &\frac{1}{2} C_{ijkl} u_{i,j} u_{k,l} + F_{ijklmn} u_{i,j} u_{n,klm} + \frac{1}{2} G_{ijklmn} u_{k,ij} u_{n,lm} \\ &+ \frac{1}{2} I_{ijklmnpq} u_{l,ijk} u_{q,mnp} + B^{\circ}_{ijkl} u_{l,ijk}. \end{aligned} \tag{2.5}$$

For isotropic materials, the components of the fourth order tensor, $C_{ijkl}$ are written in terms of the usual Lamé constants $\lambda$ and $\mu$ as $C_{1111} = \lambda + 2\mu$, $C_{1122} = \lambda$, and $C_{1212} = \mu$. The not-mentioned nonzero components are obtained via cyclic permutation of indices. The nonzero components of the higher order elastic tensors, $F_{ijklmn}$, $G_{ijklmn}$, and $I_{ijklmnpq}$ for isotropic materials are related to Mindlin's additional constants $a_i$'s, $i = 1, \ldots, 5$, $b_i$'s, $i = 1, \ldots, 7$, and $c_i$'s, $i = 1, 2, 3$ as displayed in Tables 1 and 2. The other nonzero components which are not displayed in the table can be obtained through the cyclic permutation of indices of the presented components.

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Table 1: The relations between the higher order tensors, $F_{ijklmn}$ and $G_{ijklmn}$ and Mindlin's additional constants, $a_i$'s and $c_i$'s.

| | |
|---|---|
| $F_{111122} = F_{111133} = \dfrac{c_1 + c_2}{3}$ | $F_{111221} = F_{111331} = \dfrac{c_1 + c_3}{3}$ |
| $F_{112222} = F_{113333} = c_1$ | $F_{112233} = F_{113322} = \dfrac{c_1}{3}$ |
| $F_{121112} = F_{131113} = \dfrac{c_3}{2}$ | $F_{121332} = F_{122331} = \dfrac{c_3}{6}$ |
| $F_{121121} = F_{131131} = \dfrac{c_2}{3} + \dfrac{c_3}{6}$ | $F_{111111} = c_1 + c_2 + c_3$ |
| $F_{121233} = \dfrac{c_2}{6}$ | |
| $G_{112233} = G_{113322} = \dfrac{a_2}{2}$ | $G_{111122} = G_{111133} = \dfrac{2a_1 + a_2}{2}$ |
| $G_{221111} = G_{331111} = a_2 + 2a_3$ | $G_{112112} = G_{113113} = 2(a_3 + a_4)$ |
| $G_{112211} = G_{113311} = \dfrac{a_2 + 2a_5}{2}$ | $G_{122122} = G_{133133} = \dfrac{a_1 + 2a_4 + a_5}{2}$ |
| $G_{112332} = 2a_3$ | $G_{123123} = a_4$ |
| $G_{111111} = \bar{a}$ | $G_{122133} = \dfrac{a_1}{2}$ |
| $G_{123132} = \dfrac{a_5}{2}$ | |

A material is referred to as "grade $N$" if the order of the highest position gradient in its energy density function expression is equal to $N$. Mindlin [5] showed that for a grade 3 material of volume $V$ with boundary $S$, the stress-equation of motion in rectangular coordinate system has the following form

$$\tau_{ip,i} - \tau_{ijp,ij} + \tau_{ijkp,ijk} + f_p = \rho \ddot{u}_p, \tag{2.6}$$

in which $f_p$ is the body force per unit volume and $\rho$ is the mass density of the material. By substituting from Eqs. (2.3) and (2.2) into the stress-equation of

Table 2: The relations between the higher order tensor, $I_{ijklmnpq}$ and Mindlin's additional constants, $b_i$'s.

| | |
|---|---|
| $I_{11111111} = \bar{b}$ | $I_{11111122} = I_{11111133} = \dfrac{2b_1 + 2b_2 + b_3}{3}$ |
| $I_{11112222} = 2b_1$ | $I_{11111221} = I_{11111331} = \dfrac{2b_1 + b_3 + 2b_4 + 2b_5}{3}$ |
| $I_{11122221} = 2b_4$ | $I_{11112233} = I_{11113322} = \dfrac{2b_1}{3}$ |
| $I_{11121112} = I_{11131113} = 2(b_5 + b_6)$ | $I_{11121121} = I_{11131131} = \dfrac{b_3 + 2b_4 + 2b_7}{3}$ |
| $I_{11121222} = I_{11131333} = \dfrac{b_3 + 2b_5}{3}$ | $I_{11212331} = I_{11232333} = \dfrac{b_3 + 2b_5}{9}$ |
| $I_{11121233} = I_{11131232} = \dfrac{b_3}{6}$ | $I_{11121332} = I_{11131223} = \dfrac{2b_5}{3}$ |
| $I_{11122331} = I_{11133221} = \dfrac{2b_4}{3}$ | $I_{11211121} = I_{11311131} = \dfrac{2(2b_2 + b_3 + b_5 + 3b_6 + 2b_7)}{9}$ |
| $I_{11211222} = \dfrac{2(2b_2 + b_3 + b_4)}{9}$ | $I_{11211233} = I_{11311322} = \dfrac{4b_2 + b_3}{18}$ |
| $I_{11211332} = I_{11311223} = \dfrac{b_3 + 2b_4}{9}$ | $I_{11221122} = I_{11331133} = \dfrac{2(b_1 + b_2 + b_4 + b_5 + 3b_6 + b_7)}{9}$ |
| $I_{11221133} = \dfrac{2(b_1 + b_2)}{9}$ | $I_{11221221} = \dfrac{2(b_1 + b_3 + 2b_7)}{9}$ |
| $I_{11221331} = I_{11331221} = \dfrac{2b_1 + b_3}{9}$ | $I_{11222332} = \dfrac{2(b_1 + b_4 + b_5)}{9}$ |
| $I_{11231123} = I_{11321132} = \dfrac{2(b_5 + 3b_6)}{9}$ | $I_{11231132} = \dfrac{2(b_4 + b_7)}{9}$ |
| $I_{11231231} = I_{11321231} = \dfrac{b_3 + 4b_7}{18}$ | $I_{12311231} = \dfrac{b_2 + 3b_6 + b_7}{9}$ |

motion, the displacement-equation of motion is derived as below

$$\rho \ddot{u}_i = C_{jikl} u_{k,lj} + \left(F_{pqjkli} + F_{liqjkp} - G_{klijqp}\right) u_{p,qjkl} + I_{jklimnrs} u_{s,mnrjkl} + f_i. \tag{2.7}$$

For isotropic materials, the equation of motion is written in terms of Lamé constants and Mindlin's additional parameters as below

$$
\begin{aligned}
&(\lambda+2\mu)\left(1-\ell_{11}^2\nabla^2\right)\left(1-\ell_{12}^2\nabla^2\right)u_{j,ji} \\
&-\mu\left(1-\ell_{21}^2\nabla^2\right)\left(1-\ell_{22}^2\nabla^2\right)e_{ijk}e_{kml}u_{l,mj}+f_i=\rho\ddot{u}_i,
\end{aligned}
\tag{2.8}
$$

where $e_{ijk}$ is the permutation tensor and

$$2(\lambda+2\mu)\,\ell_{1p}^2=\bar{a}-2\bar{c}\pm\left[(\bar{a}-2\bar{c})^2-4\bar{b}(\lambda+2\mu)\right]^{\frac{1}{2}}, \tag{2.9a}$$

$$2\mu\,\ell_{2p}^2=\bar{a}'-c_3\pm\left[(\bar{a}'-c_3)^2-4\bar{b}'\mu\right]^{\frac{1}{2}}, \tag{2.9b}$$

for $p=1$ and 2 pertinent to the positive and negative signs, respectively, and

$$\bar{a}=2(a_1+a_2+a_3+a_4+a_5), \tag{2.10a}$$

$$\bar{b}=2(b_1+b_2+b_3+b_4+b_5+b_6+b_7), \tag{2.10b}$$

$$\bar{c}=c_1+c_2+c_3, \tag{2.10c}$$

$$\bar{a}'=2(a_3+a_4), \tag{2.10d}$$

$$\bar{b}'=2(b_5+b_6). \tag{2.10e}$$

$\ell_{11}$, $\ell_{12}$, $\ell_{21}$, and $\ell_{22}$ are the so-called "bulk characteristic lengths" which are related to Lamé constants and the additional parameters as given by Eqs. (2.9). Moreover, Mindlin's second strain gradient theory gives rise to another physically important length scale defined as

$$\ell_{10}^2=\frac{\bar{c}}{\lambda+2\mu}. \tag{2.11}$$

This length scale appears in the surface energy [5, 7] and surface residual stress formula and thus, it is referred to as surface characteristic length [7].

Suppose that the outward unit normal at any point along $S$ is defined by $\boldsymbol{n}(\boldsymbol{x})$. Then the generalized surface tractions, $\overset{1}{\boldsymbol{t}}$, $\overset{2}{\boldsymbol{t}}$, and $\overset{3}{\boldsymbol{t}}$ on $S$ are derived as

$$\overset{1}{t}_i = n_j(\tau_{ji} - \tau_{kji,k} + \tau_{klji,kl}) + L_k(n_j\tau_{jki} - n_j\tau_{mjki,m}) + L_k(L_j(n_m\tau_{mjki}))$$

$$- L_t(n_t n_k n_m n_j n_{p,p}\tau_{mjki} - L_t(n_k)n_m n_j\tau_{mjki}), \tag{2.12a}$$

$$\overset{2}{t}_i = n_j n_k(\tau_{jki} - \tau_{ljki,l}) + n_l L_k(n_j\tau_{jkli}) + L_l(n_k n_j\tau_{jkli}), \tag{2.12b}$$

$$\overset{3}{t}_i = n_j n_k n_l\tau_{jkli}, \tag{2.12c}$$

in which $L_i = n_i n_{p,p} - \nabla_i + n_i n_j\nabla_j$ and $\nabla_i = \partial/\partial x_i$.

## 3. Second strain gradient theory in orthogonal curvilinear coordinates

In this section, the stress-equation of equilibrium as well as the boundary conditions of second strain gradient theory described in the previous section is derived in the framework of orthogonal curvilinear coordinates. To this end, consider a set of orthogonal curvilinear coordinates $x^i$, $i = 1, 2, 3$ with base vectors $\boldsymbol{g}_i$ and metric tensor $g_{ij} = \boldsymbol{g}_i.\boldsymbol{g}_j$ [10]. The base vectors of the curvilinear coordinate system are obtained via $\boldsymbol{g}_i = \partial\boldsymbol{r}/\partial x^i$ where $\boldsymbol{r} = x_k\boldsymbol{i}^k$ is the position vector in Cartesian coordinate system; $\boldsymbol{i}^k, k = 1, 2, 3$ are the Cartesian coordinate base vectors. So the square of the element of the arc length $ds$ in terms of the curvilinear coordinates is written as below

$$ds^2 = d\boldsymbol{r}.d\boldsymbol{r} = g_{km}dx^k dx^m. \tag{3.1}$$

In formulations within curvilinear coordinates, Einstein summation convention is applied for repeated indices on diagonal positions. Representation of the position vector $\boldsymbol{r} = x_k \boldsymbol{i}^k$ in the above discussion is based on this convention. The unit base vectors of the curvilinear coordinates are obtained as

$$\boldsymbol{e}_i = \frac{\boldsymbol{g}_i}{|\boldsymbol{g}_i|} \qquad \text{(no sum)}. \tag{3.2}$$

Now, consider the displacement vector $\boldsymbol{u}$ and express it via the curvilinear coordinates as

$$\boldsymbol{u} = u^i \boldsymbol{g}_i = \boldsymbol{u}^{(i)} \boldsymbol{e}_i, \tag{3.3}$$

where the physical components of $\boldsymbol{u}$ denoted by $u^{(k)}$ is obtained as

$$u^{(i)} = |\boldsymbol{g}_i| u^i, \qquad \text{(no sum)}. \tag{3.4}$$

According to Eringen [10], the passage from rectangular coordinates to curvilinear coordinates is made by replacing the usual partial differentiation by the covariant partial derivative which is indicated by the symbol ";". Hence, the stress-equation of equilibrium in the curvilinear coordinates is displayed as follows

$$\sigma^i{}_{p;i} + f_p = 0, \tag{3.5}$$

in which

$$\sigma^i{}_p = \tau^i{}_p - \tau^{ij}{}_{p;j} + \tau^{ijk}{}_{p;jk}. \tag{3.6}$$

In the above relation, $\tau^i{}_p$, $\tau^{ij}{}_p$, and $\tau^{ijk}{}_p$ are the mixed components of the stress tensors of second-, third-, and fourth-order, respectively. According to Eringen [10], $A^{i\ldots j}{}_{k\ldots l}$ is called a mixed tensor if it changes under coordinate transformation through the following rule

$$A'^{i\dots j}{}_{k\dots l}(\boldsymbol{x}') = A^{m\dots n}{}_{p\dots q}(\boldsymbol{x})\frac{\partial x'^i}{\partial x^m}\cdots\frac{\partial x'^j}{\partial x^n}\frac{\partial x^p}{\partial x'^k}\cdots\frac{\partial x^q}{\partial x'^l}, \tag{3.7}$$

where $x'^i$ denotes the component of the new coordinate system. Moreover, covariant partial differentiation of the second-, third-, and fourth-order mixed tensors is defined as below

$$\sigma^i{}_{p;j} = \sigma^i{}_{p,j} + \sigma^t{}_p\begin{Bmatrix} i \\ jt \end{Bmatrix} - \sigma^i{}_t\begin{Bmatrix} t \\ jp \end{Bmatrix}, \tag{3.8a}$$

$$\sigma^{ij}{}_{p;k} = \sigma^{ij}{}_{p,k} + \sigma^{tj}{}_p\begin{Bmatrix} i \\ kt \end{Bmatrix} + \sigma^{it}{}_p\begin{Bmatrix} j \\ kt \end{Bmatrix} - \sigma^{ij}{}_t\begin{Bmatrix} t \\ kp \end{Bmatrix}, \tag{3.8b}$$

$$\begin{aligned}\sigma^{ijk}{}_{p;m} = {}& \sigma^{ijk}{}_{p,m} + \sigma^{tjk}{}_p\begin{Bmatrix} i \\ km \end{Bmatrix} + \sigma^{itk}{}_p\begin{Bmatrix} j \\ mt \end{Bmatrix} + \sigma^{ijt}{}_p\begin{Bmatrix} k \\ mt \end{Bmatrix} \\ & - \sigma^{ijk}{}_t\begin{Bmatrix} t \\ mp \end{Bmatrix},\end{aligned} \tag{3.8c}$$

where $\left\{ {i \atop jk} \right\}$ is the Christoffel symbol of the second kind defined as follows

$$\begin{Bmatrix} i \\ jk \end{Bmatrix} = \frac{\partial^2 x_n}{\partial x^j \partial x^k}\frac{\partial x^i}{\partial x_n}. \tag{3.9}$$

It can easily be shown that the physical components of the second-, third-, and fourth-order tensors are obtained as follows

$$\tau^{(i)}{}_{(j)} = \frac{|\boldsymbol{g}_i|}{|\boldsymbol{g}_j|}\tau^i{}_j, \qquad \text{(no sum)} \tag{3.10a}$$

$$\tau^{(i)(j)}{}_{(k)} = \frac{|\boldsymbol{g}_i||\boldsymbol{g}_j|}{|\boldsymbol{g}_k|}\tau^{ij}{}_k, \qquad \text{(no sum)} \tag{3.10b}$$

$$\tau^{(i)(j)(k)}{}_{(l)} = \frac{|\boldsymbol{g}_i||\boldsymbol{g}_j||\boldsymbol{g}_k|}{|\boldsymbol{g}_l|}\tau^{ijk}{}_l. \qquad \text{(no sum)} \tag{3.10c}$$

According to the definition of covariant partial derivative (3.8), the stress equation of equilibrium (3.5) becomes

$$\sigma^i{}_{p,i} + \sigma^t{}_p\begin{Bmatrix} i \\ ti \end{Bmatrix} - \sigma^i{}_t\begin{Bmatrix} t \\ ip \end{Bmatrix} + f_p = 0, \tag{3.11}$$

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where

$$\begin{aligned}
\sigma^{i}{}_{p} &= \tau^{i}{}_{p} - \tau^{ij}{}_{p,j} - \tau^{tj}{}_{p}\begin{Bmatrix} i \\ tj \end{Bmatrix} - \tau^{it}{}_{p}\begin{Bmatrix} j \\ tj \end{Bmatrix} + \tau^{ij}{}_{t}\begin{Bmatrix} t \\ pj \end{Bmatrix} \\
&+ \left(\tau^{ijk}{}_{p,k} + \tau^{tjk}{}_{p}\begin{Bmatrix} i \\ kt \end{Bmatrix} + \tau^{itk}{}_{p}\begin{Bmatrix} j \\ kt \end{Bmatrix} + \tau^{ijt}{}_{p}\begin{Bmatrix} k \\ tk \end{Bmatrix} - \tau^{ijk}{}_{t}\begin{Bmatrix} t \\ kp \end{Bmatrix}\right)_{,j} \\
&+ \begin{Bmatrix} i \\ tj \end{Bmatrix}\left(\tau^{tjk}{}_{p,k} + \tau^{njk}{}_{p}\begin{Bmatrix} t \\ kn \end{Bmatrix} + \tau^{tnk}{}_{p}\begin{Bmatrix} j \\ nk \end{Bmatrix} + \tau^{tjn}{}_{p}\begin{Bmatrix} k \\ nk \end{Bmatrix} - \tau^{tjk}{}_{n}\begin{Bmatrix} n \\ kp \end{Bmatrix}\right) \\
&+ \begin{Bmatrix} j \\ tj \end{Bmatrix}\left(\tau^{itk}{}_{p,k} + \tau^{ntk}{}_{p}\begin{Bmatrix} i \\ kn \end{Bmatrix} + \tau^{ink}{}_{p}\begin{Bmatrix} t \\ nk \end{Bmatrix} + \tau^{itn}{}_{p}\begin{Bmatrix} k \\ nk \end{Bmatrix} - \tau^{itk}{}_{n}\begin{Bmatrix} n \\ kp \end{Bmatrix}\right) \\
&- \begin{Bmatrix} t \\ jp \end{Bmatrix}\left(\tau^{ijk}{}_{t,k} + \tau^{njk}{}_{t}\begin{Bmatrix} i \\ kn \end{Bmatrix} + \tau^{ink}{}_{t}\begin{Bmatrix} j \\ nk \end{Bmatrix} + \tau^{ijn}{}_{t}\begin{Bmatrix} k \\ nk \end{Bmatrix} - \tau^{ijk}{}_{n}\begin{Bmatrix} n \\ kt \end{Bmatrix}\right).
\end{aligned} \tag{3.12}$$

Subsequently, the equilibrium equation can be rewritten in terms of the physical components as below

$$\left(\frac{|\boldsymbol{g}_p|}{|\boldsymbol{g}_i|}\sigma^{(i)}{}_{(p)}\right)_{,i} + \sigma^{(t)}{}_{(p)}\frac{|\boldsymbol{g}_p|}{|\boldsymbol{g}_t|}\begin{Bmatrix} i \\ ti \end{Bmatrix} - \sigma^{(i)}{}_{(t)}\frac{|\boldsymbol{g}_t|}{|\boldsymbol{g}_i|}\begin{Bmatrix} t \\ ip \end{Bmatrix} + f_{(p)}|\boldsymbol{g}_p| = 0, \tag{3.13}$$

where

$$\begin{aligned}
\sigma^{i}{}_{p} &= \frac{|\boldsymbol{g}_p|}{|\boldsymbol{g}_i|}\sigma^{(i)}{}_{(p)} = \frac{|\boldsymbol{g}_p|}{|\boldsymbol{g}_i|}\tau^{(i)}{}_{(p)} - \sum_{j}\left(\frac{|\boldsymbol{g}_p|}{|\boldsymbol{g}_i||\boldsymbol{g}_j|}\tau^{(i)(j)}{}_{(p)}\right)_{,j} \\
&- \sum_{j,t}\left(\frac{|\boldsymbol{g}_p|}{|\boldsymbol{g}_t||\boldsymbol{g}_j|}\tau^{(t)(j)}{}_{(p)}\begin{Bmatrix} i \\ tj \end{Bmatrix} + \frac{|\boldsymbol{g}_p|}{|\boldsymbol{g}_i||\boldsymbol{g}_t|}\tau^{(i)(t)}{}_{(p)}\begin{Bmatrix} j \\ tj \end{Bmatrix} - \frac{|\boldsymbol{g}_t|}{|\boldsymbol{g}_i||\boldsymbol{g}_j|}\tau^{(i)(j)}{}_{(t)}\begin{Bmatrix} t \\ pj \end{Bmatrix}\right. \\
&\left. - \left(\frac{|\boldsymbol{g}_p|}{|\boldsymbol{g}_i||\boldsymbol{g}_j||\boldsymbol{g}_t|}\tau^{(i)(j)(t)}{}_{(p)}\right)_{,jt}\right) \\
&+ \sum_{j,t,k}\left(\left(\frac{|\boldsymbol{g}_p|}{|\boldsymbol{g}_t||\boldsymbol{g}_j||\boldsymbol{g}_k|}\tau^{(t)(j)(k)}{}_{(p)}\begin{Bmatrix} i \\ kt \end{Bmatrix}\right)_{,j} + \left(\frac{|\boldsymbol{g}_p|}{|\boldsymbol{g}_i||\boldsymbol{g}_t||\boldsymbol{g}_k|}\tau^{(i)(t)(k)}{}_{(p)}\begin{Bmatrix} j \\ kt \end{Bmatrix}\right)_{,j}\right. \\
&+ \left(\frac{|\boldsymbol{g}_p|}{|\boldsymbol{g}_i||\boldsymbol{g}_j||\boldsymbol{g}_t|}\tau^{(i)(j)(t)}{}_{(p)}\begin{Bmatrix} k \\ tk \end{Bmatrix}\right)_{,j} - \left(\frac{|\boldsymbol{g}_t|}{|\boldsymbol{g}_i||\boldsymbol{g}_j||\boldsymbol{g}_k|}\tau^{(i)(j)(k)}{}_{(t)}\begin{Bmatrix} t \\ kp \end{Bmatrix}\right)_{,j} \\
&+ \left(\frac{|\boldsymbol{g}_p|}{|\boldsymbol{g}_t||\boldsymbol{g}_j||\boldsymbol{g}_k|}\tau^{(t)(j)(k)}{}_{(p)}\right)_{,k}\begin{Bmatrix} i \\ tj \end{Bmatrix} + \left(\frac{|\boldsymbol{g}_p|}{|\boldsymbol{g}_i||\boldsymbol{g}_t||\boldsymbol{g}_k|}\tau^{(i)(t)(k)}{}_{(p)}\right)_{,k}\begin{Bmatrix} j \\ tj \end{Bmatrix} \\
&\left. - \left(\frac{|\boldsymbol{g}_t|}{|\boldsymbol{g}_i||\boldsymbol{g}_j||\boldsymbol{g}_k|}\tau^{(i)(j)(k)}{}_{(t)}\right)_{,k}\begin{Bmatrix} t \\ jp \end{Bmatrix}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{n,j,k,t}\Bigg(\frac{|\boldsymbol{g}_p|}{|\boldsymbol{g}_n||\boldsymbol{g}_j||\boldsymbol{g}_k|}\tau^{(n)(j)(k)}{}_{(p)}\begin{Bmatrix} t \\ kn \end{Bmatrix}\begin{Bmatrix} i \\ tj \end{Bmatrix}\\
&+\frac{|\boldsymbol{g}_p|}{|\boldsymbol{g}_t||\boldsymbol{g}_n||\boldsymbol{g}_k|}\tau^{(t)(n)(k)}{}_{(p)}\begin{Bmatrix} j \\ nk \end{Bmatrix}\begin{Bmatrix} i \\ tj \end{Bmatrix}+\frac{|\boldsymbol{g}_p|}{|\boldsymbol{g}_t||\boldsymbol{g}_j||\boldsymbol{g}_n|}\tau^{(t)(j)(n)}{}_{(p)}\begin{Bmatrix} k \\ nk \end{Bmatrix}\begin{Bmatrix} i \\ tj \end{Bmatrix}\\
&-\frac{|\boldsymbol{g}_n|}{|\boldsymbol{g}_t||\boldsymbol{g}_j||\boldsymbol{g}_k|}\tau^{(t)(j)(k)}{}_{(n)}\begin{Bmatrix} n \\ kp \end{Bmatrix}\begin{Bmatrix} i \\ tj \end{Bmatrix}+\frac{|\boldsymbol{g}_p|}{|\boldsymbol{g}_n||\boldsymbol{g}_t||\boldsymbol{g}_k|}\tau^{(n)(t)(k)}{}_{(p)}\begin{Bmatrix} i \\ kn \end{Bmatrix}\begin{Bmatrix} j \\ tj \end{Bmatrix}\\
&+\frac{|\boldsymbol{g}_p|}{|\boldsymbol{g}_i||\boldsymbol{g}_n||\boldsymbol{g}_k|}\tau^{(i)(n)(k)}{}_{(p)}\begin{Bmatrix} t \\ nk \end{Bmatrix}\begin{Bmatrix} j \\ tj \end{Bmatrix}+\frac{|\boldsymbol{g}_p|}{|\boldsymbol{g}_i||\boldsymbol{g}_t||\boldsymbol{g}_n|}\tau^{(i)(t)(n)}{}_{(p)}\begin{Bmatrix} k \\ nk \end{Bmatrix}\begin{Bmatrix} j \\ tj \end{Bmatrix}\\
&-\frac{|\boldsymbol{g}_n|}{|\boldsymbol{g}_i||\boldsymbol{g}_t||\boldsymbol{g}_k|}\tau^{(i)(t)(k)}{}_{(n)}\begin{Bmatrix} n \\ kp \end{Bmatrix}\begin{Bmatrix} j \\ tj \end{Bmatrix}-\frac{|\boldsymbol{g}_t|}{|\boldsymbol{g}_n||\boldsymbol{g}_j||\boldsymbol{g}_k|}\tau^{(n)(j)(k)}{}_{(t)}\begin{Bmatrix} i \\ kn \end{Bmatrix}\begin{Bmatrix} t \\ jp \end{Bmatrix}\\
&-\frac{|\boldsymbol{g}_t|}{|\boldsymbol{g}_i||\boldsymbol{g}_n||\boldsymbol{g}_k|}\tau^{(i)(n)(k)}{}_{(t)}\begin{Bmatrix} j \\ nk \end{Bmatrix}\begin{Bmatrix} t \\ jp \end{Bmatrix}-\frac{|\boldsymbol{g}_t|}{|\boldsymbol{g}_i||\boldsymbol{g}_j||\boldsymbol{g}_n|}\tau^{(i)(j)(n)}{}_{(t)}\begin{Bmatrix} k \\ nk \end{Bmatrix}\begin{Bmatrix} t \\ jp \end{Bmatrix}\\
&+\frac{|\boldsymbol{g}_n|}{|\boldsymbol{g}_i||\boldsymbol{g}_j||\boldsymbol{g}_k|}\tau^{(i)(j)(k)}{}_{(n)}\begin{Bmatrix} n \\ kt \end{Bmatrix}\begin{Bmatrix} t \\ jp \end{Bmatrix}\Bigg). \qquad (3.14)
\end{aligned}
$$

Moreover, the components of the second-, third-, and fourth-order strain tensors in curvilinear coordinates are written as

$$
\epsilon^i{}_j=\frac{1}{2}\left(u^i{}_{;j}+g_{jm}g^{in}u^m{}_{;n}\right)=\frac{1}{2}\left(u^i{}_{,j}+u^t\begin{Bmatrix} i \\ tj \end{Bmatrix}+g_{jm}g^{in}\left(u^m{}_{,n}+u^t\begin{Bmatrix} m \\ nt \end{Bmatrix}\right)\right), \qquad (3.15a)
$$

$$
\begin{aligned}
\epsilon^k{}_{ij}=u^k{}_{;ij}&=\left(u^k{}_{,i}+u^t\begin{Bmatrix} k \\ it \end{Bmatrix}\right)_{;j}=\left(u^k{}_{,i}+u^t\begin{Bmatrix} k \\ it \end{Bmatrix}\right)_{,j}+\left(u^t{}_{,i}+u^r\begin{Bmatrix} t \\ ir \end{Bmatrix}\right)\begin{Bmatrix} k \\ tj \end{Bmatrix}\\
&-\left(u^k{}_{,t}+u^r\begin{Bmatrix} k \\ tr \end{Bmatrix}\right)\begin{Bmatrix} t \\ ij \end{Bmatrix} \qquad (3.15b)
\end{aligned}
$$

$$
\begin{aligned}
\epsilon^l{}_{ijk}&=u^l{}_{;ijk}\\
&=\left(\left(u^l{}_{,i}+u^t\begin{Bmatrix} l \\ it \end{Bmatrix}\right)_{,j}+\left(u^t{}_{,i}+u^r\begin{Bmatrix} t \\ ir \end{Bmatrix}\right)\begin{Bmatrix} l \\ tj \end{Bmatrix}-\left(u^l{}_{,t}+u^r\begin{Bmatrix} l \\ tr \end{Bmatrix}\right)\begin{Bmatrix} t \\ ij \end{Bmatrix}\right)_{;k}\\
&=\left(\left(u^l{}_{,i}+u^t\begin{Bmatrix} l \\ it \end{Bmatrix}\right)_{,j}+\left(u^t{}_{,i}+u^r\begin{Bmatrix} t \\ ir \end{Bmatrix}\right)\begin{Bmatrix} l \\ tj \end{Bmatrix}-\left(u^l{}_{,t}+u^r\begin{Bmatrix} l \\ tr \end{Bmatrix}\right)\begin{Bmatrix} t \\ ij \end{Bmatrix}\right)_{,k}\\
&+\left(\left(u^t{}_{,i}+u^r\begin{Bmatrix} t \\ ir \end{Bmatrix}\right)_{,j}+\left(u^r{}_{,i}+u^s\begin{Bmatrix} r \\ is \end{Bmatrix}\right)\begin{Bmatrix} t \\ rj \end{Bmatrix}-\left(u^t{}_{,r}+u^s\begin{Bmatrix} t \\ rs \end{Bmatrix}\right)\begin{Bmatrix} r \\ ij \end{Bmatrix}\right)\begin{Bmatrix} l \\ tk \end{Bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\left(u^l{}_{,t}+u^r\begin{Bmatrix} l \\ tr \end{Bmatrix}\right)_{,j}+\left(u^r{}_{,t}+u^s\begin{Bmatrix} r \\ ts \end{Bmatrix}\right)\begin{Bmatrix} l \\ rj \end{Bmatrix}-\left(u^l{}_{,r}+u^s\begin{Bmatrix} l \\ rs \end{Bmatrix}\right)\begin{Bmatrix} r \\ tj \end{Bmatrix}\right)\begin{Bmatrix} t \\ ik \end{Bmatrix}\\
&-\left(\left(u^l{}_{,i}+u^r\begin{Bmatrix} l \\ ir \end{Bmatrix}\right)_{,t}+\left(u^r{}_{,i}+u^s\begin{Bmatrix} r \\ is \end{Bmatrix}\right)\begin{Bmatrix} l \\ rt \end{Bmatrix}-\left(u^l{}_{,r}+u^s\begin{Bmatrix} l \\ rs \end{Bmatrix}\right)\begin{Bmatrix} r \\ it \end{Bmatrix}\right)\begin{Bmatrix} t \\ kj \end{Bmatrix}.
\end{aligned}
\tag{3.15c}
$$

Using Eqs. (3.4) and (3.10), the physical components of the second-, third-, and fourth-order strain tensors are derived.

$$
\epsilon^{(i)}{}_{(j)}=\frac{1}{2}\left(\frac{|\boldsymbol{g}_i|}{|\boldsymbol{g}_j|}\left(\frac{u^{(i)}}{|\boldsymbol{g}_i|}\right)_{,j}+\frac{|\boldsymbol{g}_j|}{|\boldsymbol{g}_i|}\left(\frac{u^{(j)}}{|\boldsymbol{g}_j|}\right)_{,i}+\sum_t\frac{u^{(t)}}{|\boldsymbol{g}_t|}\left(\frac{|\boldsymbol{g}_i|}{|\boldsymbol{g}_j|}\begin{Bmatrix} i \\ tj \end{Bmatrix}+\frac{|\boldsymbol{g}_j|}{|\boldsymbol{g}_i|}\begin{Bmatrix} j \\ it \end{Bmatrix}\right)\right),
\tag{3.16a}
$$

$$
\begin{aligned}
\epsilon^{(k)}{}_{(i)(j)}=&\frac{|\boldsymbol{g}_k|}{|\boldsymbol{g}_i||\boldsymbol{g}_j|}\Bigg(\left(\frac{u^{(k)}}{|\boldsymbol{g}_k|}\right)_{,ij}+\sum_t\Bigg(\left(\frac{u^{(t)}}{|\boldsymbol{g}_t|}\begin{Bmatrix} k \\ it \end{Bmatrix}\right)_{,j}+\left(\frac{u^{(t)}}{|\boldsymbol{g}_t|}\right)_{,i}\begin{Bmatrix} k \\ tj \end{Bmatrix}\\
&-\left(\frac{u^{(k)}}{|\boldsymbol{g}_k|}\right)_{,t}\begin{Bmatrix} t \\ ij \end{Bmatrix}\Bigg)+\sum_{t,\,r}\left(\frac{u^{(r)}}{|\boldsymbol{g}_r|}\begin{Bmatrix} t \\ ir \end{Bmatrix}\begin{Bmatrix} k \\ tj \end{Bmatrix}-\frac{u^{(r)}}{|\boldsymbol{g}_r|}\begin{Bmatrix} k \\ tr \end{Bmatrix}\begin{Bmatrix} t \\ ij \end{Bmatrix}\right)\Bigg)
\end{aligned}
\tag{3.16b}
$$

$$
\begin{aligned}
\epsilon^{(l)}{}_{(i)(j)(k)}=&\frac{|\boldsymbol{g}_l|}{|\boldsymbol{g}_i||\boldsymbol{g}_j||\boldsymbol{g}_k|}\Bigg(\left(\frac{u^{(l)}}{|\boldsymbol{g}_l|}\right)_{,ijk}+\sum_t\Bigg(\left(\frac{u^{(t)}}{|\boldsymbol{g}_t|}\begin{Bmatrix} l \\ it \end{Bmatrix}\right)_{,jk}+\left(\left(\frac{u^{(t)}}{|\boldsymbol{g}_t|}\right)_{,i}\begin{Bmatrix} l \\ tj \end{Bmatrix}\right)_{,k}\\
&-\left(\left(\frac{u^{(l)}}{|\boldsymbol{g}_l|}\right)_{,t}\begin{Bmatrix} t \\ ij \end{Bmatrix}\right)_{,k}+\left(\frac{u^{(t)}}{|\boldsymbol{g}_t|}\right)_{,ij}\begin{Bmatrix} l \\ tk \end{Bmatrix}-\left(\frac{u^{(l)}}{|\boldsymbol{g}_l|}\right)_{,tj}\begin{Bmatrix} t \\ ik \end{Bmatrix}-\left(\frac{u^{(l)}}{|\boldsymbol{g}_l|}\right)_{,it}\begin{Bmatrix} t \\ kj \end{Bmatrix}\Bigg)\\
&+\sum_{r,\,t}\Bigg(\left(\frac{u^{(r)}}{|\boldsymbol{g}_r|}\begin{Bmatrix} t \\ ir \end{Bmatrix}\begin{Bmatrix} l \\ tj \end{Bmatrix}\right)_{,k}-\left(\frac{u^{(r)}}{|\boldsymbol{g}_r|}\begin{Bmatrix} l \\ tr \end{Bmatrix}\begin{Bmatrix} t \\ ij \end{Bmatrix}\right)_{,k}+\left(\frac{u^{(r)}}{|\boldsymbol{g}_r|}\begin{Bmatrix} t \\ ir \end{Bmatrix}\right)_{,j}\begin{Bmatrix} l \\ tk \end{Bmatrix}\\
&+\left(\frac{u^{(r)}}{|\boldsymbol{g}_r|}\right)_{,i}\begin{Bmatrix} t \\ rj \end{Bmatrix}\begin{Bmatrix} l \\ tk \end{Bmatrix}-\left(\frac{u^{(t)}}{|\boldsymbol{g}_t|}\right)_{,r}\begin{Bmatrix} r \\ ij \end{Bmatrix}\begin{Bmatrix} l \\ tk \end{Bmatrix}-\left(\frac{u^{(r)}}{|\boldsymbol{g}_r|}\begin{Bmatrix} l \\ tr \end{Bmatrix}\right)_{,j}\begin{Bmatrix} t \\ ik \end{Bmatrix}\\
&-\left(\frac{u^{(r)}}{|\boldsymbol{g}_r|}\right)_{,t}\begin{Bmatrix} l \\ rj \end{Bmatrix}\begin{Bmatrix} t \\ ik \end{Bmatrix}+\left(\frac{u^{(l)}}{|\boldsymbol{g}_l|}\right)_{,r}\begin{Bmatrix} r \\ tj \end{Bmatrix}\begin{Bmatrix} t \\ ik \end{Bmatrix}-\left(\frac{u^{(r)}}{|\boldsymbol{g}_r|}\begin{Bmatrix} l \\ ir \end{Bmatrix}\right)_{,t}\begin{Bmatrix} t \\ kj \end{Bmatrix}\\
&-\left(\frac{u^{(r)}}{|\boldsymbol{g}_r|}\right)_{,i}\begin{Bmatrix} l \\ rt \end{Bmatrix}\begin{Bmatrix} t \\ kj \end{Bmatrix}+\left(\frac{u^{(l)}}{|\boldsymbol{g}_l|}\right)_{,r}\begin{Bmatrix} r \\ it \end{Bmatrix}\begin{Bmatrix} t \\ kj \end{Bmatrix}\Bigg)+\sum_{r,\,s,\,t}\Bigg(\frac{u^{(s)}}{|\boldsymbol{g}_s|}\begin{Bmatrix} r \\ is \end{Bmatrix}\begin{Bmatrix} t \\ rj \end{Bmatrix}\begin{Bmatrix} l \\ tk \end{Bmatrix}\\
&-\frac{u^{(s)}}{|\boldsymbol{g}_s|}\begin{Bmatrix} t \\ rs \end{Bmatrix}\begin{Bmatrix} r \\ ij \end{Bmatrix}\begin{Bmatrix} l \\ tk \end{Bmatrix}-\frac{u^{(s)}}{|\boldsymbol{g}_s|}\begin{Bmatrix} r \\ ts \end{Bmatrix}\begin{Bmatrix} l \\ rj \end{Bmatrix}\begin{Bmatrix} t \\ ik \end{Bmatrix}+\frac{u^{(s)}}{|\boldsymbol{g}_s|}\begin{Bmatrix} l \\ rs \end{Bmatrix}\begin{Bmatrix} r \\ tj \end{Bmatrix}\begin{Bmatrix} t \\ ik \end{Bmatrix}\\
&-\frac{u^{(s)}}{|\boldsymbol{g}_s|}\begin{Bmatrix} r \\ is \end{Bmatrix}\begin{Bmatrix} l \\ rt \end{Bmatrix}\begin{Bmatrix} t \\ kj \end{Bmatrix}+\frac{u^{(s)}}{|\boldsymbol{g}_s|}\begin{Bmatrix} l \\ rs \end{Bmatrix}\begin{Bmatrix} r \\ it \end{Bmatrix}\begin{Bmatrix} t \\ kj \end{Bmatrix}\Bigg)\Bigg).
\end{aligned}
\tag{3.16c}
$$

Next, the traction boundary conditions in curvilinear coordinates are derived as below. To this end, define the reciprocal base vectors $\boldsymbol{g}^k$ such that $\boldsymbol{g}^k.\boldsymbol{g}_l = \delta^k{}_l$ where $\delta^k{}_l$ is the Kronecker delta. Moreover, $g^{km} = \boldsymbol{g}^k.\boldsymbol{g}^m$. Among the tractions, $\overset{3}{\boldsymbol{t}}$ has the simplest representations, and so we come up with its curvilinear representation first.

$$\overset{3}{t}{}^{i} = g^{is} g_{jp} g_{kq} g_{lr} n^p n^q n^r \tau^{jkl}{}_s, \tag{3.17}$$

and in terms of the physical components

$$\frac{\overset{3}{t}{}^{(i)}}{|\boldsymbol{g}_i|} = \sum_{\substack{j,k,l\\p,q,r,s}} g^{is} g_{jp} g_{kq} g_{lr} \frac{n^{(p)}}{|\boldsymbol{g}_p|} \frac{n^{(q)}}{|\boldsymbol{g}_q|} \frac{n^{(r)}}{|\boldsymbol{g}_r|} \tau^{(j)(k)(l)}{}_{(s)} \frac{|\boldsymbol{g}_s|}{|\boldsymbol{g}_j||\boldsymbol{g}_k||\boldsymbol{g}_l|}. \tag{3.18}$$

If orthogonal curvilinear coordinates are used, then it simplifies to

$$\overset{3}{t}{}^{(i)} = n^{(j)} n^{(k)} n^{(l)} \tau^{(j)(k)(l)}{}_{(i)}. \tag{3.19}$$

The second traction boundary condition is similarly written as follows

$$\begin{aligned} \overset{2}{t}{}^{m} g_{mi} &= n^p n^q g_{pj} g_{qk} (\tau^{kj}{}_i - \tau^{ljk}{}_{i;l}) + n^s n^r n^x g_{sl} g_{rk} g_{xj} n^p{}_{;p} \tau^{jkl}{}_i - n^s g_{sl} (n^r g_{rj} \tau^{jkl}{}_i)_{;k} \\ &+ n^q n^s n^p g_{lq} g_{sk} (n^x g_{xj} \tau^{jkl}{}_i)_{;p} + n^p{}_{;p} n^q n^r n^s g_{ql} g_{rk} g_{sj} \tau^{jkl}{}_i - (n^r n^s g_{rk} g_{js} \tau^{jkl}{}_i)_{;l} \\ &+ n^p n^t g_{pl} (n^r n^s g_{rk} g_{sj} \tau^{jkl}{}_i)_{;t}, \end{aligned} \tag{3.20}$$

and in terms of the physical components we have

$$\begin{aligned} \sum_m \frac{\overset{2}{t}{}^{(m)}}{|\boldsymbol{g}_m|} g_{mi} &= \sum_{\substack{p,\,q\\j,\,k,\,l}} \frac{n^{(p)}}{|\boldsymbol{g}_p|} \frac{n^{(q)}}{|\boldsymbol{g}_q|} g_{pj} g_{qk} (\tau^{(k)(j)}{}_{(i)} - \tau^{(l)(j)(k)}{}_{(i);(l)}) \frac{|\boldsymbol{g}_i|}{|\boldsymbol{g}_k||\boldsymbol{g}_j|} \\ &+ \sum_{\substack{r,\,s,\,x\\j,\,k,\,l,\,p}} \frac{n^{(s)}}{|\boldsymbol{g}_s|} \frac{n^{(r)}}{|\boldsymbol{g}_r|} \frac{n^{(x)}}{|\boldsymbol{g}_x|} g_{sl} g_{rk} g_{xj} n^{(p)}{}_{;(p)} \tau^{(j)(k)(l)}{}_{(i)} \frac{|\boldsymbol{g}_i|}{|\boldsymbol{g}_j||\boldsymbol{g}_k||\boldsymbol{g}_l|} \\ &- \sum_{\substack{r,\,s\\j,\,k,\,l}} \frac{n^{(s)}}{|\boldsymbol{g}_s|} g_{sl} (n^{(r)} g_{rj} \tau^{(j)(k)(l)}{}_{(i)})_{;(k)} \frac{|\boldsymbol{g}_i|}{|\boldsymbol{g}_j||\boldsymbol{g}_l||\boldsymbol{g}_r|} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{\substack{q,\,r,\,s,\,x\\ j,\,k,\,l,\,p}} \frac{n^{(s)}}{|\boldsymbol{g}_s|}\frac{n^{(p)}}{|\boldsymbol{g}_p|}\frac{n^{(q)}}{|\boldsymbol{g}_q|} g_{sk}g_{ql}\big(n^{(x)}g_{xj}\tau^{(j)(k)(l)}{}_{(i)}\big)_{;(p)}\frac{|\boldsymbol{g}_i||\boldsymbol{g}_p|}{|\boldsymbol{g}_x||\boldsymbol{g}_j||\boldsymbol{g}_k||\boldsymbol{g}_l|}\\
&+\sum_{\substack{q,\,r,\,s\\ j,\,k,\,l,\,p}} n^{(p)}{}_{;(p)}\frac{n^{(s)}}{|\boldsymbol{g}_s|}\frac{n^{(r)}}{|\boldsymbol{g}_r|}\frac{n^{(q)}}{|\boldsymbol{g}_q|} g_{sj}g_{rk}g_{ql}\tau^{(j)(k)(l)}{}_{(i)}\frac{|\boldsymbol{g}_i|}{|\boldsymbol{g}_j||\boldsymbol{g}_k||\boldsymbol{g}_l|}\\
&-\sum_{\substack{r,\,s\\ j,\,k,\,l}} \big(\frac{n^{(s)}}{|\boldsymbol{g}_s|}\frac{n^{(r)}}{|\boldsymbol{g}_r|} g_{sj}g_{rk}\tau^{(j)(k)(l)}{}_{(i)}\big)_{;(l)}\frac{|\boldsymbol{g}_i|}{|\boldsymbol{g}_j||\boldsymbol{g}_k|}\\
&+\sum_{\substack{p,\,q,\,r,\,s\\ j,\,k,\,l,\,t}} \frac{n^{(p)}}{|\boldsymbol{g}_p|}\frac{n^{(t)}}{|\boldsymbol{g}_t|} g_{pl}\big(\frac{n^{(r)}}{|\boldsymbol{g}_r|}\frac{n^{(s)}}{|\boldsymbol{g}_s|} g_{rk}g_{sj}\tau^{(j)(k)(l)}{}_{(i)}\big)_{;(t)}\frac{|\boldsymbol{g}_i||\boldsymbol{g}_t|}{|\boldsymbol{g}_j||\boldsymbol{g}_k||\boldsymbol{g}_l|}.
\end{aligned}
\tag{3.21}
$$

In orthogonal curvilinear coordinates, it can further be simplified to

$$
\begin{aligned}
\overset{2}{t}{}^{(i)} &= n^{(j)}n^{(k)}\big(\tau^{(k)(j)}{}_{(i)} - \tau^{(l)(j)(k)}{}_{(i);(l)}\big) + n^{(l)}n^{(k)}n^{(j)}n^{(p)}{}_{;(p)}\tau^{(j)(k)(l)}{}_{(i)}\\
&- n^{(l)}\big(n^{(j)}\tau^{(j)(k)(l)}{}_{(i)}\big)_{;(k)} + n^{(k)}n^{(p)}n^{(l)}\big(n^{(j)}\tau^{(j)(k)(l)}{}_{(i)}\big)_{;(p)}\\
&+ n^{(p)}{}_{;(p)}n^{(j)}n^{(k)}n^{(l)}\tau^{(j)(k)(l)}{}_{(i)} - \big(n^{(j)}n^{(k)}\tau^{(j)(k)(l)}{}_{(i)}\big)_{;(l)}\\
&+ n^{(l)}n^{(t)}\big(n^{(k)}n^{(j)}\tau^{(j)(k)(l)}{}_{(i)}\big)_{;(t)},
\end{aligned}
\tag{3.22}
$$

where for an arbitrary tensor of any order

$$
\begin{aligned}
A^{(i)\cdots(j)}{}_{(k);(m)} &= \frac{|\boldsymbol{g}_i|\cdots|\boldsymbol{g}_j|}{|\boldsymbol{g}_k||\boldsymbol{g}_m|}A^{i\cdots j}{}_{k;m} = \frac{|\boldsymbol{g}_i|\cdots|\boldsymbol{g}_j|}{|\boldsymbol{g}_k||\boldsymbol{g}_m|}\Bigg(A^{i\cdots j}{}_{k,m} + A^{t\cdots j}{}_{k}\begin{Bmatrix} i\\ tm\end{Bmatrix}\\
&\quad + \cdots + A^{i\cdots t}{}_{k}\begin{Bmatrix} j\\ tm\end{Bmatrix} - A^{i\cdots j}{}_{t}\begin{Bmatrix} t\\ km\end{Bmatrix}\Bigg)\\
&= \frac{|\boldsymbol{g}_i|\cdots|\boldsymbol{g}_j|}{|\boldsymbol{g}_k||\boldsymbol{g}_m|}\Bigg(\bigg(\frac{|\boldsymbol{g}_k|}{|\boldsymbol{g}_i|\cdots|\boldsymbol{g}_j|}A^{(i)\cdots(j)}{}_{(k)}\bigg)_{,m}\\
&\quad + \frac{|\boldsymbol{g}_k|}{|\boldsymbol{g}_t|\cdots|\boldsymbol{g}_j|}A^{(t)\cdots(j)}{}_{(k)}\begin{Bmatrix} i\\ tm\end{Bmatrix} + \cdots + \frac{|\boldsymbol{g}_k|}{|\boldsymbol{g}_i|\cdots|\boldsymbol{g}_t|}A^{(i)\cdots(t)}{}_{(k)}\begin{Bmatrix} j\\ tm\end{Bmatrix}\\
&\quad - \frac{|\boldsymbol{g}_t|}{|\boldsymbol{g}_i|\cdots|\boldsymbol{g}_j|}A^{(i)\cdots(j)}{}_{(t)}\begin{Bmatrix} t\\ km\end{Bmatrix}\Bigg).
\end{aligned}
\tag{3.23}
$$

Likewise, it can be shown that the first traction boundary condition in curvilinear coordinates which is more involved than the second and third traction types has the following representation

$$
\begin{aligned}
g_{mi}\overset{1}{t}{}^{m} &= n^t g_{tj}(\tau^j{}_i - \tau^{kj}{}_{i;k} + \tau^{klj}{}_{i;kl}) + n^p{}_{;p} n^t g_{tk}(n^s g_{sj}\tau^{jk}{}_i - n^s g_{sj}\tau^{mjk}{}_{i;m}) \\
&+ n^q g_{qk} n^p (n^t g_{tj}\tau^{jk}{}_i - n^s g_{sj}\tau^{mjk}{}_{i;m})_{;p} - g_{mj} n^m{}_{;k}(\tau^{jk}{}_i - \tau^{pjk}{}_{i;p}) \\
&- g_{tj} n^t (\tau^{jk}{}_{i;k} - \tau^{mjp}{}_{i;mp}) + n^t g_{tk} n^s{}_{;s} n^l g_{lj} n^p{}_{;p} n^n g_{nm}\tau^{mjk}{}_i \\
&- n^p g_{pk} n^s{}_{;s}(n^n g_{nm}\tau^{mjk}{}_i)_{;j} + n^r g_{rk} n^s{}_{;s} n^l g_{lj} n^p (n^t g_{tm}\tau^{mjk}{}_i)_{;p} \\
&- (n^t g_{tj} n^p{}_{;p} n^s g_{sm}\tau^{mjk}{}_i)_{;k} + (n^t g_{tm}\tau^{mjk}{}_i)_{;jk} - (n^t g_{tj} n^p (n^r g_{rm}\tau^{mjk}{}_i)_{;p})_{;k} \\
&+ n^r g_{rk} n^s (n^l g_{lj} n^p{}_{;p} n^x g_{xm}\tau^{mjk}{}_i)_{;s} - n^p g_{pk} n^s (n^q g_{qm}\tau^{mjk}{}_i)_{;js} \\
&+ n^q g_{qk} n^s (n^t g_{tj} n^p (n^r g_{rm}\tau^{mjk}{}_i)_{;p})_{;s} - n^t n^s{}_{;s} n^p{}_{;t} g_{pk} n^q g_{qm} n^r g_{rj}\tau^{mjk}{}_i \\
&+ (n^p{}_{;h} g_{pk} g^{ht} n^q n^r g_{qm} g_{rj}\tau^{mjk}{}_i)_{;t} - n^t n^r (n^p{}_{;t} g_{pk} n^q n^x g_{qm} g_{xj}\tau^{mjk}{}_i)_{;r} \\
&+ n^s{}_{;s} n^p n^x g_{xm} n^t g_{tj} n^q{}_{;p} g_{qk}\tau^{mjk}{}_i - (n^t n^p n^q g_{qm} n^x g_{xj} n^s{}_{;p} g_{sk}\tau^{mjk}{}_i)_{;t} \\
&+ n^x g_{tx} n^s (n^t n^p n^q g_{mq} n^r g_{rj} n^z{}_{;p} g_{zk}\tau^{mjk}{}_i)_{;s},
\end{aligned}
\tag{3.24}
$$

which in terms of the physical components within the orthogonal curvilinear coordinate system has the following form

$$
\begin{aligned}
\overset{1}{t}{}^{(i)} &= n^{(j)}(\tau^{(j)}{}_{(i)} - \tau^{(k)(j)}{}_{(i);(k)} + \tau^{(k)(l)(j)}{}_{(i);(k)(l)}) + n^{(p)}{}_{;(p)} n^{(k)} n^{(j)}(\tau^{(j)(k)}{}_{(i)} \\
&- \tau^{(m)(j)(k)}{}_{(i);(m)}) + n^{(k)} n^{(p)}(n^{(j)}\tau^{(j)(k)}{}_{(i)} - n^{(j)}\tau^{(m)(j)(k)}{}_{(i);(m)})_{;(p)} \\
&- n^{(j)}{}_{;(k)}(\tau^{(j)(k)}{}_{(i)} - \tau^{(p)(j)(k)}{}_{(i);(p)}) - n^{(j)}(\tau^{(j)(k)}{}_{(i);(k)} - \tau^{(m)(j)(p)}{}_{(i);(m)(p)}) \\
&+ n^{(k)} n^{(s)}{}_{;(s)} n^{(j)} n^{(p)}{}_{;(p)} n^{(m)}\tau^{(m)(j)(k)}{}_{(i)} - n^{(k)} n^{(s)}{}_{;(s)}(n^{(m)}\tau^{(m)(j)(k)}{}_{(i)})_{;(j)} \\
&+ n^{(k)} n^{(s)}{}_{;(s)} n^{(j)} n^{(p)}(n^{(m)}\tau^{(m)(j)(k)}{}_{(i)})_{;(p)} - (n^{(j)} n^{(p)}{}_{;(p)} n^{(m)}\tau^{(m)(j)(k)}{}_{(i)})_{;(k)} \\
&+ (n^{(m)}\tau^{(m)(j)(k)}{}_{(i)})_{;(j)(k)} - (n^{(j)} n^{(p)}(n^{(m)}\tau^{(m)(j)(k)}{}_{(i)})_{;(p)})_{;(k)} \\
&+ n^{(k)} n^{(s)}(n^{(j)} n^{(p)}{}_{;(p)} n^{(m)}\tau^{(m)(j)(k)}{}_{(i)})_{;(s)} - n^{(k)} n^{(s)}(n^{(m)}\tau^{(m)(j)(k)}{}_{(i)})_{;(j)(s)}
\end{aligned}
$$

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$$
\begin{aligned}
&+ n^{(k)}n^{(s)}(n^{(j)}n^{(p)}(n^{(m)}\tau^{(m)(j)(k)}{}_{(i)})_{;(p)})_{;(s)} - n^{(t)}n^{(s)}{}_{;(s)}n^{(k)}{}_{;(t)}n^{(m)}n^{(j)}\tau^{(m)(j)(k)}{}_{(i)} \\
&+ (n^{(k)}{}_{;(t)}n^{(m)}n^{(j)}\tau^{(m)(j)(k)}{}_{(i)})_{;(t)} - n^{(t)}n^{(r)}(n^{(k)}{}_{;(t)}n^{(m)}n^{(j)}\tau^{(m)(j)(k)}{}_{(i)})_{;(r)} \\
&+ n^{(s)}{}_{;(s)}n^{(p)}n^{(m)}n^{(j)}n^{(k)}{}_{;(p)}\tau^{(m)(j)(k)}{}_{(i)} - (n^{(t)}n^{(p)}n^{(m)}n^{(j)}n^{(k)}{}_{;(p)}\tau^{(m)(j)(k)}{}_{(i)})_{;(t)} \\
&+ n^{(t)}n^{(s)}(n^{(t)}n^{(p)}n^{(m)}n^{(j)}n^{(k)}{}_{;(p)}\tau^{(m)(j)(k)}{}_{(i)})_{;(s)}. \qquad (3.25)
\end{aligned}
$$

Finally, by substituting the components of the elastic tensors in Eqs. (2.3) in terms of Lamé constants and Mindlin's additional parameters, the constitutive relations for isotropic materials in the curvilinear coordinates are obtained as follows

$$
\begin{aligned}
\tau^{(p)}{}_{(q)} &= \lambda\epsilon^{(i)}{}_{(i)}\delta^{(p)}{}_{(q)} + 2\mu\epsilon^{(p)}{}_{(q)} + c_1\epsilon^{(j)}{}_{(i)(i)(j)}\delta^{(p)}{}_{(q)} + c_2\epsilon^{(i)}{}_{(p)(q)(i)} \\
&\quad + \frac{c_3}{2}(\epsilon^{(q)}{}_{(i)(i)(p)} + \epsilon^{(p)}{}_{(i)(i)(q)}), \qquad (3.26a) \\
\tau^{(p)(q)}{}_{(r)} &= a_1(\epsilon^{(i)}{}_{(p)(i)}\delta^{(q)}{}_{(r)} + \epsilon^{(i)}{}_{(q)(i)}\delta^{(p)}{}_{(r)}) + \frac{a_2}{2}(\epsilon^{(p)}{}_{(i)(i)}\delta^{(q)}{}_{(r)} \\
&\quad + 2\epsilon^{(i)}{}_{(r)(i)}\delta^{(q)}{}_{(p)} + \epsilon^{(q)}{}_{(i)(i)}\delta^{(p)}{}_{(r)}) + 2a_3\epsilon^{(r)}{}_{(i)(i)}\delta^{(p)}{}_{(q)} \\
&\quad + 2a_4\epsilon^{(r)}{}_{(q)(p)} + a_5(\epsilon^{(p)}{}_{(r)(q)} + \epsilon^{(q)}{}_{(r)(p)}), \qquad (3.26b) \\
\tau^{(p)(q)(r)}{}_{(s)} &= \frac{2}{3}b_1\epsilon^{(j)}{}_{(i)(i)(j)}(\delta^{(p)}{}_{(q)}\delta^{(r)}{}_{(s)} + \delta^{(p)}{}_{(r)}\delta^{(q)}{}_{(s)} + \delta^{(q)}{}_{(r)}\delta^{(p)}{}_{(s)}) \\
&\quad + \frac{2}{3}b_2\epsilon^{(i)}{}_{(j)(k)(i)}(\delta^{(j)}{}_{(p)}\delta^{(k)}{}_{(q)}\delta^{(r)}{}_{(s)} + \delta^{(j)}{}_{(p)}\delta^{(k)}{}_{(r)}\delta^{(q)}{}_{(s)} \\
&\quad + \delta^{(j)}{}_{(q)}\delta^{(k)}{}_{(r)}\delta^{(p)}{}_{(s)}) + \frac{1}{6}b_3(\epsilon^{(k)}{}_{(i)(i)(j)}(\delta^{(j)}{}_{(p)}\delta^{(k)}{}_{(q)}\delta^{(r)}{}_{(s)} \\
&\quad + \delta^{(j)}{}_{(p)}\delta^{(k)}{}_{(r)}\delta^{(q)}{}_{(s)} + \delta^{(j)}{}_{(q)}\delta^{(k)}{}_{(r)}\delta^{(p)}{}_{(s)}) + \epsilon^{(j)}{}_{(i)(i)(k)}(\delta^{(j)}{}_{(p)}\delta^{(k)}{}_{(q)}\delta^{(r)}{}_{(s)} \\
&\quad + \delta^{(j)}{}_{(p)}\delta^{(k)}{}_{(r)}\delta^{(q)}{}_{(s)} + \delta^{(j)}{}_{(q)}\delta^{(k)}{}_{(r)}\delta^{(p)}{}_{(s)}) + 2\epsilon^{(i)}{}_{(j)(s)(i)}(\delta^{(j)}{}_{(p)}\delta^{(q)}{}_{(r)} \\
&\quad + \delta^{(j)}{}_{(q)}\delta^{(p)}{}_{(r)} + \delta^{(j)}{}_{(r)}\delta^{(p)}{}_{(q)})) + \frac{2}{3}b_4\epsilon^{(j)}{}_{(i)(i)(s)}(\delta^{(j)}{}_{(p)}\delta^{(q)}{}_{(r)} \\
&\quad + \delta^{(j)}{}_{(q)}\delta^{(p)}{}_{(r)} + \delta^{(j)}{}_{(r)}\delta^{(p)}{}_{(q)}) + \frac{2}{3}b_5\epsilon^{(s)}{}_{(i)(i)(j)}(\delta^{(j)}{}_{(p)}\delta^{(q)}{}_{(r)} + \delta^{(j)}{}_{(q)}\delta^{(p)}{}_{(r)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \delta^{(j)}{}_{(r)}\delta^{(p)}{}_{(q)}) + 2b_6\epsilon^{(s)}{}_{(p)(q)(r)} + \frac{2}{3}b_7(\epsilon^{(p)}{}_{(q)(r)(s)} + \epsilon^{(q)}{}_{(r)(s)(p)} \\
&+ \epsilon^{(r)}{}_{(s)(p)(q)}) + \frac{1}{3}c_1\epsilon^{(i)}{}_{(i)}(\delta^{(p)}{}_{(q)}\delta^{(r)}{}_{(s)} + \delta^{(p)}{}_{(r)}\delta^{(q)}{}_{(s)} + \delta^{(p)}{}_{(s)}\delta^{(q)}{}_{(r)}) \\
&+ \frac{1}{3}c_2\epsilon^{(i)}{}_{(j)}(\delta^{(i)}{}_{(p)}\delta^{(j)}{}_{(q)}\delta^{(r)}{}_{(s)} + \delta^{(i)}{}_{(p)}\delta^{(j)}{}_{(r)}\delta^{(q)}{}_{(s)} + \delta^{(i)}{}_{(q)}\delta^{(j)}{}_{(r)}\delta^{(p)}{}_{(s)}) \\
&+ \frac{1}{3}c_3\epsilon^{(i)}{}_{(s)}(\delta^{(i)}{}_{(p)}\delta^{(q)}{}_{(r)} + \delta^{(i)}{}_{(q)}\delta^{(p)}{}_{(r)} + \delta^{(i)}{}_{(r)}\delta^{(p)}{}_{(q)}) \\
&+ \frac{1}{3}b_0(\delta^{(p)}{}_{(q)}\delta^{(r)}{}_{(s)} + \delta^{(p)}{}_{(r)}\delta^{(q)}{}_{(s)} + \delta^{(p)}{}_{(s)}\delta^{(q)}{}_{(r)}). \qquad (3.26c)
\end{aligned}
$$

## 4. Second strain gradient theory in spherical coordinates

In the spherical coordinate system shown in Fig. 1, the independent curvilinear variables $x^1 = r$, $x^2 = \theta$, and $x^3 = \phi$ are related to the Cartesian coordinates as

$$
\begin{aligned}
x_1 &= r\sin\phi\cos\theta, \\
x_2 &= r\sin\phi\sin\theta, \\
x_3 &= r\cos\phi.
\end{aligned}
$$

The corresponding base vectors are obtained via $\boldsymbol{g}_i = \partial\boldsymbol{r}/\partial x^i$, $i = 1, 2, 3$. Thus, letting $\boldsymbol{g}_r \equiv \boldsymbol{g}_1$, $\boldsymbol{g}_\theta \equiv \boldsymbol{g}_2$, and $\boldsymbol{g}_\phi \equiv \boldsymbol{g}_3$ we obtain

$$
\begin{aligned}
\boldsymbol{g}_r &= \sin\phi\cos\theta\boldsymbol{i}^1 + \sin\phi\sin\theta\boldsymbol{i}^2 + \cos\phi\boldsymbol{i}^3, \\
\boldsymbol{g}_\theta &= -r\sin\phi\sin\theta\boldsymbol{i}^1 + r\sin\phi\cos\theta\boldsymbol{i}^2, \\
\boldsymbol{g}_\phi &= r\cos\phi\cos\theta\boldsymbol{i}^1 + r\cos\phi\sin\theta\boldsymbol{i}^2 - r\sin\phi\boldsymbol{i}^3. \qquad (4.1)
\end{aligned}
$$

The unit base vectors are obtained by the normalization of the above base vectors as

$$
\boldsymbol{e}_r = \sin\phi\cos\theta\boldsymbol{i}^1 + \sin\phi\sin\theta\boldsymbol{i}^2 + \cos\phi\boldsymbol{i}^3,
$$

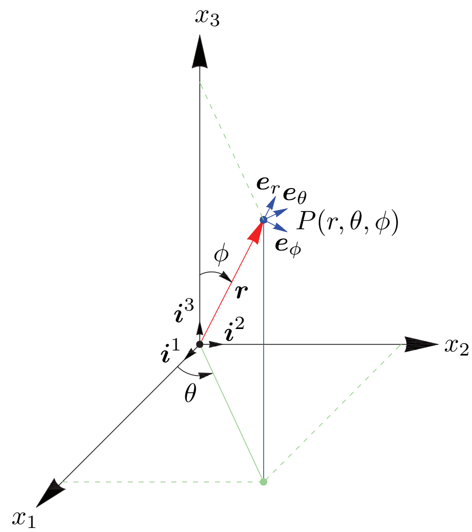


Figure 1: Geometrical representation of the Cartesian and spherical coordinate systems and the pertinent unit base vectors

$$\boldsymbol{e}_\theta = -\sin\theta\boldsymbol{i}^1 + \cos\theta\boldsymbol{i}^2,$$

$$\boldsymbol{e}_\phi = \cos\phi\cos\theta\boldsymbol{i}^1 + \cos\phi\sin\theta\boldsymbol{i}^2 - \sin\phi\boldsymbol{i}^3. \tag{4.2}$$

Since the base vectors given in Eqs. (4.1) are orthogonal, then the nonzero diagonal components of the pertinent metric tensor and their corresponding reciprocal components are as below

$$g_{rr} = 1, \qquad g_{\theta\theta} = r^2\sin^2\phi, \qquad g_{\phi\phi} = r^2, \tag{4.3a}$$

$$g^{rr} = 1, \qquad g^{\theta\theta} = \frac{1}{r^2\sin^2\phi}, \qquad g^{\phi\phi} = \frac{1}{r^2}. \tag{4.3b}$$

Moreover, the nonzero components of the Christoffel symbol of the second kind in spherical coordinates are as below

$$\begin{Bmatrix} r \\ \theta\theta \end{Bmatrix} = -r\sin^2\phi, \qquad \begin{Bmatrix} \phi \\ \theta\theta \end{Bmatrix} = -\sin\phi\cos\phi, \qquad \begin{Bmatrix} r \\ \phi\phi \end{Bmatrix} = -r,$$
$$\begin{Bmatrix} \theta \\ r\theta \end{Bmatrix} = \begin{Bmatrix} \theta \\ \theta r \end{Bmatrix} = \begin{Bmatrix} \phi \\ r\phi \end{Bmatrix} = \begin{Bmatrix} \phi \\ \phi r \end{Bmatrix} = \frac{1}{r}, \qquad \begin{Bmatrix} \theta \\ \theta\phi \end{Bmatrix} = \begin{Bmatrix} \theta \\ \phi\theta \end{Bmatrix} = \cot\phi. \tag{4.4}$$

**Remark 1.** In section 2, the physical components of a quantity within a curvilinear coordinate system were indicated by embracing the pertinent superscripts

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and subscripts by parentheses. In what follows, for convenience we drop out the parentheses and, subsequently, move the superscript to the subscript. For example, the physical strain component $\epsilon^{(i)}{}_{(j)}$ with respect to the spherical coordinates will be presented as $\epsilon_{rr}$, $\epsilon_{r\theta}$, $\epsilon_{r\phi}$, $\epsilon_{\theta\theta}$, $\epsilon_{\phi\phi}$, and $\epsilon_{\phi\theta}$.

Using Eqs. (3.16), the strain field and its first and second gradients in the spherical coordinates, after some manipulations, are derived in terms of the components of the displacement field ($u_r$, $u_\theta$, $u_\phi$) as below

$$\epsilon_{rr} = u_{r,r}, \qquad \epsilon_{r\theta} = \frac{1}{2r}\left(ru_{\theta,r} + u_{r,\theta}\csc\phi - u_\theta\right), \qquad \epsilon_{r\phi} = \frac{1}{2r}\left(u_{r,\phi} + ru_{\phi,r} - u_\phi\right),$$
$$\epsilon_{\theta\theta} = \frac{1}{r}\left(u_r + u_{\theta,\theta}\csc\phi + u_\phi\cot\phi\right), \qquad \epsilon_{\theta\phi} = \frac{1}{2r}\left(u_{\theta,\phi} + u_{\phi,\theta}\csc\phi - u_\theta\cot\phi\right),$$
$$\epsilon_{\phi\phi} = \frac{1}{r}\left(u_r + u_{\phi,\phi}\right), \tag{4.5a}$$

$$\epsilon_{rrr} = u_{r,rr}, \qquad \epsilon_{rr\theta} = u_{\theta,rr}, \qquad \epsilon_{rr\phi} = u_{\phi,rr},$$
$$\epsilon_{r\theta r} = \frac{1}{r^2}\left(u_\theta - u_{r,\theta}\csc\phi\right) + \frac{1}{r}\left(u_{r,r\theta}\csc\phi - u_{\theta,r}\right),$$
$$\epsilon_{r\theta\theta} = \frac{1}{r}\left(u_{r,r} + u_{\theta,r\theta}\csc\phi + u_{\phi,r}\cot\phi\right) - \frac{1}{r^2}\left(u_r + u_{\theta,\theta}\csc\phi + u_\phi\cot\phi\right),$$
$$\epsilon_{r\theta\phi} = \frac{1}{r}\left(u_{\phi,r\theta}\csc\phi - u_{\theta,r}\cot\phi\right) + \frac{1}{r^2}\left(u_\theta\cot\phi - u_{\phi,\theta}\csc\phi\right),$$
$$\epsilon_{r\phi r} = \frac{1}{r}\left(u_{r,r\phi} - u_{\phi,r}\right) + \frac{1}{r^2}\left(u_\phi - u_{r,\phi}\right), \qquad \epsilon_{r\phi\theta} = \frac{1}{r}u_{\theta,r\phi} - \frac{1}{r^2}u_{\theta,\phi},$$
$$\epsilon_{r\phi\phi} = \frac{1}{r}\left(u_{\phi,r\phi} + u_{r,r}\right) - \frac{1}{r^2}\left(u_r + u_{\phi,\phi}\right),$$
$$\epsilon_{\theta\theta r} = \frac{1}{r^2}\left(u_{r,\theta\theta}\csc^2\phi - 2u_{\theta,\theta}\csc\phi + u_{r,\phi}\cot\phi - 2u_\phi\cot\phi - u_r\right) + \frac{1}{r}u_{r,r},$$
$$\epsilon_{\theta\theta\theta} = \frac{1}{r^2}\left(u_{\theta,\theta\theta}\csc^2\phi + 2u_{\phi,\theta}\cot\phi\csc\phi + 2u_{r,\theta}\csc\phi + u_{\theta,\phi}\cot\phi - u_\theta\csc^2\phi\right) + \frac{1}{r}u_{\theta,r},$$
$$\epsilon_{\theta\theta\phi} = \frac{1}{r^2}\left(u_{\phi,\theta\theta}\csc^2\phi - 2u_{\theta,\theta}\cot\csc\phi + u_{\phi,\phi}\cot\phi - u_\phi\cot^2\phi\right) + \frac{1}{r}u_{\phi,r},$$
$$\epsilon_{\theta\phi r} = \frac{1}{r^2}\left(u_{r,\theta\phi}\csc\phi - u_{\phi,\theta}\csc\phi - u_{r,\theta}\csc\phi\cot\phi - u_{\theta,\phi} + u_\theta\cot\phi\right),$$
$$\epsilon_{\theta\phi\theta} = \frac{1}{r^2}\left(u_{r,\phi} + u_{\theta,\theta\phi}\csc\phi - u_{\theta,\theta}\cot\phi\csc\phi + u_{\phi,\phi}\cot\phi - u_\phi\csc^2\phi\right),$$

$$
\begin{aligned}
&\epsilon_{\theta\phi\phi} = \frac{1}{r^2}\left(u_{\phi,\theta\phi}\csc\phi - u_{\phi,\theta}\csc\phi\cot\phi + u_{r,\theta}\csc\phi - u_{\theta,\phi}\cot\phi + u_\theta\cot^2\phi\right),\\
&\epsilon_{\phi\phi r} = \frac{1}{r}u_{r,r} + \frac{1}{r^2}\left(u_{r,\phi\phi} - 2u_{\phi,\phi} - u_r\right), \qquad \epsilon_{\phi\phi\theta} = \frac{1}{r}u_{\theta,r} + \frac{1}{r^2}u_{\theta,\phi\phi},\\
&\epsilon_{\phi\phi\phi} = \frac{1}{r}u_{\phi,r} + \frac{1}{r^2}\left(u_{\phi,\phi\phi} + 2u_{r,\phi} - u_\phi\right), \qquad (4.5\text{b})\\
&\epsilon_{rrrr} = u_{r,rrr}, \qquad \epsilon_{rrr\theta} = u_{\theta,rrr}, \qquad \epsilon_{rrr\phi} = u_{\phi,rrr},\\
&\epsilon_{rr\theta r} = \frac{2}{r^3}\left(u_{r,\theta}\csc\phi - u_\theta\right) + \frac{2}{r^2}\left(u_{\theta,r} - u_{r,r\theta}\csc\phi\right) + \frac{1}{r}\left(u_{r,rr\theta}\csc\phi - u_{\theta,rr}\right),\\
&\epsilon_{rr\theta\theta} = \frac{2}{r^3}\left(u_r + u_{\theta,\theta}\csc\phi + u_\phi\cot\phi\right) - \frac{2}{r^2}\left(u_{r,r} + u_{\theta,r\theta}\csc\phi + u_{\phi,r}\cot\phi\right)\\
&+ \frac{1}{r}\left(u_{r,rr\theta} + u_{\theta,rr\theta}\csc\phi + u_{\phi,rr}\cot\phi\right)\\
&\epsilon_{rr\theta\phi} = \frac{2}{r^3}\left(u_{\phi,\theta}\csc\phi - u_\theta\cot\phi\right) - \frac{2}{r^2}\left(u_{\phi,r\theta}\csc\phi - u_{\theta,r}\cot\phi\right) - \frac{1}{r}\Big(u_{\theta,rr}\cot\phi\\
&- u_{\phi,rr\theta}\csc\phi\Big),\\
&\epsilon_{rr\phi r} = \frac{2}{r^3}\left(u_{r,\phi} - u_\phi\right) + \frac{2}{r^2}\left(u_{\phi,r} - u_{r,r\phi}\right) + \frac{1}{r}\left(u_{r,rr\phi} - u_{\phi,rr}\right),\\
&\epsilon_{rr\phi\theta} = \frac{2}{r^3}u_{\theta,\phi} - \frac{2}{r^2}u_{\theta,r\phi} + \frac{1}{r}u_{\theta,rr\phi},\\
&\epsilon_{rr\phi\phi} = \frac{2}{r^3}\left(u_r + u_{\phi,\phi}\right) - \frac{2}{r^2}\left(u_{r,r} + u_{\phi,r\phi}\right) + \frac{1}{r}\left(u_{r,rr} + u_{\phi,rr\phi}\right),\\
&\epsilon_{r\theta\theta r} = \frac{2}{r^3}\left(u_r + 2u_\phi\cot\phi - u_{r,\phi}\cot\phi + 2u_{\theta,\theta}\csc\phi - u_{r,\theta\theta}\csc^2\phi\right)\\
&+ \frac{1}{r^2}\left(u_{r,r\theta\theta}\csc^2\phi - 2u_{r,r} - 2u_{\theta,r\theta}\csc\phi - 2u_{\phi,r}\cot\phi + u_{r,r\phi}\cot\phi\right) + \frac{1}{r}u_{r,rr},\\
&\epsilon_{r\theta\theta\theta} = \frac{2}{r^3}\left(u_\theta\csc^2\phi - u_{\theta,\phi}\cot\phi - 2u_{r,\theta}\csc\phi - 2u_{\phi,\theta}\csc\phi\cot\phi - u_{\theta,\theta\theta}\csc^2\phi\right)\\
&+ \frac{1}{r^2}\left(u_{\theta,r\theta\theta}\csc^2\phi - 2u_{\theta,r} + 2u_{r,r\theta}\csc\phi + 2u_{\phi,r\theta}\cot\phi\csc\phi + u_{\theta,r\phi}\cot\phi - u_{\theta,r}\cot^2\phi\right)\\
&+ \frac{1}{r}u_{\theta,rr},\\
&\epsilon_{r\theta\theta\phi} = \frac{2}{r^3}\left(u_\phi\cot^2\phi - u_{\phi,\phi}\cot\phi + 2u_{\theta,\theta}\cot\phi\csc\phi - u_{\phi,\theta\theta}\csc^2\phi\right)\\
&+ \frac{1}{r^2}\left(u_{\phi,r\theta\theta}\csc^2\phi - 2u_{\theta,r\theta}\cot\phi\csc\phi + u_{\phi,r\phi}\cot\phi - u_{\phi,r}\cot^2\phi - u_{\phi,r}\right) + \frac{1}{r}u_{\phi,rr},\\
&\epsilon_{r\theta\phi r} = \frac{2}{r^3}\left(-u_{r,\theta\phi}\csc\phi + u_{\phi,\theta}\csc\phi + u_{r,\theta}\cot\phi\csc\phi + u_{\theta,\phi} - u_\theta\cot\phi\right)
\end{aligned}
$$

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$$+\frac{1}{r^2}\left(u_{\theta,r}\cot\phi-u_{\theta,r\phi}-u_{r,r\theta}\cot\phi\csc\phi-u_{\phi,r\theta}\csc\phi+u_{r,r\theta\phi}\csc\phi\right),$$

$$\epsilon_{r\theta\phi\theta}=\frac{2}{r^3}\left(u_\phi\csc^2\phi-u_{r,\phi}-u_{\phi,\phi}\cot\phi+u_{\theta,\theta}\cot\phi\csc\phi-u_{\theta,\theta\phi}\csc\phi\right)$$

$$+\frac{1}{r^2}\left(-u_{\phi,r}\csc^2\phi+u_{r,r\phi}+u_{\phi,r\phi}\cot\phi-u_{\theta,r\theta}\cot\phi\csc\phi+u_{\theta,r\theta\phi}\csc\phi\right),$$

$$\epsilon_{r\theta\phi\phi}=\frac{2}{r^3}\left(-u_\theta\cot^2\phi+u_{\theta,\phi}\cot\phi-u_{r,\theta}\csc\phi+u_{\phi,\theta}\cot\phi\csc\phi-u_{\phi,\theta\phi}\csc\phi\right)$$

$$+\frac{1}{r^2}\left(u_{\theta,r}\cot^2\phi-u_{\theta,r\phi}\cot\phi+u_{r,r\theta}\csc\phi-u_{\phi,r\theta}\cot\phi\csc\phi+u_{\phi,r\theta\phi}\csc\phi\right),$$

$$\epsilon_{r\phi\phi r}=\frac{2}{r^3}\left(u_r+2u_{\phi,\phi}-u_{r,\phi\phi}\right)-\frac{1}{r^2}\left(2u_{r,r}+2u_{\phi,r\phi}-u_{r,r\phi\phi}\right)+\frac{1}{r}u_{r,rr},$$

$$\epsilon_{r\phi\phi\theta}=-\frac{2}{r^3}u_{\theta,\phi\phi}+\frac{1}{r^2}\left(u_{\theta,r\phi\phi}-u_{\theta,r}\right)+\frac{1}{r}u_{\theta,rr},$$

$$\epsilon_{r\phi\phi\phi}=\frac{2}{r^3}\left(u_\phi-2u_{r,\phi}-u_{\phi,\phi\phi}\right)-\frac{2}{r^2}\left(2u_{\phi,r}-2u_{r,r\phi}-u_{\phi,r\phi\phi}\right)+\frac{1}{r}u_{\phi,rr},$$

$$\epsilon_{\theta\theta r}=\frac{1}{r^3}\big(3u_\theta\csc^2\phi-3u_{\theta,\phi}\cot\phi-5u_{r,\theta}\csc\phi-2u_{r,\theta}\cot^2\phi\csc\phi-6u_{\phi,\theta}\cot\phi\csc\phi$$

$$+3u_{r,\theta\phi}\cot\phi\csc\phi-3u_{\theta,\theta\theta}\csc^2\phi+u_{r,\theta\theta\theta}\csc^3\phi\big)+\frac{3}{r^2}\left(u_{r,r\theta}\csc\phi-u_{\theta,r}\right),$$

$$\epsilon_{\theta\theta\theta}=\frac{1}{r^3}\Big(-3u_r+\frac{3}{4}\left(-5\cos\phi+\cos3\phi\right)\csc^3\phi u_\phi+3u_{r,\phi}\cot\phi+3u_{\phi,\phi}\cot^2\phi$$

$$-5u_{\theta,\theta}\csc^3\phi+3u_{\theta,\theta\phi}\cot\phi\csc\phi+3u_{r,\theta\theta}\csc^2\phi+3u_{\phi,\theta\theta}\cot\phi\csc^2\phi+u_{\theta,\theta\theta\theta}\csc^3\phi\Big)$$

$$+\frac{3}{r^2}\left(u_{r,r}+u_{\phi,r}\cot\phi+u_{\theta,r\theta}\csc\phi\right),$$

$$\epsilon_{\theta\theta\theta\phi}=\frac{1}{r^3}\Big(3u_\theta\cot\phi\csc^2\phi-3u_{\theta,\phi}\cot^2\phi-2u_{\phi,\theta}\csc\phi-5u_{\phi,\theta}\cot^2\phi\csc\phi$$

$$+3u_{\phi,\theta\phi}\cot\phi\csc\phi-3u_{\theta,\theta\theta}\cot\phi\csc^2\phi+u_{\phi,\theta\theta\theta}\csc^3\phi\Big)+\frac{3}{r^2}\left(u_{\phi,r\theta}\csc\phi-u_{\theta,r}\cot\phi\right),$$

$$\epsilon_{\theta\theta\phi r}=\frac{1}{r^3}\Big(\left(2+3\cot^2\phi\right)u_\phi-\left(2+\cot^2\phi\right)u_{r,\phi}-3u_{\phi,\phi}\cot\phi+u_{r,\phi\phi}\cot\phi$$

$$+4u_{\theta,\theta}\cot\phi\csc\phi-2u_{\theta,\theta\phi}\csc\phi-2u_{r,\theta\theta}\cot\phi\csc^2\phi-u_{\phi,\theta\theta}\csc^2\phi+u_{r,\theta\theta\phi}\csc^2\phi\Big)$$

$$+\frac{1}{r^2}\left(u_{r,r\phi}-u_{\phi,r}\right),$$

$$\epsilon_{\theta\theta\phi\theta}=\frac{1}{r^3}\Big(2\cot\phi\csc^2\phi u_\theta-2u_{\theta,\phi}\csc^2\phi+u_{\theta,\phi\phi}\cot\phi-2u_{r,\theta}\cot\phi\csc\phi$$

$$-2u_{\phi,\theta}\csc\phi\cot^2\phi-2u_{\phi,\theta}\csc^3\phi+2\csc\phi u_{r,\theta\phi}+2\cot\phi\csc\phi u_{\phi,\theta\phi}$$

$$- 2\cot\phi\csc^2\phi u_{\theta,\theta\theta} + \csc^2\phi u_{\theta,\theta\theta\phi}\Big) + \frac{1}{r^2}u_{\theta,r\phi},$$

$$\epsilon_{\theta\theta\phi\phi} = \frac{1}{r^3}\Big( - u_r + 2u_\phi\cot^3\phi + u_{r,\phi}\cot\phi - u_{\phi,\phi} - 2u_{\phi,\phi}\cot^2\phi + u_{\phi,\phi\phi}\cot\phi$$

$$+ 4u_{\theta,\theta}\cot^2\phi\csc\phi - 2u_{\theta,\theta\phi}\cot\phi\csc\phi + u_{r,\theta\theta}\csc^2\phi$$

$$- 2u_{\phi,\theta\theta}\cot\phi\csc^2\phi + u_{\phi,\theta\theta\phi}\csc^2\phi\Big) + \frac{1}{r^2}\left(u_{r,r} + u_{\phi,r\phi}\right),$$

$$\epsilon_{\theta\phi\phi r} = \frac{1}{r^3}\Big( - 2\cot^2\phi u_\theta + 2u_{\theta,\phi}\cot\phi - u_{\theta,\phi\phi} - 2u_{r,\theta}\csc\phi + u_{r,\theta}\csc\phi\cot^2\phi$$

$$+ u_{r,\theta}\csc^3\phi + 2u_{\phi,\theta}\cot\phi\csc\phi - 2u_{r,\theta\phi}\cot\phi\csc\phi - 2u_{\phi,\theta\phi}\csc\phi$$

$$+ u_{r,\theta\phi\phi}\csc\phi + \frac{1}{r^2}\left(u_{r,r\theta}\csc\phi - u_{\theta,r}\right),$$

$$\epsilon_{\theta\phi\phi\theta} = \frac{1}{r^3}\Big( - u_r + \cot\phi(2\csc^2\phi - 1)u_\phi - 2u_{\phi,\phi}\csc^2\phi + u_{r,\phi\phi} + u_{\phi,\phi\phi}\cot\phi$$

$$+ 2u_{\theta,\theta}\cot^2\phi\csc\phi - 2\cot\phi\csc\phi u_{\theta,\theta\phi} + u_{\theta,\theta\phi\phi}\csc\phi\Big) + \frac{1}{r^2}\big(u_{r,r} + u_{\phi,r}\cot\phi$$

$$+ u_{\theta,r\theta}\csc\phi\big),$$

$$\epsilon_{\theta\phi\phi\phi} = \frac{1}{r^3}\Big( - 2\cot^3\phi u_\theta + 2u_{\theta,\phi}\cot^2\phi - u_{\theta,\phi\phi}\cot\phi - 2u_{r,\theta}\cot\phi\csc\phi - 2u_{\phi,\theta}\csc\phi$$

$$+ u_{\phi,\theta}\csc\phi\cot^2\phi + u_{\phi,\theta}\csc^3\phi + 2u_{r,\theta\phi}\csc\phi - 2u_{\phi,\theta\phi}\cot\phi\csc\phi$$

$$+ u_{\phi,\theta\phi\phi}\csc\phi\Big) + \frac{1}{r^2}\left(u_{\phi,r\theta}\csc\phi - u_{\theta,r}\cot\phi\right),$$

$$\epsilon_{\phi\phi\phi r} = \frac{1}{r^3}\left(3u_\phi - 5u_{r,\phi} - 3u_{\phi,\phi\phi} + u_{r,\phi\phi\phi}\right) + \frac{3}{r^2}\left(u_{r,r\phi} - u_{\phi,r}\right),$$

$$\epsilon_{\phi\phi\phi\theta} = \frac{1}{r^3}\left(u_{\theta,\phi\phi\phi} - 2u_{\theta,\phi}\right) + \frac{3}{r^2}u_{\theta,r\phi},$$

$$\epsilon_{\phi\phi\phi\phi} = \frac{1}{r^3}\left(-3u_r - 5u_{\phi,\phi} + 3u_{r,\phi\phi} + u_{\phi,\phi\phi\phi}\right) + \frac{3}{r^2}\left(u_{r,r} + u_{\phi,r\phi}\right). \tag{4.5c}$$

Finally, the set of equilibrium equations in spherical coordinate system is derived as follows

$$\bar{\tau}_{rr,r} + \frac{1}{r\sin\phi}\bar{\tau}_{\theta r,\theta} + \frac{1}{r}\bar{\tau}_{\phi r,\phi} + \frac{1}{r}\left(2\bar{\tau}_{rr} + \bar{\tau}_{\phi r}\cot\phi - \bar{\tau}_{\theta\theta} - \bar{\tau}_{\phi\phi}\right) + f_r = 0, \tag{4.6a}$$

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$$\bar{\tau}_{r\theta,r} + \frac{1}{r\sin\phi}\bar{\tau}_{\theta\theta,\theta} + \frac{1}{r}\bar{\tau}_{\phi\theta,\phi} + \frac{1}{r}\left(2\bar{\tau}_{r\theta} + \bar{\tau}_{\theta r} + (\bar{\tau}_{\theta\phi} + \bar{\tau}_{\phi\theta})\cot\phi\right) + f_\theta = 0, \tag{4.6b}$$

$$\bar{\tau}_{r\phi,r} + \frac{1}{r\sin\phi}\bar{\tau}_{\theta\phi,\theta} + \frac{1}{r}\bar{\tau}_{\phi\phi,\phi} + \frac{1}{r}\left(2\bar{\tau}_{r\phi} + \bar{\tau}_{\phi r} + (\bar{\tau}_{\phi\phi} - \bar{\tau}_{\theta\theta})\cot\phi\right) + f_\phi = 0, \tag{4.6c}$$

where the expressions for $\bar{\tau}_{ij}$, $i$, $j = r$, $\theta$, $\phi$ are given in Appendix A.

## 5. Application to problems involving Spherical symmetry

As a three-dimensional useful application, consider a spherical surface of radius $a$ bounding an isotropic body. In the absence of any type of external loading and under the assumption of central symmetry, it is expected that due to the effect of surface, the components of the displacement field along $\theta$ and $\phi$ be zero, $u_\theta = u_\phi = 0$ and hence

$$\boldsymbol{u} = (u_r(r),\, 0,\, 0). \tag{5.1}$$

Based on the formulations given in Section 4 and using the relations (4.6), the only nontrivial equilibrium equation governing the body has the following form

$$r^2\ell_{11}^2\ell_{12}^2 u_r'''''' + 6r\ell_{11}^2\ell_{12}^2 u_r''''' - (r^2(\ell_{11}^2 + \ell_{12}^2) + 6\ell_{11}^2\ell_{12}^2)u_r'''' - 4r(\ell_{11}^2 + \ell_{12}^2)u_r'''$$
$$+ (4(\ell_{11}^2 + \ell_{12}^2) + r^2)u_r'' + 2ru_r' - 2u_r = 0. \tag{5.2}$$

By defining the new parameters $p_1$ and $p_2$ as

$$p_1 = \frac{\ell_{11}^2 + \ell_{12}^2}{\ell_{11}^2\ell_{12}^2}, \qquad p_2 = \frac{1}{\ell_{11}^2\ell_{12}^2}, \tag{5.3}$$

the equilibrium equation may be rewritten as

$$r^6 u_r'''''' + 6r^5 u_r''''' - (6r^4 + p_1 r^6)u_r'''' - 4r^5 p_1 u_r''' + (4p_1 r^4 + p_2 r^6)u_r'' + 2p_2 r^5 u_r'$$

$$- 2p_2 r^4 u_r = 0. \tag{5.4}$$

In the above equation, $r = 0$ is a regular singular point. Its solution may be obtained by employing the Frobenius series

$$u_r = \sum_{n=0}^{\infty} a_n r^{n+s}. \tag{5.5}$$

Its substitution into Eq. (5.4) yields

$$\begin{aligned}&\sum_{n=0}^{\infty}(n+s)(n+s-1)(n+s-2)(n+s-3)(n+s-5)(n+s+2)a_n r^{n+s}\\&- p_1\sum_{n=2}^{\infty}(n+s-2)(n+s-3)(n+s-5)(n+s)a_{n-2}r^{n+s}\\&+ p_2\sum_{n=4}^{\infty}(n+s-5)(n+s-2)a_{n-4}r^{n+s} = 0.\end{aligned} \tag{5.6}$$

The indicial polynomial, $P(s)$ corresponding to $n = 0$ becomes

$$P(s) = s(s-1)(s-2)(s-3)(s-5)(s+2). \tag{5.7}$$

By the assumption $a_0 \neq 0$, $s$ is required to be a root of the indicial polynomial. It is observed that $P(s)$ has six integer roots as $s = -2,\ 0,\ 1,\ 2,\ 3,$ and $5$. In Eq. (5.6), by equating the coefficients of $r^{n+s}$ for $n \geq 1$, the following relations are resulted

$$n = 1: \qquad a_1(s+1)s(s-1)(s-2)(s-4)(s+3) = 0, \tag{5.8a}$$

$$n = 2: \qquad s(s-1)(s-3)(s+2)\left[(s+1)(s+4)a_2 - p_1 a_0\right] = 0, \tag{5.8b}$$

$$n = 3: \qquad s(s+1)(s-2)(s+3)\left[(s+2)(s+5)a_3 - p_1 a_1\right] = 0, \tag{5.8c}$$

$$\begin{aligned}n \geq 4: \qquad &a_n P(n+s) - p_1 a_{n-2}(n+s-2)(n+s-3)(n+s-5)(n+s)\\&+ p_2 a_{n-4}(n+s-5)(n+s-2) = 0.\end{aligned} \tag{5.8d}$$

A close scrutiny of the above relations reveals that by considering the roots, $s = 2, -2, 0$ the complete list of the independent solutions will be obtained. First of all by considering $s = 2$ relations (5.8a)-(5.8d) result in $a_1 \neq 0$, $a_2 = \dfrac{p_1 a_0}{3*6}$, $a_3 \neq 0$, and

$$a_{2n} = \frac{-2a_0}{p_2(n+1)(n+2)(2n+1)!} \sum_{k=1}^{\frac{n+2}{2}} (-1)^k p_1^{n+2-2k} p_2^k \binom{n+1-k}{k-1}, \quad n \geq 2 \tag{5.9a}$$

$$\begin{aligned} a_{2n+1} = &-\frac{1680a_3}{p_2} \frac{n+2}{(2n+5)!} \sum_{k=1}^{\frac{n+1}{2}} (-1)^k p_1^{n+1-2k} p_2^k \binom{n-k}{k-1} \\ &+ \frac{60a_1(n+2)}{(2n+5)!} \sum_{k=1}^{n/2} (-1)^k p_1^{n-2k} p_2^k \binom{n-1-k}{k-1}, \quad n \geq 2. \end{aligned} \tag{5.9b}$$

Subsequently, employing the relation (5.5), the corresponding solution reduces to

$$\begin{aligned} u_r = &-30(\ell_{11}^2 + \ell_{12}^2)a_1 \left( \frac{u_{r_1}}{3} - \frac{\ell_{11}^6}{\ell_{11}^4 - \ell_{12}^4} u_{r_2} + \frac{\ell_{12}^6}{\ell_{11}^4 - \ell_{12}^4} u_{r_3} \right) \\ &+ \frac{840a_3}{p_2} \left( \frac{u_{r_1}}{3} - \frac{\ell_{11}^4}{\ell_{11}^2 - \ell_{12}^2} u_{r_2} + \frac{\ell_{12}^4}{\ell_{11}^2 - \ell_{12}^2} u_{r_3} \right) \\ &- \frac{2a_0}{p_2} \left( 4u_{r_4} + \frac{4\ell_{11}}{\ell_{11}^2 - \ell_{12}^2} u_{r_5} - \frac{4\ell_{11}}{\ell_{11}^2 - \ell_{12}^2} u_{r_2} - \frac{4\ell_{12}}{\ell_{11}^2 - \ell_{12}^2} u_{r_6} + \frac{4\ell_{12}}{\ell_{11}^2 - \ell_{12}^2} u_{r_3} \right), \end{aligned} \tag{5.10}$$

where $a_0$, $a_1$, and $a_3$ are arbitrary and

$$u_{r_1} = r, \tag{5.11a}$$

$$u_{r_2} = \frac{\cosh \frac{r}{\ell_{11}}}{r} - \frac{\ell_{11} \sinh \frac{r}{\ell_{11}}}{r^2}, \tag{5.11b}$$

$$u_{r_3} = \frac{\cosh \frac{r}{\ell_{12}}}{r} - \frac{\ell_{12} \sinh \frac{r}{\ell_{12}}}{r^2}, \tag{5.11c}$$

$$u_{r_4} = \frac{1}{r^2}, \tag{5.11d}$$

$$u_{r_5} = \frac{(\ell_{11}+r)e^{-\frac{r}{\ell_{11}}}}{r^2}, \tag{5.11e}$$

$$u_{r_6} = \frac{(\ell_{12}+r)e^{-\frac{r}{\ell_{12}}}}{r^2}. \tag{5.11f}$$

Likewise, for $s=-2$ it can be shown that

$$\begin{aligned} u_r = a_0 &\left(-3u_{r_4} + \frac{4\ell_{11}}{\ell_{11}^2-\ell_{12}^2}u_{r_5} - \frac{4\ell_{11}}{\ell_{11}^2-\ell_{12}^2}u_{r_2} - \frac{4\ell_{12}}{\ell_{11}^2-\ell_{12}^2}u_{r_6} + \frac{4\ell_{12}}{\ell_{11}^2-\ell_{12}^2}u_{r_3}\right) \\ &+ \frac{2a_2\ell_{11}\ell_{12}}{\ell_{11}^2-\ell_{12}^2}\left(\ell_{12}u_{r_5} - \ell_{12}u_{r_2} - \ell_{11}u_{r_6} + \ell_{11}u_{r_3}\right) - \frac{6a_3}{2p_2(\ell_{11}^2-\ell_{12}^2)}\left(u_{r_2}-u_{r_3}\right), \end{aligned} \tag{5.12}$$

for arbitrary $a_0$, $a_2$, and $a_3$. Finally, if one assumes $s=0$, the series solution will collapse to the following form

$$\begin{aligned} u_r = a_0 &\left(2(\ell_{11}^2+\ell_{12}^2)u_{r_4} - \frac{2}{\ell_{11}^2-\ell_{12}^2}\left(\ell_{11}^3u_{r_5} - \ell_{11}^3u_{r_2} - \ell_{12}^3u_{r_6} + \ell_{12}^3u_{r_3}\right)\right) \\ &+ a_2\left(4u_{r_4} + \frac{4\ell_{11}}{\ell_{11}^2-\ell_{12}^2}u_{r_5} - \frac{4\ell_{11}}{\ell_{11}^2-\ell_{12}^2}u_{r_2} - \frac{4\ell_{12}}{\ell_{11}^2-\ell_{12}^2}u_{r_6} + \frac{4\ell_{12}}{\ell_{11}^2-\ell_{12}^2}u_{r_3}\right) \\ &+ \frac{3a_1}{\ell_{11}^2-\ell_{12}^2}\left(\ell_{11}^4u_{r_2} - \ell_{12}^4u_{r_3}\right) + \frac{60a_3}{2p_2(\ell_{11}^2-\ell_{12}^2)}\left(\ell_{11}^2u_{r_2} - \ell_{12}^2u_{r_3}\right), \end{aligned} \tag{5.13}$$

with arbitrary $a_0$, $a_1$, $a_2$ and $a_3$. In view of Eqs. (5.10), (5.12), and (5.13) it is concluded that the solution to Eq. (5.4) associated with spherical geometry problem having central symmetry can be represented as

$$u_r = \mathscr{L}\{u_{r_1}, u_{r_2}, u_{r_3}, u_{r_4}, u_{r_5}, u_{r_6}\}. \tag{5.14}$$

Next, the traction boundary conditions associated with the proposed spherical problem are extracted from the general treatment given in the earlier sections in curvilinear coordinates. Recall that the spherical body is traction free and deforms just under the surface effect. The nontrivial components of the tractions in second strain gradient theory for this special problem which are derived using the relations (4.5), (3.26), (3.25), (3.22), and (3.19) must be equalized to zero on the spherical free surface ($r = a$) as below

$$\begin{aligned}\overset{1}{t}_r &= -\frac{2b_0}{r^2} + \left(\frac{2\lambda}{r} - \frac{4(a_4 + a_5 + \bar{a} - 2c_2 - 2c_3)}{r^3} - \frac{8(\bar{b} - 2b_1 + 4b_6 + 4b_7)}{r^5}\right) u_r \\ &+ \left(\lambda + 2\mu + \frac{4(a_4 + a_5 + \bar{a} + c_1) - 10\bar{c}}{r^2} + \frac{8(\bar{b} - 2b_1 + 4b_6 + 4b_7)}{r^4}\right) u_{r,r} \\ &+ \left(\frac{4\bar{c} - 2\bar{a} + 2c_1}{r} + \frac{8(b_2 + b_3 + b_4 + b_5)}{r^3}\right) u_{r,rr} + \left(2\bar{c} - \bar{a} - \frac{10\bar{b}}{r^2}\right) u_{r,rrr} \\ &+ \frac{4\bar{b}u_{r,rrrr}}{r} + \bar{b}u_{r,rrrrr} = 0, \quad \text{on} \quad r = a, \end{aligned} \tag{5.15a}$$

$$\begin{aligned}\overset{2}{t}_r &= \frac{2b_0}{r} + \left(\frac{4(a_1 + a_2 + a_3) + 4c_1 + 2\bar{c}}{r^2} + \frac{8(\bar{b} - 2b_1 + 4b_6 + 4b_7)}{r^4}\right) u_r \\ &+ \left(\frac{4(a_1 + a_2 + a_3)}{r} - \frac{8(\bar{b} - 2b_1 + 4b_6 + 4b_7)}{r^3}\right) u_{r,r} \\ &+ \left(\bar{a} - \bar{c} + \frac{12\bar{b} - 8(b_2 + b_3 + b_4 + b_5)}{r^2}\right) u_{r,rr} \\ &- \frac{2\bar{b}}{r} u_{r,rrr} - \bar{b}u_{r,rrrr} = 0, \quad \text{on} \quad r = a, \end{aligned} \tag{5.15b}$$

$$\begin{aligned}\overset{3}{t}_r &= b_0 + \left(\frac{2c_1}{r} + \frac{8(b_2 + b_3 + b_4 + b_5)}{r^3}\right) u_r + \left(\bar{c} - \frac{8(b_2 + b_3 + b_4 + b_5)}{r^2}\right) u_{r,r} \\ &+ \left(\frac{2\bar{b} + 4b_1 - 4b_6 - 4b_7}{r}\right) u_{r,rr} + \bar{b}u_{r,rrr} = 0 \quad \text{on} \quad r = a. \end{aligned} \tag{5.15c}$$

For illustration, the formulations for spherical geometries are specialized to an isotropic solid nanosphere in Section 5.1 and isotropic infinite body with spherical nanocavity in Section 5.2. The corresponding numerical examples will be given in Section 7.

### 5.1. Isotropic solid nanosphere

For an elastic isotropic spherical solid considered under the surface effect, the solution must be bounded at its center, Thus, from the solutions presented by Eqs. (5.11), the solution of the problem od interest may be written as

$$\begin{aligned} u_r &= A_1 u_{r_1} + A_2 u_{r_2} + A_3 u_{r_3} \\ &= A_1 r + A_2 \left( \frac{\cosh \dfrac{r}{\ell_{11}}}{r} - \frac{\ell_{11} \sinh \dfrac{r}{\ell_{11}}}{r^2} \right) + A_3 \left( \frac{\cosh \dfrac{r}{\ell_{12}}}{r} - \frac{\ell_{12} \sinh \dfrac{r}{\ell_{12}}}{r^2} \right), \end{aligned} \tag{5.16}$$

The unknown coefficients $A_1$ through $A_3$ are determined using the boundary conditions given by Eqs. (5.15).

### 5.2. Infinite isotropic domain with spherical nanocavity

For an infinite isotropic domain containing a spherical void under the surface effect, $u_{r_1}$, $u_{r_2}$, and $u_{r_3}$ given in Eqs. (5.11) are not suitable since they do not diminish as $r \to \infty$, and thus the displacement field has the following form

$$\begin{aligned} u_r &= A_1 u_{r_4} + A_2 u_{r_5} + A_3 u_{r_6} \\ &= \frac{A_1}{r^2} + A_2 \frac{(\ell_{11} + r)\, e^{-\dfrac{r}{\ell_{11}}}}{r^2} + A_3 \frac{(\ell_{12} + r)\, e^{-\dfrac{r}{\ell_{12}}}}{r^2}. \end{aligned} \tag{5.17}$$

Again, in a similar manner to the previous case, the unknown coefficients $A_1$ through $A_3$ are determined using the boundary conditions given by Eqs. (5.15).

## 6. Comparison to Gurtin-Murdoch surface elasticity solution

One can obtain nontrivial solutions for two problems of isotropic spherical solid and infinite isotropic domain containing a spherical void under the surface effect in the framework of Gurtin-Murdoch surface elasticity [9], as well.

According to Gurtin and Murdoch [9], in the absence of body forces, the system of governing equilibrium equations in the bulk volume $V$ of the solid can be simplified as below

$$\begin{cases} \tau_{pq,p} = 0, \\ \tau_{pq} = \lambda \epsilon_{ii} \delta_{pq} + 2\mu \epsilon_{pq}, \end{cases} \tag{6.1}$$

where $\epsilon_{ij}$ follows definition given by Eq. (2.2a). In Gurtin-Murdoch theory, these equations are coupled with the following governing equilibrium equations on the surface of the volume, $\partial V$

$$\begin{cases} \mathsf{I}_{jM} \mathsf{S}_{iM,j} = n_p \tau_{pi}, \\ \mathsf{E}_{MN} = \dfrac{1}{2} \left( P_{Mi}\, u_{i,j}\, \mathsf{I}_{jN} + P_{Mj}\, u_{i,j}\, \mathsf{I}_{iN} \right), \\ \mathsf{S}_{iM} = \sigma_0 \mathsf{I}_{iM} + \lambda_0 \mathsf{E}_{NN}\, \mathsf{I}_{iM} + 2\mu_0 \mathsf{I}_{iN}\, \mathsf{E}_{NM} + \sigma_0 u_{i,j}\, \mathsf{I}_{jM}. \end{cases} \tag{6.2}$$

In the above equations, upper-case indices belong to the two-dimensional subspace of the three-dimensional Euclidean space. Consider a three-dimensional vector space $\mathscr{V}$ and its two-dimensional subspace $\mathscr{T}_{\boldsymbol{x}}$ through which the structure of a surface $\mathfrak{s}$ at each $\boldsymbol{x} \in \mathfrak{s}$ is defined. According to the definition given by Gurtin and Murdoch [9], $\mathsf{I}_{iM}$ is an inclusion map that linearly transforms any vector in the two-dimensional subspace ($\mathscr{T}_{\boldsymbol{x}}$) to its corresponding vector in the three-dimensional space ($\mathscr{V}$), while $P_{Mi}$ is the perpendicular projection from $\mathscr{V}$ onto $\mathscr{T}_{\boldsymbol{x}}$. In the above relations, $\mathsf{E}_{MN}$ is the tangential surface strain tensor, $\mathsf{S}_{iM}$ is the first Piola-Kirchhoff surface stress tensor, $n_i$ is the outward unit normal vector of the surface $\mathfrak{s}$, and $\sigma_0$, $\lambda_0$, and $\mu_0$ are, respectively, residual surface tension and surface Lamé moduli for the isotropic surface $\mathfrak{s}$. It may be noteworthy to mention that $\mathsf{I}_{jM}\mathsf{S}_{iM,j}$ in the first of Eqs. (6.2) represents the surface divergence of the surface stress tensor $\mathsf{S}_{iM}$ as defined by Gurtin and Murdoch [9, 19].

For the problems involving spherical symmetry, as discussed in Section 5, the displacement field has the form (5.1). The displacement form of the governing equilibrium equation for the bulk is derived using Eq. (6.1)

$$r^2 u_{r,rr} + 2r u_{r,r} - 2u_r = 0, \tag{6.3}$$

with the general solution

$$u_r = A\,r + \frac{B}{r^2}. \tag{6.4}$$

The unknown constants $A$ and $B$ will be obtained using the surface boundary conditions given by Eqs. (6.2).

*6.1. Isotropic solid nanosphere*

In this case, since $r = 0$ is a field point it requires that $B = 0$, and the unknown $A$ is determined via the first relation of Eqs. (6.2) on $r = a$. Thus, the displacement field will take the following form

$$u_r = \frac{-2\sigma_0\, r}{2(2\lambda_0 + 2\mu_0 + \sigma_0) + (3\lambda + 2\mu)a}. \tag{6.5}$$

It is interesting to note that comparison of the above solution with that of Mindlin's second strain gradient theory given by (5.16) reveals that the two solutions share the linear term, but Mindlin's theory gives rise to two additional terms. The effect of these additional terms will be numerically demonstrated in Section 7.

### *6.2. Infinite isotropic domain with spherical nanocavity*

Since the displacement field induced by the surface due to the spherical nanocavity should diminish at infinity, then $A = 0$. By employing the relations (6.2) on $r = a$, the displacement field will have the following form

$$u_r = \frac{-\sigma_0\, a^3}{(2\mu a + 2\lambda_0 + 2\mu_0 + \sigma_0)\, r^2}. \tag{6.6}$$

As it is seen, in this case surface elasticity theory recovers only the first term of the solution obtained via Mindlin's second strain gradient theory. In fact, the Mindlin's solution (5.17) contains two additional terms which are absent in (6.6).

## 7. Numerical results

For the illustration of the current theoretical developments, surface relaxation of spherical domains are considered. More specifically, the displacement field of a nano-spherical medium as well as the displacement field within an infinite domain containing a spherical nanocavity is examined. To the end of comparison, the numerical results are given in the framework of both Mindlin's second strain gradient elasticity and Gurtin-Murdoch surface elasticity. The results are presented for some face-centered cubic crystals (fcc) as Ag, Au, and Pt. First, a brief explanation for the determination of the numerical values of the

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material parameters in second strain gradient elasticity and Gurtin-Murdoch surface elasticity is given in Section 7.1.

### *7.1. Evaluation of Mindlin's material parameters via lattice dynamics and ab-initio calculations*

In order to present the numeric solution of problems involving spherical symmetry through second strain gradient elasticity, one should first determine the pertinent numerical values of the components of the fourth, sixth, and eighth order elastic moduli tensors of the crystals of interest. To this end, a short introduction to the atomistic description of materials via lattice dynamics is given here.

Consider the bulk of a centrosymmetric crystal with perfect lattice of infinite extension in space and denote the position of an arbitrary primitive unit cell of volume $v$ within by the vector $\boldsymbol{x}$. Suppose that the distance between the $\alpha^{\text{th}}$ primitive unit cell from the reference unit cell at $\boldsymbol{x}$ is indicated as $\boldsymbol{R}_\alpha$. Moreover, let $K_{ij}^{\alpha}$ present the atomic force constant between the unit cells with location vectors $\boldsymbol{x}$ and $\boldsymbol{x}+\boldsymbol{R}_\alpha$.

For any perturbation of the atomistic configuration from the equilibrium, the potential energy density function pertinent to the one-atom unit cell at $\boldsymbol{x}$ to within a harmonic approximation may be expressed as

$$W=-\frac{1}{4v}\sum_{\alpha}K_{ij}^{\alpha}\left(u_i(\boldsymbol{x}+\boldsymbol{R}_\alpha)-u_i(\boldsymbol{x})\right)\left(u_j(\boldsymbol{x}+\boldsymbol{R}_\alpha)-u_j(\boldsymbol{x})\right). \tag{7.1}$$

By writing Taylor's expansion of $\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{R}_\alpha)$ about $\boldsymbol{x}$ and based on the fact that for centrosymmetric crystals the odd-ranked elastic moduli tensors are equal to zero, the potential energy density function may be written as

$$W = \frac{1}{2}\tilde{C}_{ijmn}u_{i,m}u_{j,n} + \tilde{C}_{ijmnpq}u_{i,m}u_{j,npq} + \frac{1}{2}\tilde{\tilde{C}}_{ijmnpq}u_{i,mn}u_{j,pq} + \frac{1}{2}\tilde{C}_{ijmntpqr}u_{i,mnt}u_{j,pqr}. \tag{7.2}$$

The coefficients $\tilde{\boldsymbol{C}}$ appearing in the above relation depend on the atomic force constants and the equilibrium positions of the atoms as follows

$$\tilde{C}_{ijmn} = -\frac{1}{2v}\sum_{\alpha} K_{ij}^{\alpha} R_{\alpha_m} R_{\alpha_n}, \tag{7.3a}$$

$$\tilde{C}_{ijmnpq} = -\frac{1}{12v}\sum_{\alpha} K_{ij}^{\alpha} R_{\alpha_m} R_{\alpha_n} R_{\alpha_p} R_{\alpha_q}, \tag{7.3b}$$

$$\tilde{C}_{ijmnpqrs} = -\frac{1}{72v}\sum_{\alpha} K_{ij}^{\alpha} R_{\alpha_m} R_{\alpha_n} R_{\alpha_p} R_{\alpha_q} R_{\alpha_r} R_{\alpha_s}, \tag{7.3c}$$

and $\tilde{\tilde{C}}_{ijmnpq} = \frac{3}{2}\tilde{C}_{ijmnpq}$. Employing the Hamilton's principle as described by Ojaghnezhad and Shodja [7] and Shodja et al. [20], the equations of motion for centrosymmetric crystals are obtained as

$$\rho\ddot{u}_i = \tilde{C}_{ijmn}u_{j,mn} + \tilde{C}_{ijmnpq}u_{j,mnpq} + \tilde{C}_{ijmnpqrs}u_{j,mnpqrs}, \tag{7.4}$$

in which $\rho$ is the ratio of the mass of the atom in one primitive unit cell to its volume, and "," in the subscript denotes differentiation with respect to $\boldsymbol{x}$.

The atomic force constants, $K_{ij}^{\alpha}$ are equivalent to the components of the Hessian matrix which are in turn equal to the value of the second derivative of the total potential energy with respect to the corresponding atomic positions at the equilibrium. The Hessian matrix is obtained from the first principles density functional theory (DFT) and, subsequently, the fourth, sixth, and eighth order constants given by relations (7.3) are evaluated. From comparison of Eqs. (2.7) and (7.4) in the absence of body forces $f_i$ the following relations are obtained

$$C_{imjn} = \tilde{C}_{ijmn} + \tilde{C}_{mjin} - \tilde{C}_{mijn}, \tag{7.5a}$$

$$\begin{aligned}\tilde{C}_{ijklmn} = &\frac{1}{4}\big(F_{jkmnli} + F_{jlmnki} + F_{jmknli} + F_{jnmkli} + F_{likmnj} + F_{kilmnj} + F_{miklnj} \\ &+ F_{nikmlj}\big) - \frac{1}{6}\big(G_{nlimkj} + G_{nkimlj} + G_{klimnj} + G_{nmilkj} + G_{mlinkj} \\ &+ G_{mkinlj}\big),\end{aligned} \tag{7.5b}$$

$$\begin{aligned}\tilde{C}_{ijklmnpq} = &\frac{1}{20}\big(I_{qklimnpj} + I_{qmliknpj} + I_{qnlimkpj} + I_{qplimnkj} + I_{qkmilnpj} + I_{qknimlpj} \\ &+ I_{qkpimnlj} + I_{mkliqnpj} + I_{nklimqpj} + I_{pklimnqj} + I_{qmniklpj} + I_{qmpiklnj} \\ &+ I_{qnpimklj} + I_{mkniqlpj} + I_{mkpiqnlj} + I_{nkpimqlj} + I_{mnliqkpj} + I_{mpliqnkj} \\ &+ I_{nplimqkj} + I_{mnpiqklj}\big).\end{aligned} \tag{7.5c}$$

Using the above relations, a set of equations for $\bar{a} - 2\bar{c}$, $\bar{a}' - c_3$, $\bar{b}$, $\bar{b}'$, $\lambda$, and $\mu$, pertinent to isotropic materials, in terms of the components of the tensors $\tilde{\boldsymbol{C}}$ is obtained. Subsequently, the bulk characteristic lengths are readily available via Eqs. (2.9).

To obtain the other additional parameters of Mindlin's theory, one may equalize the higher order terms of the strain energy density functions pertinent to the continuum model (2.5) and the pertinent lattice dynamics formulation (7.2) as below

$$\tilde{\tilde{C}}_{knijlm} u_{k,ij} u_{n,lm} = G_{ijklmn} u_{k,ij} u_{n,lm}, \tag{7.6a}$$

$$\tilde{C}_{injklm} u_{i,j} u_{n,klm} = F_{ijklmn} u_{i,j} u_{n,klm}, \tag{7.6b}$$

$$\tilde{C}_{lqijkmnp} u_{l,ijk} u_{q,mnp} = I_{ijklmnpq} u_{l,ijk} u_{q,mnp}. \tag{7.6c}$$

Using the above equalities, one may obtain the following relations between the

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additional parameters

$$a_1 = a_2 = a_5, \qquad 2a_3 = a_4, \tag{7.7a}$$

$$2c_1 = c_2 = c_3, \tag{7.7b}$$

$$4b_1 = 2b_2 = b_3 = 4b_4 = 2b_7, \qquad b_5 = \frac{3}{2}b_6. \tag{7.7c}$$

Based on the numerical values of Lamé constants, bulk and surface characteristic lengths and modulus of cohesion for Ag, Au, and Pt given by Ojaghnezhad and Shodja [7] and summarized in Table 3, one may determine all the material parameters in Mindlin's second strain gradient theory. Moreover, Ojaghnezhad and Shodja [7] have also provided the surface residual stress and surface elastic constants for Ag, Au, and Pt within Gurtin-Murdoch surface elasticity. For convenience, the numerical values of these are displayed in Table 4.

Table 3: Lamé constants in units of eV/Å$^3$, bulk and surface characteristic lengths in units of Å, and modulus of cohesion in units of eV/Å for Ag, Au, and Pt.

| element | $\lambda$ (eV/Å$^3$) | $\mu$ (eV/Å$^3$) | $\ell_{11}, \ell_{12}$ (Å) | $\ell_{21}, \ell_{22}$ (Å) | $\ell_{10}$ (Å) | $b_0$ (eV/Å) |
|---|---|---|---|---|---|---|
| Ag | 0.56 | 0.24 | $0.91 \pm 1.03i$ | $1.37 \pm 1,66i$ | $2.16i$ | -1.87 |
| Au | 1.08 | 0.26 | $0.56 \pm 0.63i$ | $0.69 \pm 0.69i$ | $0.259{+}0.420i$ | $-0.157{+}0.314i$ |
| Pt | 1.55 | 0.60 | $0.81 \pm 0.91i$ | $1.44 \pm 1.45i$ | $0.271{+}0.936i$ | $-0.984{+}0.622i$ |

*7.2. Descriptive examples*

Based on the numerical values of the material parameters and characteristic lengths as determined in the previous section, one may evaluate the displacement and stress field through the domains of the spherical symmetry in the framework

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Table 4: Surface residual stress and surface elastic constants in units of eV/Å$^2$ for Ag, Au, and Pt.

| element | $\sigma_0$ | $\lambda_0$ | $\mu_0$ |
|---|---|---|---|
| Ag | 0.088 | -0.047 | -0.044 |
| Au | 0.073 | -0.028 | -0.036 |
| Pt | 0.160 | -0.083 | -0.081 |

of Mindlin's second strain gradient theory. Let $a_0$ denote the lattice constant for the element under consideration, then the normalized parameter, $\alpha = \dfrac{a}{a_0}$ provides a sense on the size of the spherical domain as compared to the lattice constant of the element. Exploiting the given numerical data for the material constants, the variation of the displacement field $u_r$ in units of Å versus the normalized variable $r/a$ for nanosphere and spherical nanocavity of radius $a$ pertinent to various values of $\alpha = \dfrac{a}{a_0}$ is plotted in Figs. 2 and 3, respectively. These plots are given for fcc metals of (a) Ag, (b) Au, and (c) Pt according to (1) Mindlin's second strain gradient elasticity and (2) Gurtin-Murdoch surface elasticity. Since Mindlin's bulk characteristic lengths are complex numbers, the displacement fields pertinent to Mindlin's second strain gradient theory represent oscillatory behavior with increasing $\dfrac{r}{a}$ (Figs. 2 and 3). Moreover, it is observed that the Gurtin-Murdoch solutions for nanosphere and spherical nanocavity are less sensitive to the sphere radius in comparison to Mindlin's solutions. Additionally, Mindlin's second strain gradient theory provides values

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of $u_r$ on the boundary of the nanosphere much larger than Gurtin-Murdoch surface elasticity.

The normalized change of the radius $\left(\frac{\Delta a}{a_0}\right)$ versus the normalized radius of the sphere $\left(\frac{a}{a_0}\right)$ is also plotted for the fcc crystals of (a) Ag, (b) Au, and (c) Pt in Figs. 4(a)-(c) according to both Mindlin's second strain gradient elasticity (MSGE) and Gurtin-Murdoch surface elasticity (GMSE). For the sake of comparison, the relaxation of the nanosphere is also calculated by simulating the spherical domain via the molecular dynamics package LAMMPS at absolute temperature using Embedded-Atom-Method (EAM) functions reported by Foiles et al. [21]. It is observed that the result obtained from Gurtin-Murdoch surface elasticity is approximately size independent while the results of Mindlin's strain gradient theory and LAMMPS are size-dependent. Likewise, the normalized change of the radius $\left(\frac{\Delta a}{a_0}\right)$ of an embedded nano-sized spherical cavity is plotted versus the normalized cavity radius $\left(\frac{a}{a_0}\right)$ for Ag, Au, and Pt in Figs. 5(a)-(c), respectively, using Mindlin's strain gradient theory and Gurtin-Murdoch surface elasticity. As it is observed the phenomenon of relaxation captured within MSGT is remarkably affected by the size of the spherical cavity, whereas GMSE remains nearly size independent.

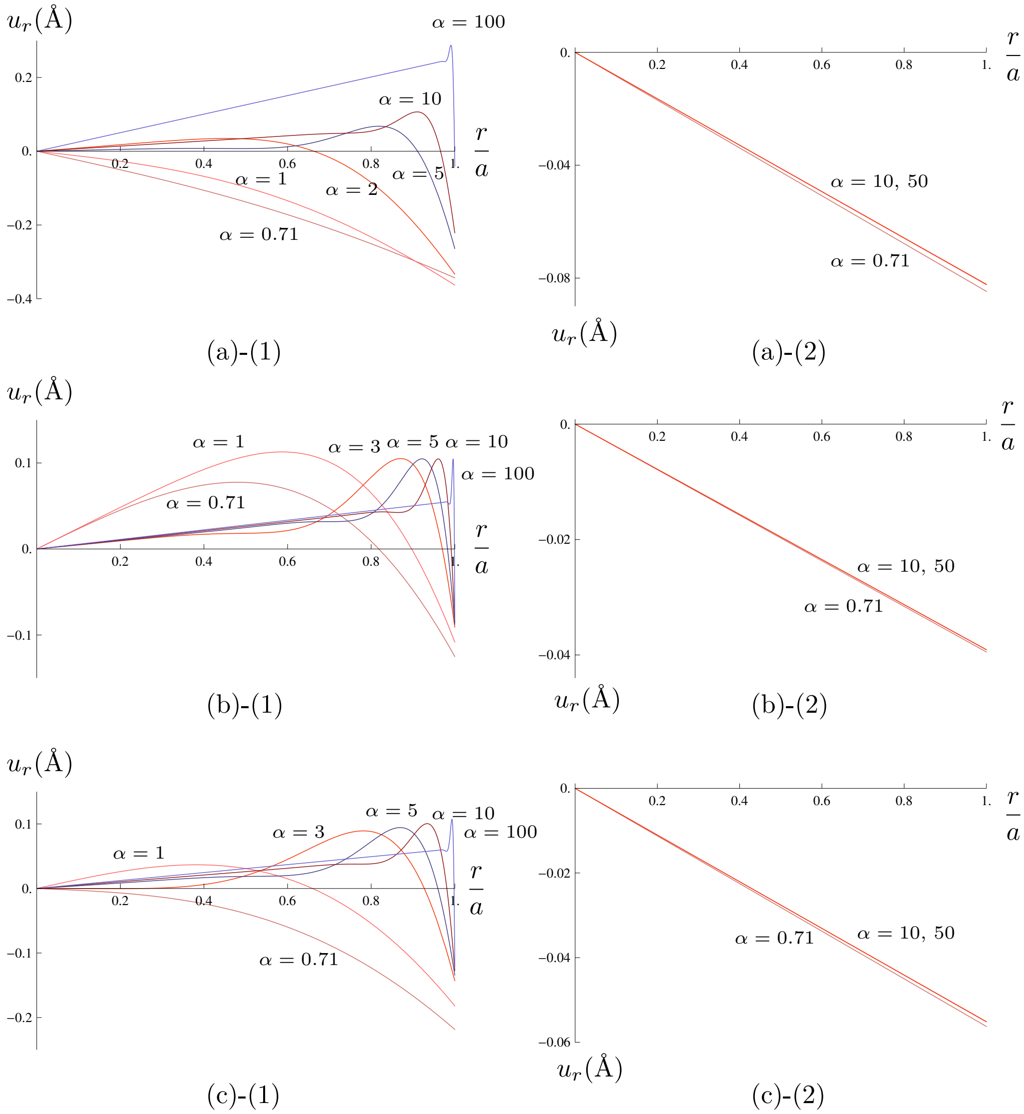


Figure 2: Variation of the displacement field $u_r$ in Å versus $r/a$ in the (a) Ag, (b) Au, and (c) Pt nanospheres of radius $a$ according to (1) Mindlin's second strain gradient theory and (2) Gurtin-Murdoch surface elasticity for different ratios of $\alpha = \dfrac{a}{a_0}$.

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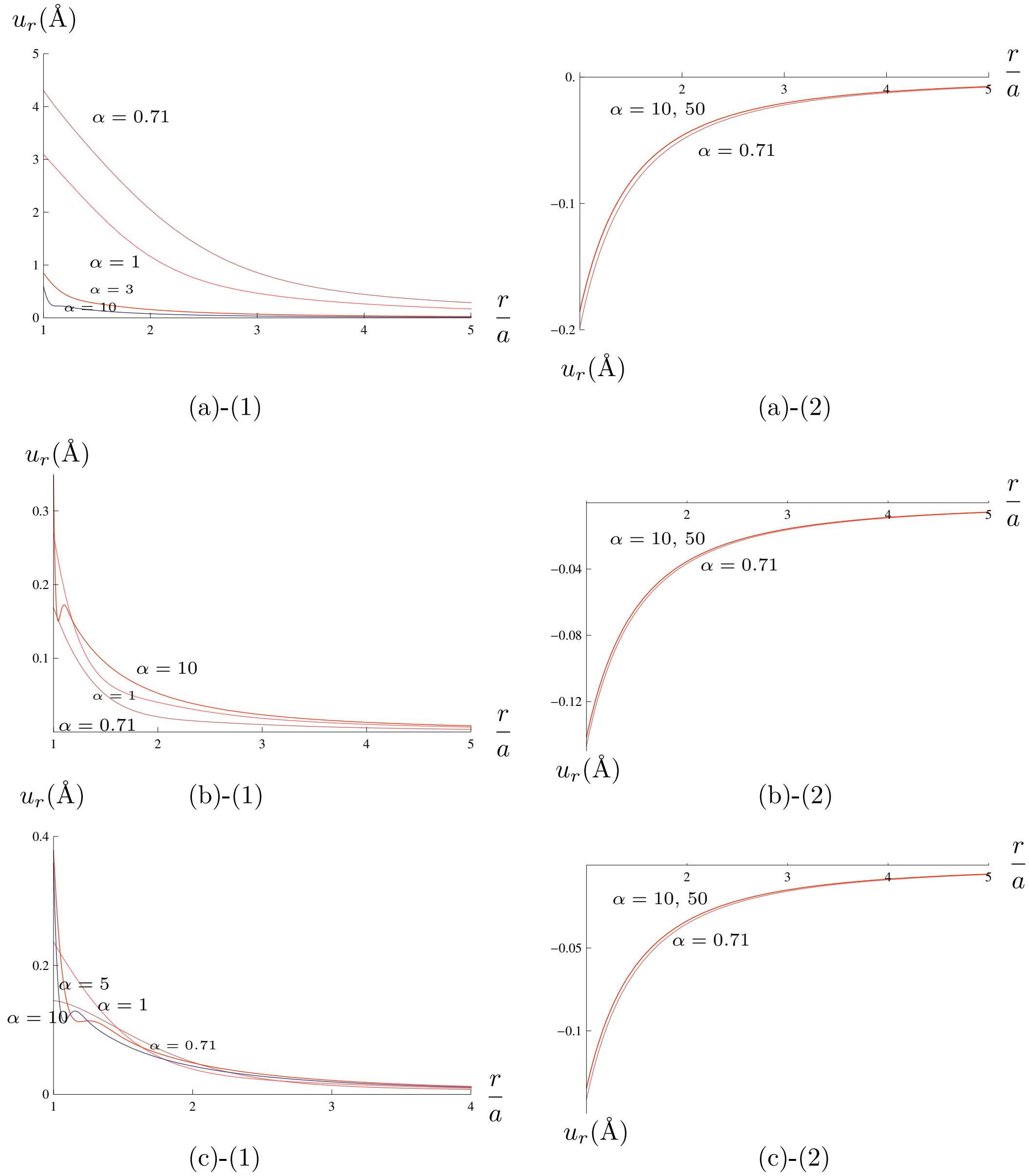


Figure 3: Variation of the displacement field $u_r$ in Å versus $r/a$ in the (a) Ag, (b) Au, and (c) Pt infinite domain with spherical cavity of radius $a$ according to (1) Mindlin's second strain gradient theory and (2) Gurtin-Murdoch surface elasticity for different ratios of $\alpha = \dfrac{a}{a_0}$.

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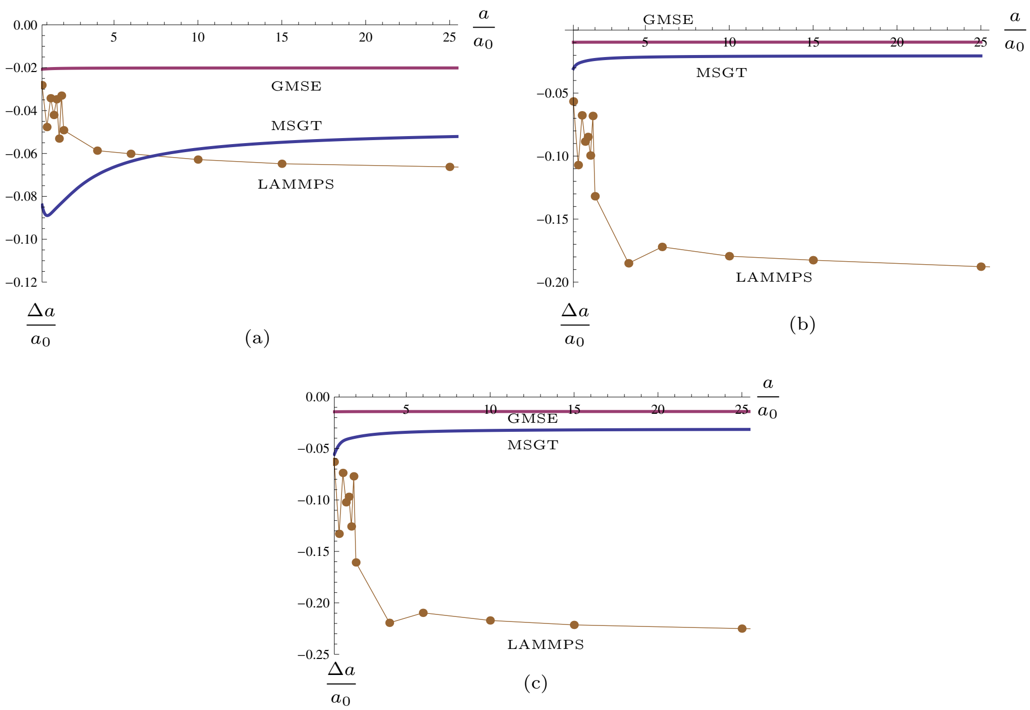


Figure 4: Variation of the normalized change in the sphere radius ($\dfrac{\Delta a}{a_0}$) with the normalized radius of the sphere ($\dfrac{a}{a_0}$) obtained via Mindlin's second strain gradient theory (MSGT), Gurtin-Murdoch surface elasticity (GMSE), and LAMMPS simulation for (a) Ag, (b) Au, and (c) Pt.

In order to compare the results on relaxation of a spherical solid predicted by the current continuum theories (MSGT and GMSE) with those of atomistic simulations, the atomic displacements calculated for Ag spherical domains via LAMMPS using different EAMs [21, 22, 23, 24] as well as the corresponding results of the thories of interest are inserted in a common picture, Fig. 6. In Figs. 6(a)-(c), three different sizes of solid nanospheres with ratios $\alpha = 0.71$, 1, and 2 are considered, respectively. In Figs. 6(a)-(b), it was feasible to indicate the atoms in a common radial distance from the center atom by letters. The

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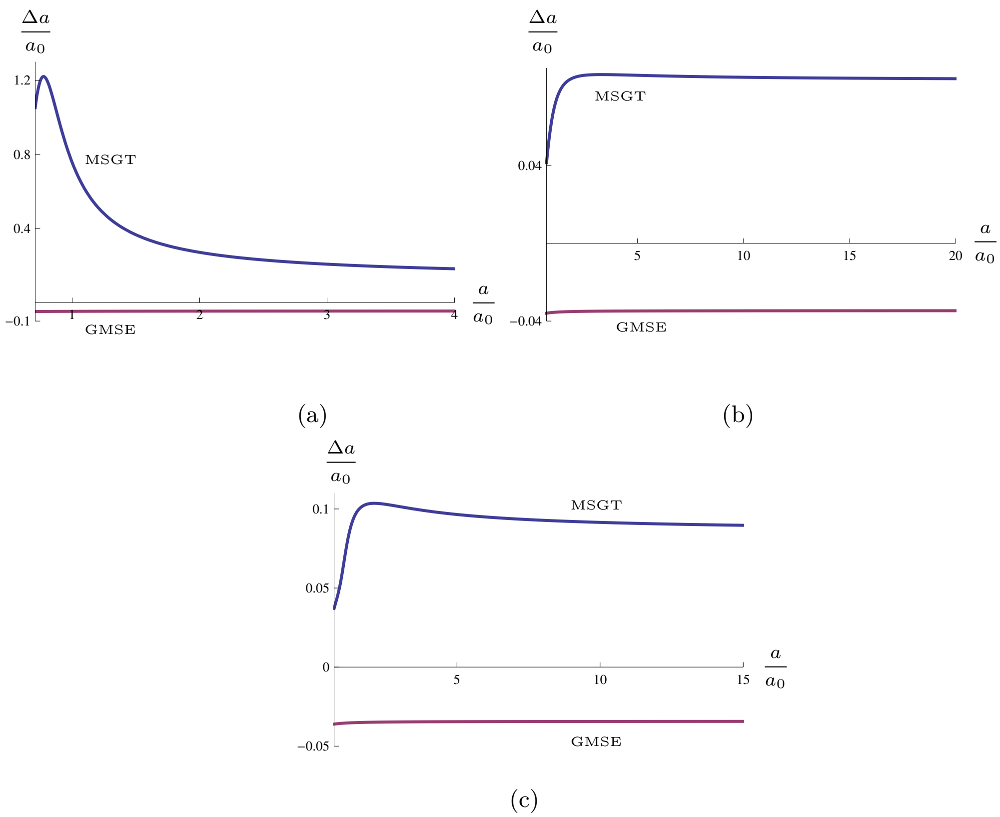


Figure 5: Variation of the normalized change in the embedded spherical void radius ($\frac{\Delta a}{a_0}$) with the normalized radius of the sphere ($\frac{a}{a_0}$) obtained via Mindlin's second strain gradient theory (MSGT) and Gurtin-Murdoch surface elasticity (GMSE) for (a) Ag, (b) Au, and (c) Pt.

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computed radial displacement for each atom is also given in Å in this figure.

Under the surface effect in the above-discussed problems involving spherical symmetry, the non-trivial displacement field induces non-trivial stress field with non-zero components $\tau_{rr}(r)$ and $\tau_{\theta\theta}(r) = \tau_{\phi\phi}(r)$. Figs. 7(a)-(c) represent the variation of $\tau_{rr}$ and $\tau_{\theta\theta} = \tau_{\phi\phi}$ in eV/Å$^3$ versus $\dfrac{r}{a}$ for some different values of $\alpha$, respectively, within nano-spherical domains of Ag, Au, and Pt based on Mindlin's second strain gradient theory. It is observed that the phenomenon of relaxation for larger spheres ($\alpha$ larger) has little or no effect on the stress field distribution as the center of the sphere is approached, whereas steep variations in the stresses near its surface occur and attain notable values just beneath the surface. Through Gurtin-Murdoch surface elasticity, however, all the non-zero components of the induced stress field in nanosphere are constant and $\tau_{rr} = \tau_{\theta\theta} = \tau_{\phi\phi}$. The variation of these stress components in eV/Å$^3$ versus normalized sphere radius $\left(\dfrac{a}{a_0}\right)$ are plotted in Figs. 8(a)-(c) for Ag, Au, and Pt, respectively. As it is seen, the surface effect is more pronounced for smaller spheres; in all the considered cases the variation of stresses becomes sharper as $a/a_0 \rightarrow 0$. Likewise, in the case of an embedded spherical nanocavity, the variation of $\tau_{rr}$ and $\tau_{\theta\theta} = \tau_{\phi\phi}$ in eV/Å$^3$ versus $\dfrac{r}{a}$ for different values of cavity size are displayed in Figs. 9 and 10 for Ag, Au, and Pt based on Mindlin's second strain gradient theory and Gurtin-Murdoch surface elasticity, respectively. In Mindlin's solution, generally, the variation of the stress components near the boundary of the cavity have oscillatory nature, as it was the case in the case of the displacement field. The stresses attain notably large values at the cavity's

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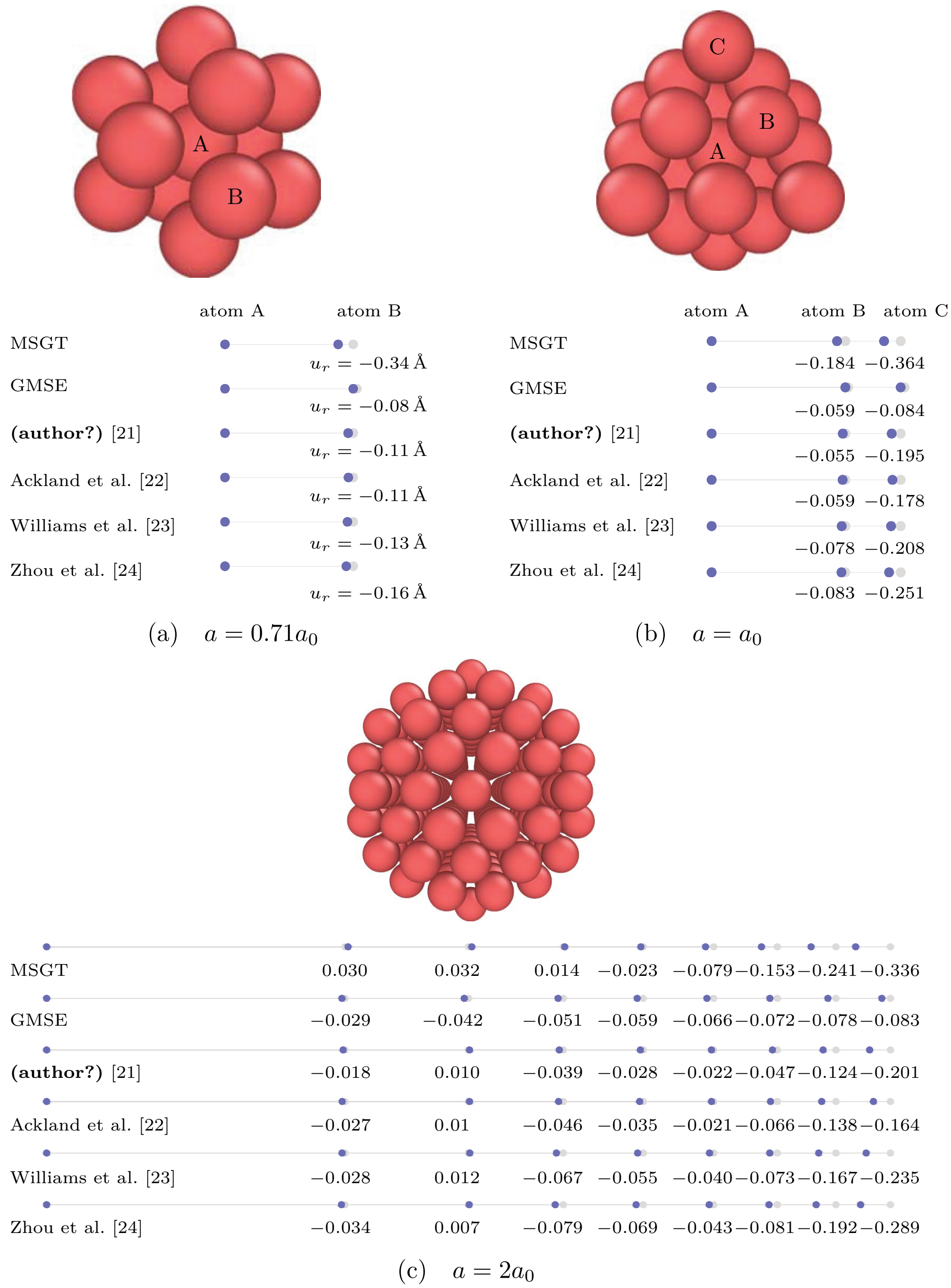
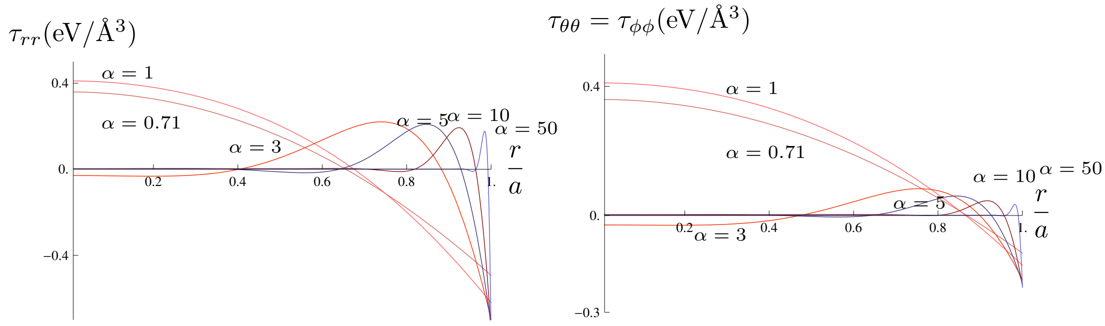
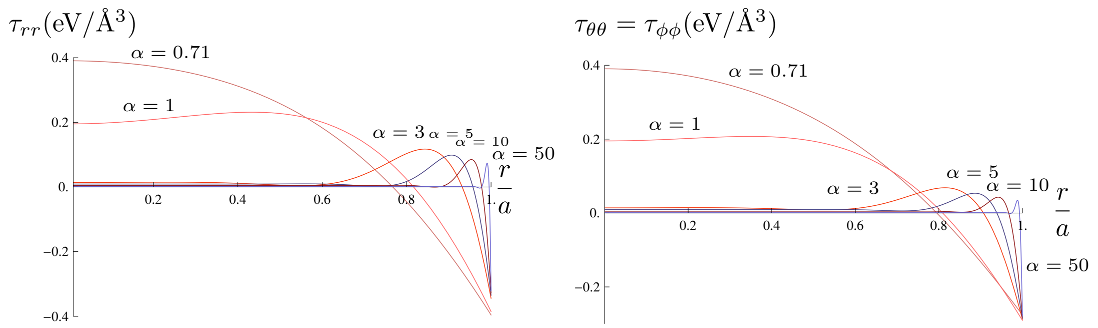



Figure 6: Relaxation phenomenon observed for the Ag spherical domains of radii (a) $a = 0.71a_0$, (b) $a = a_0$, and (c) $a = 2a_0$ via continuum theories of interest (MSGT and GMSE) as well as some EAMs.

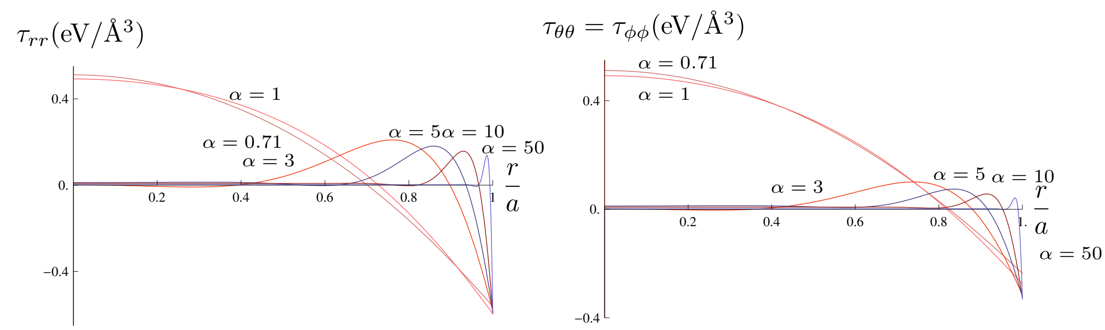
surface, just inside the matrix, and decay rapidly with distance from the cavity. A similar phenomenon is observed within Gurtin-Murdoch surface elasticity, except the solutions are not oscillatory.




(a)




(b)




(c)

Figure 7: Variation of the stress field components $\tau_{rr}$ and $\tau_{\theta\theta} = \tau_{\phi\phi}$ in eV/Å$^3$ versus $r/a$ in the (a) Ag, (b) Au, and (c) Pt solid nanosphere of radius $a$ according to Mindlin's second strain gradient theory for different ratios of $\alpha = \dfrac{a}{a_0}$.

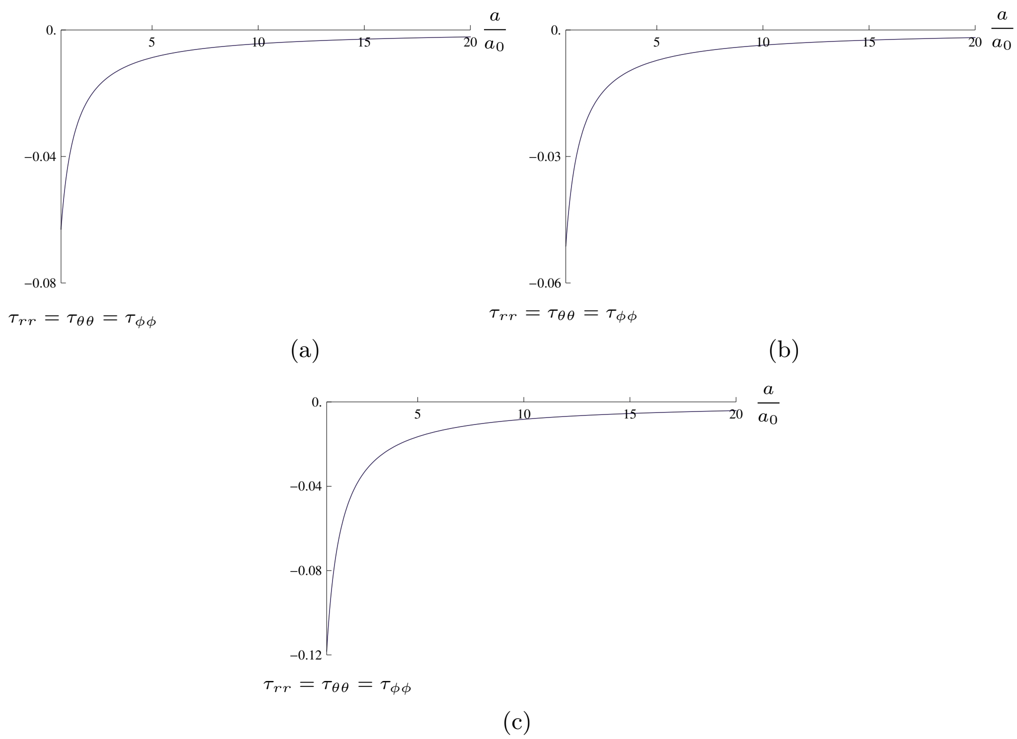



Figure 8: Stress components $\tau_{rr} = \tau_{\theta\theta} = \tau_{\phi\phi}$ in eV/Å$^3$ versus $\dfrac{a}{a_0}$ via Gurtin-Murdoch surface elasticity for solid nanospheres made of (a) Ag, (b) Au, and (c) Pt.

## 8. Conclusion

Mindlin's second strain gradient theory has been formulated in an arbitrary orthogonal curvilinear coordinate system. Using Eringen [10] mathematical tools for transformation from Cartesian coordinates to any arbitrary curvilinear coordinates, the equilibrium equations, generalized stress-strain constitutive relations, components of the generalized strain tensors, and three different types of traction boundary conditions in any orthogonal curvilinear coordinate system are derived. In continue, in order to give a highly-pragmatic example in the field

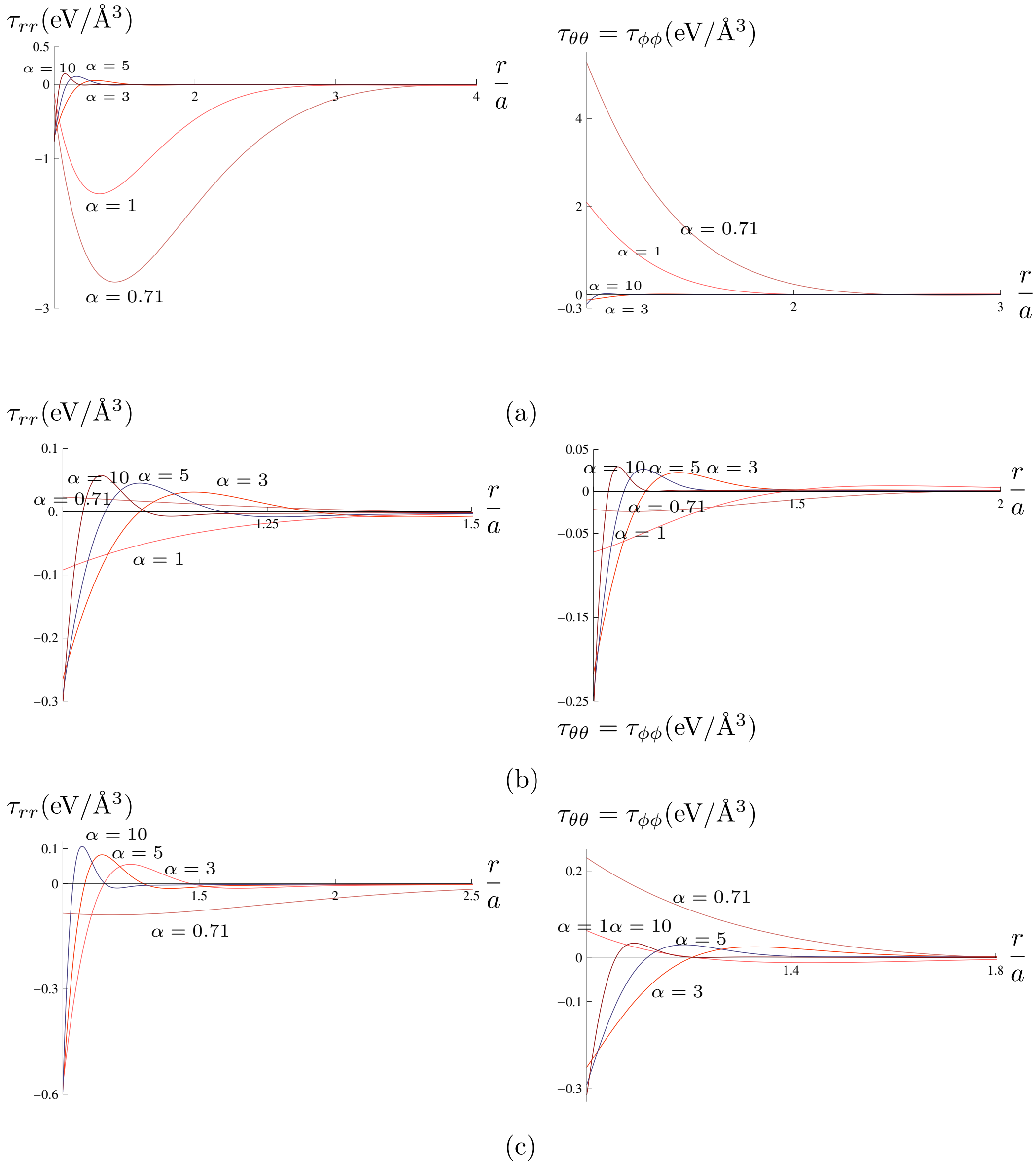
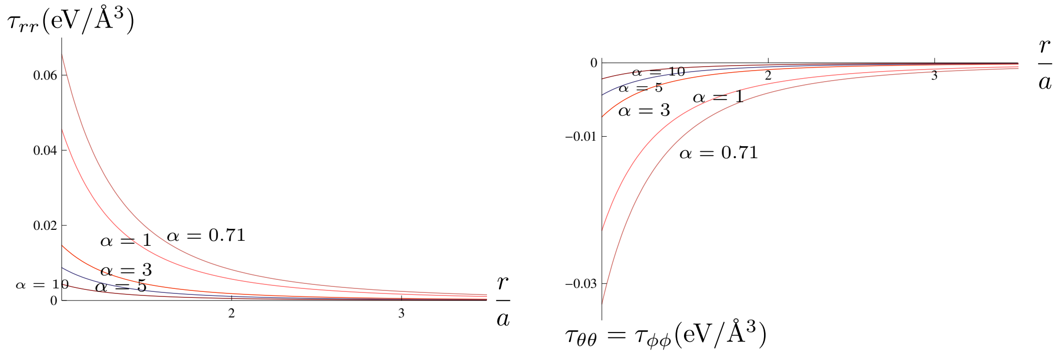
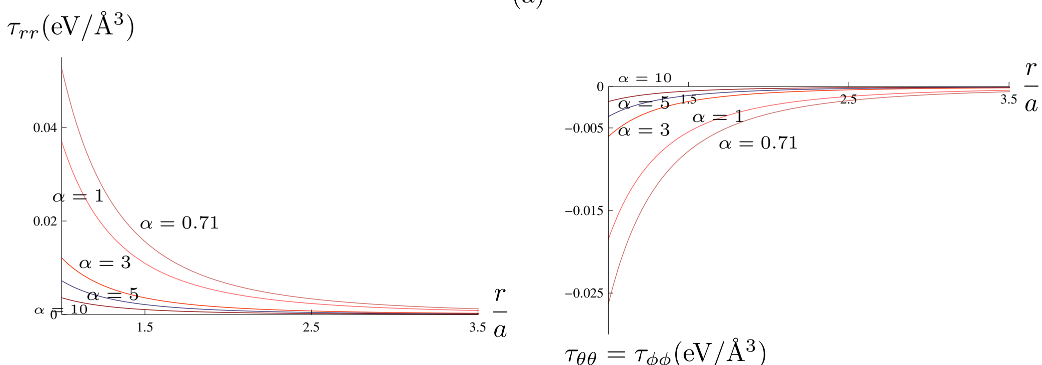


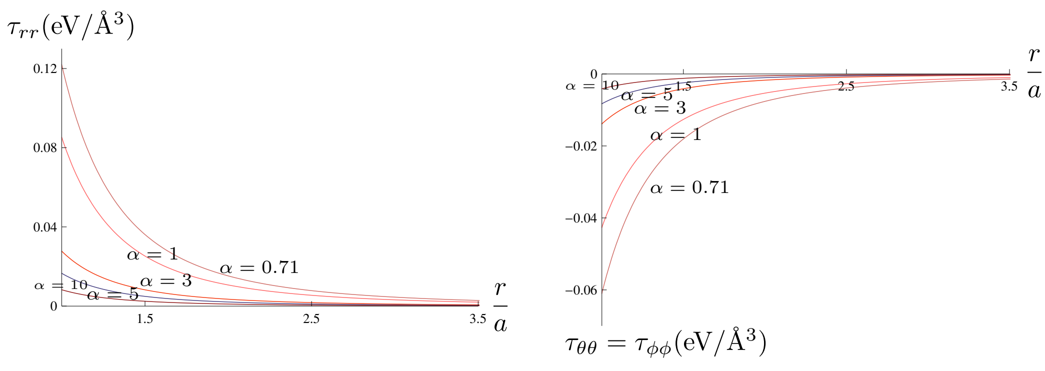

Figure 9: Variation of the stress field components $\tau_{rr}$ and $\tau_{\theta\theta} = \tau_{\phi\phi}$ in eV/Å$^3$ versus $r/a$ in the (a) Ag, (b) Au, and (c) Pt domain with spherical cavity of radius $a$ according to Mindlin's second strain gradient theory for different ratios of $\alpha = \frac{a}{a_0}$.




(a)




(b)




(c)

Figure 10: Variation of the stress field components $\tau_{rr}$ and $\tau_{\theta\theta} = \tau_{\phi\phi}$ in eV/Å$^3$ versus $r/a$ in the (a) Ag, (b) Au, and (c) Pt domain solid with spherical cavity of radius $a$ according to Gurtin-Murdoch surface elasticity for different ratios of $\alpha = \dfrac{a}{a_0}$.

of nanomechanics, Mindlin's second strain gradient theory is represented in the spherical coordinate system and surface relaxation associated with Ag, Au, and Pt nanospheres as well as nanocavities buried in Ag, Au, and Pt is examined in the framework of Mindlin's second strain gradient theory. The results are compared with those obtained using Gurtin-Murdoch surface elasticity. For further verification, Ag solid nanosphere has also been simulated using molecular dynamics with various EAMs. It is observed that the phenomenon of relaxation captured within Mindlin's second strain gradient theory is remarkably affected by the size of the nanospherical domain while that of Gurtin-Murdoch surface elasticity is nearly size independent.

## Appendix A. The expressions $\bar{\tau}_{ij}$, $i$, $j = r$, $\theta$, $\phi$ in spherical coordinate system

$$
\begin{aligned}
\bar{\tau}_{rr} = \tau_{rr} - \Bigg(&\tau_{rrr,r} + \frac{1}{r}\tau_{r\phi r,\phi} + \frac{1}{r\sin\phi}\tau_{r\theta r,\theta} + \frac{1}{r}\big(2\tau_{rrr} - \tau_{\phi\phi r} - \tau_{r\phi\phi} - \tau_{\theta\theta r} - \tau_{r\theta\theta} \\
&+ \tau_{r\phi r}\cot\phi\big)\Bigg) + \frac{1}{r^2}\Big(2\tau_{rrrr} - 4\tau_{rr\theta\theta} + 4\tau_{rr\phi r}\cot\phi - 4\tau_{rr\phi\phi} - 6\tau_{r\theta\theta r} - 2\tau_{r\theta\phi\theta}\cot\phi \\
&- 7\tau_{r\phi\phi r} + \tau_{r\phi\phi r}\cot^2\phi - \tau_{r\phi\phi r}\csc^2\phi - 2\tau_{r\phi\phi\phi}\cot\phi + 2\tau_{\theta\theta\theta\theta} - 2\tau_{\theta\theta\phi r}\cot\phi \\
&+ 2\tau_{\theta\theta\phi\phi} + 2\tau_{\theta\phi\phi\theta} - 2\tau_{\phi\phi\phi r}\cot\phi + 2\tau_{\phi\phi\phi\phi} + 4\tau_{rr\phi r,\phi} - \tau_{r\theta\theta r,\phi}\cot\phi \\
&- 2\tau_{r\theta\phi\theta,\phi} + 2\tau_{r\phi\phi r,\phi}\cot\phi - 2\tau_{r\phi\phi\phi,\phi} - 2\tau_{\theta\theta\phi r,\phi} - 2\tau_{\phi\phi\phi r,\phi} + \tau_{r\phi\phi r,\phi\phi} \\
&+ 4\tau_{rr\theta r,\theta}\csc\phi - 2\tau_{r\theta\theta\theta,\theta}\csc\phi + 2\tau_{r\theta\phi r,\theta}\cot\phi\csc\phi - 2\tau_{r\theta\phi\phi,\theta}\csc\phi - 2\tau_{\theta\theta\theta r,\theta}\csc\phi \\
&- 2\tau_{\theta\phi\phi r,\theta}\csc\phi + 2\tau_{r\theta\phi r,\theta\phi}\csc\phi + \tau_{r\theta\theta r,\theta\theta}\csc^2\phi\Big) + \frac{1}{r}\Big(4\tau_{rrrr,r} - 2\tau_{rr\theta\theta,r} \\
&+ 2\tau_{rr\phi r,r}\cot\phi - 2\tau_{rr\phi\phi,r} - 3\tau_{r\theta\theta r,r} - 3\tau_{r\phi\phi r,r} + 2\tau_{rr\phi r,r\phi} + 2\tau_{rr\theta r,r\theta}\csc\phi\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \tau_{rrrr,rr}, \\
\bar{\tau}_{r\theta} &= \tau_{r\theta} - \Big(\tau_{rr\theta,r} + \frac{1}{r}\tau_{r\phi\theta,\phi} + \frac{1}{r\sin\phi}\tau_{r\theta\theta,\theta} + \frac{1}{r}\big(2\tau_{rr\theta} + \tau_{r\theta\theta} - \tau_{\phi\phi\theta} - \tau_{\theta\theta\theta} \\
&+ (\tau_{r\phi\theta} + \tau_{r\theta\phi})\cot\phi\big)\Big) + \frac{1}{r^2}\Big(2\tau_{rrr\theta} + 4\tau_{rr\theta r} + 4\tau_{rr\theta\phi}\cot\phi + 4\tau_{rr\phi\theta}\cot\phi - 6\tau_{r\theta\theta\theta} \\
&- \tau_{r\theta\theta\theta}\cot^2\phi + 4\tau_{r\theta\phi r}\cot\phi - 3\tau_{r\theta\phi\phi} + \tau_{r\theta\phi\phi}\cot^2\phi + \tau_{r\theta\phi\phi}\csc^2\phi - 7\tau_{r\phi\phi\theta} \\
&- 2\tau_{\theta\theta\theta r} - 2\tau_{\theta\theta\theta\phi}\cot\phi - 2\tau_{\theta\theta\phi\theta}\cot\phi - 2\tau_{\theta\phi\phi r} - 2\tau_{\theta\phi\phi\phi}\cot\phi - 2\tau_{\phi\phi\phi\theta}\cot\phi \\
&+ 4\tau_{rr\phi\theta,\phi} - \tau_{r\theta\theta\theta,\phi}\cot\phi + 2\tau_{r\theta\phi r,\phi} + 2\tau_{r\theta\phi\phi,\phi}\cot\phi + 2\tau_{r\phi\phi\theta,\phi}\cot\phi - 2\tau_{\theta\theta\phi\theta,\phi} \\
&- 2\tau_{\phi\phi\phi\theta,\phi} + \tau_{r\phi\phi\theta,\phi\phi} + 4\tau_{rr\theta\theta,\theta}\csc\phi + 2\tau_{r\theta\theta r,\theta}\csc\phi + 2\tau_{r\theta\theta\phi,\theta}\cot\phi\csc\phi \\
&+ 2\tau_{r\theta\phi\theta,\theta}\cot\phi\csc\phi - 2\tau_{\theta\theta\theta\theta,\theta}\csc\phi - 2\tau_{\theta\phi\phi\theta,\theta}\csc\phi + 2\tau_{r\theta\phi\theta,\theta\phi}\csc\phi \\
&+ \tau_{r\theta\theta\theta,\theta\theta}\csc^2\phi\Big) + \frac{1}{r}\Big(4\tau_{rrr\theta,r} + 2\tau_{rr\theta r,r} + 2\tau_{rr\theta\phi,r}\cot\phi + 2\tau_{rr\phi\theta,r}\cot\phi - 3\tau_{r\theta\theta\theta,r} \\
&- 3\tau_{r\phi\phi\theta,r} + 2\tau_{rr\phi\theta,r\phi} + 2\tau_{rr\theta\theta,r\theta}\csc\phi\Big) + \tau_{rrr\theta,rr}, \\
\bar{\tau}_{r\phi} &= \tau_{r\phi} - \Big(\tau_{rr\phi,r} + \frac{1}{r}\tau_{r\phi\phi,\phi} + \frac{1}{r\sin\phi}\tau_{r\theta\phi,\theta} + \frac{1}{r}\big(2\tau_{rr\phi} + \tau_{r\phi r} - \tau_{\phi\phi\phi} - \tau_{\theta\theta\phi} \\
&+ (\tau_{r\phi\phi} - \tau_{r\theta\theta})\cot\phi\big)\Big) + \frac{1}{2r^2}\Big(4\tau_{rrr\phi} - 8\tau_{rr\theta\theta}\cot\phi + 8\tau_{rr\phi r} + 8\tau_{rr\phi\phi}\cot\phi \\
&- 4\tau_{r\theta\theta r}\cot\phi - 10\tau_{r\theta\theta\phi} - 2\tau_{r\theta\theta\phi}\cot^2\phi + \tau_{r\theta\phi\theta} - 3\tau_{r\theta\phi\theta}\cot^2\phi - \tau_{r\theta\phi\theta}\csc^2\phi \\
&+ 4\tau_{r\phi\phi r}\cot\phi - 15\tau_{r\phi\phi\phi} + \tau_{r\phi\phi\phi}\cot^2\phi - \tau_{r\phi\phi\phi}\csc^2\phi + 4\tau_{\theta\theta\theta\theta}\cot\phi - 4\tau_{\theta\theta\phi r} \\
&- 4\tau_{\theta\theta\phi\phi}\cot\phi + 4\tau_{\theta\phi\phi\theta}\cot\phi - 4\tau_{\phi\phi\phi r} - 4\tau_{\phi\phi\phi\phi}\cot\phi + 8\tau_{rr\phi\phi,\phi} - 2\tau_{r\theta\theta\phi,\phi}\cot\phi \\
&- 4\tau_{r\theta\phi\theta,\phi}\cot\phi + 4\tau_{r\phi\phi r,\phi} + 4\tau_{r\phi\phi\phi,\phi}\cot\phi - 4\tau_{\theta\theta\phi\phi,\phi} - 4\tau_{\phi\phi\phi\phi,\phi} + 2\tau_{r\phi\phi\phi,\phi\phi} \\
&+ 8\tau_{rr\theta\phi,\theta}\csc\phi - 4\tau_{r\theta\theta\theta,\theta}\cot\phi\csc\phi + 4\tau_{r\theta\phi r,\theta}\csc\phi + 4\tau_{r\theta\phi\phi,\theta}\cot\phi\csc\phi \\
&- 4\tau_{\theta\theta\theta\phi,\theta}\csc\phi - 4\tau_{\theta\phi\phi\phi,\theta}\csc\phi + 4\tau_{r\theta\phi\phi,\theta\phi}\csc\phi + 2\tau_{r\theta\theta\phi,\theta\theta}\csc^2\phi\Big) + \frac{1}{r}\Big(4\tau_{rrr\phi,r} \\
&- 2\tau_{rr\theta\theta,r}\cot\phi + 2\tau_{rr\phi r,r} + 2\tau_{rr\phi\phi,r}\cot\phi - 3\tau_{r\theta\theta\phi,r} - 3\tau_{r\phi\phi\phi,r} + 2\tau_{rr\phi\phi,r\phi} \\
&+ 2\tau_{rr\theta\phi,r\theta}\csc\phi\Big) + \tau_{rrr\phi,rr},
\end{aligned}
$$

$$
\begin{aligned}
\bar{\tau}_{\theta r} &= \tau_{\theta r} - \left(\tau_{\theta rr,r} + \frac{1}{r}\tau_{\theta\phi r,\phi} + \frac{1}{r\sin\phi}\tau_{\theta\theta r,\theta} + \frac{1}{r}\left(3\tau_{\theta rr} - \tau_{\theta\phi\phi} - \tau_{\theta\theta\theta} + 2\tau_{\theta\phi r}\cot\phi\right)\right) \\
&+ \frac{1}{r^2}\Big(6\tau_{rr\theta r} - 6\tau_{r\theta\theta\theta} + 12\tau_{r\theta\phi r}\cot\phi - 6\tau_{r\theta\phi\phi} - (2+\cot^2\phi)\tau_{\theta\theta\theta r} - 4\tau_{\theta\theta\phi\theta}\cot\phi \\
&- 2\tau_{\theta\phi\phi r} + 5\tau_{\theta\phi\phi r}\cot^2\phi - 3\tau_{\theta\phi\phi r}\csc^2\phi - 4\tau_{\theta\phi\phi\phi}\cot\phi + 6\tau_{r\theta\phi r,\phi} - \tau_{\theta\theta\theta r,\phi}\cot\phi \\
&- 2\tau_{\theta\theta\phi\theta,\phi} + 4\tau_{\theta\phi\phi r,\phi}\cot\phi - 2\tau_{\theta\phi\phi\phi,\phi} + \tau_{\theta\phi\phi r,\phi\phi} + 6\tau_{r\theta\theta r,\theta}\csc\phi - 2\tau_{\theta\theta\theta\theta,\theta}\csc\phi \\
&+ 4\tau_{\theta\theta\phi r,\theta}\cot\phi\csc\phi - 2\tau_{\theta\theta\phi\phi,\theta}\csc\phi + 2\tau_{\theta\theta\phi r,\theta\phi}\csc\phi + \tau_{\theta\theta\theta r,\theta\theta}\csc^2\phi\Big) \\
&+ \frac{1}{r}\Big(6\tau_{rr\theta r,r} - 2\tau_{r\theta\theta\theta,r} + 4\tau_{r\theta\phi r,r}\cot\phi - 2\tau_{r\theta\phi\phi,r} - \tau_{\theta\theta\theta r,r} - \tau_{\theta\phi\phi r,r} + 2\tau_{r\theta\phi r,r\phi} \\
&+ 2\tau_{r\theta\theta r,r\theta}\csc\phi\Big) + \tau_{rr\theta r,rr}, \\
\bar{\tau}_{\theta\theta} &= \tau_{\theta\theta} - \left(\tau_{\theta r\theta,r} + \frac{1}{r}\tau_{\theta\phi\theta,\phi} + \frac{1}{r\sin\phi}\tau_{\theta\theta\theta,\theta} + \frac{1}{r}\left(3\tau_{r\theta\theta} + \tau_{\theta\theta r} + (2\tau_{\phi\theta\theta} + \tau_{\theta\theta\phi})\cot\phi\right)\right) \\
&+ \frac{1}{r^2}\Big(6\tau_{rr\theta\theta} + 6\tau_{r\theta\theta r} + 6\tau_{r\theta\theta\phi}\cot\phi + 12\tau_{r\theta\phi\theta}\cot\phi - 2\tau_{\theta\theta\theta\theta}\csc^2\phi - 7/2\tau_{\theta\theta\phi\phi} \\
&+ 3/2\tau_{\theta\theta\phi\phi}\csc^2\phi - 4\tau_{\theta\phi\phi\theta} + 6\tau_{r\theta\phi\theta,\phi} + 2\tau_{\theta\theta\phi r,\phi} + 1/2\cot\phi(12\tau_{\theta\theta\phi r} + 5\tau_{\theta\theta\phi\phi}\cot\phi \\
&+ 4\tau_{\theta\phi\phi\theta}\cot\phi - 2\tau_{\theta\theta\theta\theta,\phi} + 4\tau_{\theta\theta\phi\phi,\phi} + 8\tau_{\theta\phi\phi\theta,\phi}) + \tau_{\theta\phi\phi\theta,\phi\phi} + 6\tau_{r\theta\theta\theta,\theta}\csc\phi \\
&+ 2\tau_{\theta\theta\theta r,\theta}\csc\phi + 2\tau_{\theta\theta\theta\phi,\theta}\cot\phi\csc\phi + 4\tau_{\theta\theta\phi\theta,\theta}\cot\phi\csc\phi + 2\tau_{\theta\theta\phi\theta,\theta\phi}\csc\phi \\
&+ \tau_{\theta\theta\theta\theta,\theta\theta}\csc^2\phi\Big) + \frac{1}{r}\Big(6\tau_{rr\theta\theta,r} + 2\tau_{r\theta\theta r,r} + 2\tau_{r\theta\theta\phi,r}\cot\phi + 4\tau_{r\theta\phi\theta,r}\cot\phi - \tau_{\theta\theta\theta\theta,r} \\
&- \tau_{\theta\phi\phi\theta,r} + 2\tau_{r\theta\phi\theta,r\phi} + 2\tau_{r\theta\theta\theta,r\theta}\csc\phi\Big) + \tau_{rr\theta\theta,rr}, \\
\bar{\tau}_{\theta\phi} &= \tau_{\theta\phi} - \left(\tau_{\theta r\phi,r} + \frac{1}{r}\tau_{\theta\phi\phi,\phi} + \frac{1}{r\sin\phi}\tau_{\theta\theta\phi,\theta} + \frac{1}{r}\Big(3\tau_{r\theta\phi} + \tau_{\theta\phi r} + \big(\tau_{\phi\theta\phi} + \tau_{\theta\phi\phi}\right. \\
&\left. - \tau_{\theta\theta\theta}\big)\cot\phi\Big)\right) + \frac{1}{r^2}\Big(6\tau_{rr\theta\phi} - 6\tau_{r\theta\theta\theta}\cot\phi + 6\tau_{r\theta\phi r} + 12\tau_{r\theta\phi\phi}\cot\phi - 2\tau_{\theta\theta\theta r}\cot\phi \\
&- \tau_{\theta\theta\theta\phi} - 2\tau_{\theta\theta\theta\phi}\cot^2\phi - \tau_{\theta\theta\phi\theta} - 5\tau_{\theta\theta\phi\theta}\cot^2\phi + \tau_{\theta\theta\phi\theta}\csc^2\phi + 4\tau_{\theta\phi\phi r}\cot\phi \\
&- 2\tau_{\theta\phi\phi\phi} + 5\tau_{\theta\phi\phi\phi}\cot^2\phi - 3\tau_{\theta\phi\phi\phi}\csc^2\phi + 6\tau_{r\theta\phi\phi,\phi} - \tau_{\theta\theta\theta\phi,\phi}\cot\phi - 2\tau_{\theta\theta\phi\theta,\phi} \\
&+ 2\tau_{\theta\phi\phi r,\phi} + 4\tau_{\theta\phi\phi\phi,\phi}\cot\phi + \tau_{\theta\phi\phi\phi,\phi\phi} + 6\tau_{r\theta\theta\phi,\theta}\csc\phi - 2\tau_{\theta\theta\theta\theta,\theta}\cot\phi\csc\phi \\
&+ 2\tau_{\theta\theta\phi r,\theta}\csc\phi + 4\tau_{\theta\theta\phi\phi,\theta}\cot\phi\csc\phi + 2\tau_{\theta\theta\phi\phi,\theta\phi}\csc\phi + \tau_{\theta\theta\theta\phi,\theta\theta}\csc^2\phi\Big)
\end{aligned}
$$

$$+\frac{1}{r}\Big(6\tau_{rr\theta\phi,r}-2\tau_{r\theta\theta\theta,r}\cot\phi+2\tau_{r\theta\phi r,r}+4\tau_{r\theta\phi\phi,r}\cot\phi-\tau_{\theta\theta\theta\phi,r}-\tau_{\theta\phi\phi\phi,r}$$

$$+2\tau_{r\theta\phi\phi,r\phi}+2\tau_{r\theta\theta\phi,r\theta}\csc\phi\Big)+\tau_{rr\theta\phi,rr},$$

$$\bar{\tau}_{\phi r}=\tau_{\phi r}-\Bigg(\tau_{\phi rr,r}+\frac{1}{r}\tau_{\phi\phi r,\phi}+\frac{1}{r\sin\phi}\tau_{\phi\theta r,\theta}+\frac{1}{r}\big(2\tau_{\phi rr}+\tau_{r\phi r}-\tau_{\phi\phi\phi}-\tau_{\phi\theta\theta}$$

$$+\big(\tau_{\phi\phi r}-\tau_{\theta\theta r}\big)\cot\phi\big)\Bigg)+\frac{1}{r^2}\Big(6\tau_{rr\phi r}-6\tau_{r\theta\theta r}\cot\phi-6\tau_{r\theta\phi\theta}+6\tau_{r\phi\phi r}\cot\phi$$

$$-6\tau_{r\phi\phi\phi}+2\tau_{\theta\theta\theta\theta}\cot\phi-3\tau_{\theta\theta\phi r}-5\tau_{\theta\theta\phi r}\cot^2\phi+2\tau_{\theta\theta\phi r}\csc^2\phi+2\tau_{\theta\theta\phi\phi}\cot\phi$$

$$-2\tau_{\theta\phi\phi\theta}\cot\phi-3\tau_{\phi\phi\phi r}+\tau_{\phi\phi\phi r}\cot^2\phi-\tau_{\phi\phi\phi r}\csc^2\phi-2\tau_{\phi\phi\phi\phi}\cot\phi+6\tau_{r\phi\phi r,\phi}$$

$$-3\tau_{\theta\theta\phi r,\phi}\cot\phi-2\tau_{\theta\phi\phi\theta,\phi}+2\tau_{\phi\phi\phi r,\phi}\cot\phi-2\tau_{\phi\phi\phi\phi,\phi}+\tau_{\phi\phi\phi r,\phi\phi}+6\tau_{r\theta\phi r,\theta}\csc\phi$$

$$-2\tau_{\theta\theta\theta r,\theta}\cot\phi\csc\phi-2\tau_{\theta\theta\phi\theta,\theta}\csc\phi+2\tau_{\theta\phi\phi r,\theta}\cot\phi\csc\phi-2\tau_{\theta\phi\phi\phi,\theta}\csc\phi$$

$$+2\tau_{\theta\phi\phi r,\theta\phi}\csc\phi+\tau_{\theta\theta\phi r,\theta\theta}\csc^2\phi\Big)+\frac{1}{r}\Big(6\tau_{rr\phi r,r}-2\tau_{r\theta\theta r,r}\cot\phi-2\tau_{r\theta\phi\theta,r}$$

$$+2\tau_{r\phi\phi r,r}\cot\phi-2\tau_{r\phi\phi\phi,r}-\tau_{\theta\theta\phi r,r}-\tau_{\phi\phi\phi r,r}+2\tau_{r\phi\phi r,r\phi}+2\tau_{r\theta\phi r,r\theta}\csc\phi\Big)$$

$$+\tau_{rr\phi r,rr},$$

$$\bar{\tau}_{\phi\theta}=\tau_{\phi\theta}-\Bigg(\tau_{\phi r\theta,r}+\frac{1}{r}\tau_{\phi\phi\theta,\phi}+\frac{1}{r\sin\phi}\tau_{\phi\theta\theta,\theta}+\frac{1}{r}\big(3\tau_{r\phi\theta}+\tau_{\phi\theta r}+\big(\tau_{\phi\theta\phi}+\tau_{\phi\phi\theta}$$

$$-\tau_{\theta\theta\theta}\big)\cot\phi\big)\Bigg)+\frac{1}{r^2}\Big(6\tau_{rr\phi\theta}-6\tau_{r\theta\theta\theta}\cot\phi+6\tau_{r\theta\phi r}+6\tau_{r\theta\phi\phi}\cot\phi+6\tau_{r\phi\phi\theta}\cot\phi$$

$$-2\tau_{\theta\theta\theta r}\cot\phi-2\tau_{\theta\theta\theta\phi}\cot^2\phi-3\tau_{\theta\theta\phi\theta}-6\tau_{\theta\theta\phi\theta}\cot^2\phi+2\tau_{\theta\theta\phi\theta}\csc^2\phi+4\tau_{\theta\phi\phi r}\cot\phi$$

$$-2\tau_{\theta\phi\phi\phi}+2\tau_{\theta\phi\phi\phi}\cot^2\phi-3\tau_{\phi\phi\phi\theta}+6\tau_{r\phi\phi\theta,\phi}-3\tau_{\theta\theta\phi\theta,\phi}\cot\phi+2\tau_{\theta\phi\phi r,\phi}$$

$$+2\tau_{\theta\phi\phi\phi,\phi}\cot\phi+2\tau_{\phi\phi\phi\theta,\phi}\cot\phi+\tau_{\phi\phi\phi\theta,\phi\phi}+6\tau_{r\theta\phi\theta,\theta}\csc\phi-2\tau_{\theta\theta\theta\theta,\theta}\cot\phi\csc\phi$$

$$+2\tau_{\theta\theta\phi r,\theta}\csc\phi+2\tau_{\theta\theta\phi\phi,\theta}\cot\phi\csc\phi+2\tau_{\theta\phi\phi\theta,\theta}\cot\phi\csc\phi+2\tau_{\theta\phi\phi\theta,\theta\phi}\csc\phi$$

$$+\tau_{\theta\theta\phi\theta,\theta\theta}\csc^2\phi\Big)+\frac{1}{r}\Big(6\tau_{rr\phi\theta,r}-2\tau_{r\theta\theta\theta,r}\cot\phi+2\tau_{r\theta\phi r,r}+2\tau_{r\theta\phi\phi,r}\cot\phi$$

$$+2\tau_{r\phi\phi\theta,r}\cot\phi-\tau_{\theta\theta\phi\theta,r}-\tau_{\phi\phi\phi\theta,r}+2\tau_{r\phi\phi\theta,r\phi}+2\tau_{r\theta\phi\theta,r\theta}\csc\phi\Big)+\tau_{rr\phi\theta,rr},$$

$$\bar{\tau}_{\phi\phi}=\tau_{\phi\phi}-\Bigg(\tau_{\phi r\phi,r}+\frac{1}{r}\tau_{\phi\phi\phi,\phi}+\frac{1}{r\sin\phi}\tau_{\phi\theta\phi,\theta}+\frac{1}{r}\big(3\tau_{r\phi\phi}+\tau_{\phi\phi r}+\big(\tau_{\phi\phi\phi}-\tau_{\theta\theta\phi}$$

$$- \tau_{\phi\theta\theta}) \cot\phi)\Big) + \frac{1}{r^2}\Big(6\tau_{rr\phi\phi} - 6\tau_{r\theta\theta\phi}\cot\phi - 6\tau_{r\theta\phi\theta}\cot\phi + 6\tau_{r\phi\phi r} + 6\tau_{r\phi\phi\phi}\cot\phi$$

$$+ 2\tau_{\theta\theta\theta\theta}\cot^2\phi - 4\tau_{\theta\theta\phi r}\cot\phi - 2\tau_{\theta\theta\phi\phi} - 6\tau_{\theta\theta\phi\phi}\cot^2\phi + 2\tau_{\theta\theta\phi\phi}\csc^2\phi - \tau_{\theta\phi\phi\theta}$$

$$- 3\tau_{\theta\phi\phi\theta}\cot^2\phi + \tau_{\theta\phi\phi\theta}\csc^2\phi + 2\tau_{\phi\phi\phi r}\cot\phi - 3\tau_{\phi\phi\phi\phi} + \tau_{\phi\phi\phi\phi}\cot^2\phi - \tau_{\phi\phi\phi\phi}\csc^2\phi$$

$$+ 6\tau_{r\phi\phi\phi,\phi} - 3\tau_{\theta\theta\phi\phi,\phi}\cot\phi - 2\tau_{\theta\phi\phi\theta,\phi}\cot\phi + 2\tau_{\phi\phi\phi r,\phi} + 2\tau_{\phi\phi\phi\phi,\phi}\cot\phi + \tau_{\phi\phi\phi\phi,\phi\phi}$$

$$+ 6\tau_{r\theta\phi\phi,\theta}\csc\phi - 2\tau_{\theta\theta\theta\phi,\theta}\cot\phi\csc\phi - 2\tau_{\theta\theta\phi\theta,\theta}\cot\phi\csc\phi + 2\tau_{\theta\phi\phi r,\theta}\csc\phi$$

$$+ 2\tau_{\theta\phi\phi\phi,\theta}\cot\phi\csc\phi + 2\tau_{\theta\phi\phi\phi,\theta\phi}\csc\phi + \csc^2\phi\,\tau_{\theta\theta\phi\phi,\theta\theta}\Big) + \frac{1}{r}\Big(6\tau_{rr\phi\phi,r}$$

$$- 2\tau_{r\theta\theta\phi,r}\cot\phi - 2\tau_{r\theta\phi\theta,r}\cot\phi + 2\tau_{r\phi\phi r,r} + 2\tau_{r\phi\phi\phi,r}\cot\phi - \tau_{\theta\theta\phi\phi,r} - \tau_{\phi\phi\phi\phi,r}$$

$$+ 2\tau_{r\phi\phi\phi,r\phi} + 2\tau_{r\theta\phi\phi,r\theta}\csc\phi\Big) + \tau_{rr\phi\phi,rr}.$$

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