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Symmetric Properties of Nested Magic Squares

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Abstract

*The idea of **nested magic squares** is well known in the literature, generally known by **bordered magic squares**. In this work, **nested magic squares** are studied for the consecutive natural numbers for the orders 5 to 25. Properties like, sub-magic squares sums, total entries sums, borders entries sums, etc. are studies. Final results lead us to symmetric properties. The **nested magic squares** for the consecutive odd numbers entries refer author's another work [12].*

Contents

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1 Introduction

It is well-known that sum of **positive natural numbers** and **odd numbers series** are given by

$$N_n := 1+2+3+\ldots+n = \frac{n(n+1)}{2}, n \in N_+ \tag{1}$$

and

$$F_n := 1+3+5+\ldots+(2n-1) = n^2, n \in N_+ \tag{2}$$

Since, we are working with magic squares, let's write $n = k^2$. This gives

$$N_{k^2} := 1+2+3+\ldots+k^2 = \frac{k^2(k^2+1)}{2}, n \in N_+ \tag{3}$$

and

$$F_{k^2} = F(k) := 1+3+5+\ldots+(2k^2-1) = k^4, k \in N_+ \tag{4}$$

Since k is the order of magic square, then the magic square sum S and total entries for positive natural numbers are

$$S_k := \frac{k(k^2+1)}{2}, n \in N_+ \tag{5}$$

and

$$T_{k^2} := \frac{k^2(k^2+1)}{2}, n \in N_+ \tag{6}$$

The number k is the order of the magic square. Thus, for all $k \geq 3$, we can always have a consecutive odd numbers magic square with sum entries a perfect square.

Let's write last member of (2) as m, then $n = \frac{m+1}{2}$. This gives

$$F(m) := 1+3+5+\ldots+m = \left(\frac{m+1}{2}\right)^2,\ m \in N_+ \tag{7}$$

The magic square sum of order n is given

$$S_{n\times n} := \frac{n(n^2+1)}{2}. \tag{8}$$

The sum of all entries of a magic square of order n is given by

$$T_{n\times n} := \frac{n^2 \times \left(n^2+1\right)}{2}. \tag{9}$$

There are many types of **bordered magic squares**. Here we are working with bordered magic squares, where there are sub-magic squares one contained in another. Due to this we called this category as **nested magic squares**. For study in this direction, refer to [1, 2, 3, 4, 5, 6, 7, 10, 11].

Recently, Walkington [6] gave an idea of **concentric rings**. The **concentric rings** works with **difference among consecutive magic squares**. For simplicity, let's call it **borders** of **nested magic squares**. In this paper, we shall work with **nested magic squares** of orders 5 to 25. It includes the results even and odd orders nested magic squares. Some properties of these nested magic squares are studied Interestingly, these properties turns out to be symmetric. In [12], the author worked with consecutive odd numbers entries in nested magic squares.

2 Nested Magic Squares

This section brings study and properties of nested magic squares of orders 5 to 25. These are studied in subsections below in decreasing orders. The first two nested magic squares of orders 25 and 24 are derived from the site by H. White [1].

2.1 Nested Magic Squares of Orders 25

Let's consider following two Nested Magic Square of order 25 obtained from the site by H. White [1].

Example 2.1. *Nested magic square of order 25 with consecutive natural numbers,* $\{1,2,3,.....,624,625\}$ *is given by*

																									7825
26	625	623	621	619	617	615	613	611	609	607	605	25	27	29	31	33	35	37	39	41	43	45	47	602	7825
48	556	91	89	87	85	83	81	79	77	75	73	71	559	561	563	565	567	569	571	573	575	577	72	578	7825
46	534	512	495	497	499	501	503	505	507	509	511	513	109	107	105	103	101	99	97	95	93	112	92	580	7825
44	536	94	476	167	165	163	161	159	157	155	153	151	479	481	483	485	487	489	491	493	152	532	90	582	7825
42	538	96	458	442	199	197	195	193	191	189	187	185	445	447	449	451	453	455	457	186	168	530	88	584	7825
40	540	98	460	426	412	398	400	402	404	406	408	213	212	210	208	206	204	202	410	200	166	528	86	586	7825
38	542	100	462	428	227	240	229	231	233	235	237	385	383	381	379	377	375	384	399	198	164	526	84	588	7825
36	544	102	464	430	225	396	262	253	255	257	259	363	361	359	357	355	362	230	401	196	162	524	82	590	7825
34	546	104	466	432	223	394	372	344	274	276	278	279	342	340	338	346	254	232	403	194	160	522	80	592	7825
32	548	106	468	434	221	392	370	339	330	290	292	293	328	326	332	287	256	234	405	192	158	520	78	594	7825
30	550	108	470	436	219	390	368	341	327	322	318	303	302	320	299	285	258	236	407	190	156	518	76	596	7825
28	552	110	472	438	217	388	366	343	329	307	310	315	314	319	297	283	260	238	409	188	154	516	74	598	7825
603	69	111	149	183	215	387	365	345	331	305	317	313	309	321	295	281	261	239	411	443	477	515	557	23	7825
604	68	510	148	182	415	244	266	277	291	325	312	311	316	301	335	349	360	382	211	444	478	116	558	22	7825
606	66	508	146	180	417	246	268	275	289	306	308	323	324	304	337	351	358	380	209	446	480	118	560	20	7825
608	64	506	144	178	419	248	270	273	294	336	334	333	298	300	296	353	356	378	207	448	482	120	562	18	7825
610	62	504	142	176	421	250	272	280	352	350	348	347	284	286	288	282	354	376	205	450	484	122	564	16	7825
612	60	502	140	174	423	252	264	373	371	369	367	263	265	267	269	271	364	374	203	452	486	124	566	14	7825
614	58	500	138	172	425	242	397	395	393	391	389	241	243	245	247	249	251	386	201	454	488	126	568	12	7825
616	56	498	136	170	216	228	226	224	222	220	218	413	414	416	418	420	422	424	214	456	490	128	570	10	7825
618	54	496	134	440	427	429	431	433	435	437	439	441	181	179	177	175	173	171	169	184	492	130	572	8	7825
620	52	494	474	459	461	463	465	467	469	471	473	475	147	145	143	141	139	137	135	133	150	132	574	6	7825
622	50	514	131	129	127	125	123	121	119	117	115	113	517	519	521	523	525	527	529	531	533	114	576	4	7825
624	554	535	537	539	541	543	545	547	549	551	553	555	67	65	63	61	59	57	55	53	51	49	70	2	7825
24	1	3	5	7	9	11	13	15	17	19	21	601	599	597	595	593	591	589	587	585	583	581	579	600	7825
7825	7825	7825	7825	7825	7825	7825	7825	7825	7825	7825	7825	7825	7825	7825	7825	7825	7825	7825	7825	7825	7825	7825	7825	7825	7825

Above we have a **nested magic square** of order 25. There are total 625 numbers, and the middle number is 313. It is as a central point of a magic square and remains at the center of first magic square of order 3. It has total 9 members, the there are four each side, i.e., {*309, 310, 311, 312,* **313**, *314, 315, 316, 317*}. Based on this philosophy, the other magic squares are constructed. See below the distributions in each case:

Distribution 2.1. *In each case, the **nested magic square** of order 25 is formed by the following entries:*

$D_{3\times3} := \{309, 310, 311, 312, \mathbf{313}, 314, 315, 316, 317\}$; $0+9+0$; *Total 9 entries*

$D_{5\times5} := \{301, 302, ..., 307, 308, \mathbf{D_{3\times3}}, 318, 319, ..., 324, 325\}$; $8+9+8$; *Total 25 entries*

$D_{7\times 7} := \{289, 290, ..., 299, 300, \mathbf{D_{5\times 5}}, 326, 327, ..., 336, 337\}$; $12+25+12$; *Total 49 entries*
$D_{9\times 9} := \{273, 274, ..., 287, 288, \mathbf{D_{7\times 7}}, 338, 339, ..., 352, 353\}$; $16+49+16$; *Total 81 entries*
$D_{11\times 11} := \{253, 254, ..., 271, 272, \mathbf{D_{9\times 9}}, 354, 355, ..., 372, 373\}$; $20+81+20$; *Total 121 entries*
$D_{13\times 13} := \{229, 230, ..., 251, 252, \mathbf{D_{11\times 11}}, 374, 375, ..., 396, 397\}$; $24+121+24$; *Total 169 entries*
$D_{15\times 15} := \{201, 202, ..., 227, 228, \mathbf{D_{13\times 13}}, 398, 399, ..., 424, 425\}$; $28+169+28$; *Total 225 entries*
$D_{17\times 17} := \{169, 170, ..., 199, 200, \mathbf{D_{15\times 15}}, 426, 427, ..., 456, 457\}$; $32+225+32$; *Total 289 entries*
$D_{19\times 19} := \{133, 134, ..., 167, 168, \mathbf{D_{17\times 17}}, 458, 459, ..., 492, 493\}$; $36+289+36$; *Total 361 entries*
$D_{21\times 21} := \{93, 94, ..., 131, 132, \mathbf{D_{19\times 19}}, 494, 495, ..., 532, 533\}$; $40+361+40$; *Total 441 entries*
$D_{23\times 23} := \{49, 50, ..., 91, 92, \mathbf{D_{21\times 21}}, 534, 535, ..., 576, 577\}$; $44+441+44$; *Total 529 entries*
$D_{25\times 25} := \{1, 2, ..., 47, 48, \mathbf{D_{23\times 23}}, 578, 579, ..., 624, 625\}$; $48+529+48$; *Total 625 entries*

According to above distributions of magic squares entries, we have the following result.

Result 2.1. *The **nested magic square** of order 25 has the following properties:*

i. *The magic square sums are given by*

$$S_{3\times 3} := 939 = 3\times 313$$
$$S_{5\times 5} := 1565 = 5\times 313$$
$$S_{7\times 7} := 2191 = 7\times 313$$
$$S_{9\times 9} := 2817 = 9\times 313$$
$$S_{11\times 11} := 3443 = 11\times 313$$
$$S_{13\times 13} := 4069 = 13\times 313$$
$$S_{15\times 15} := 4695 = 15\times 313$$
$$S_{17\times 17} := 5321 = 17\times 313$$
$$S_{19\times 19} := 5947 = 19\times 313$$
$$S_{21\times 21} := 6573 = 21\times 313$$
$$S_{23\times 23} := 7199 = 23\times 313$$
$$S_{25\times 25} := 7825 = 25\times 313.$$

ii. *The total entries sums are given by*

$$T_{3\times 3} := 3\times 939 = 2817 = 3^2\times 313$$
$$T_{5\times 5} := 5\times 1565 = 7825 = 5^2\times 313$$
$$T_{7\times 7} := 7\times 2191 = 15337 = 7^2\times 313$$
$$T_{9\times 9} := 9\times 2817 = 25353 = 9^2\times 313$$
$$T_{11\times 11} := 11\times 3443 = 37873 = 11^2\times 313$$
$$T_{13\times 13} := 13\times 4069 = 52897 = 13^2\times 313$$
$$T_{15\times 15} := 15\times 4695 = 70425 = 15^2\times 313$$
$$T_{17\times 17} := 17\times 5321 = 90457 = 17^2\times 313$$
$$T_{19\times 19} := 19\times 5947 = 112993 = 19^2\times 313$$
$$T_{21\times 21} := 21\times 6573 = 138033 = 21^2\times 313$$
$$T_{23\times 23} := 23\times 7199 = 165577 = 23^2\times 313$$
$$T_{25\times 25} := 25\times 7825 = 195625 = 25^2\times 313$$

iii. *The **borders** entries sums for the **nested magic square** of order 25 are given by*

$$C_{25\times 25} := T_{25\times 25} - T_{23\times 23} = 4\times (S_{25\times 25} - S_{23\times 23}) = 30048 = 12\times 8\times 313$$

$$
\begin{aligned}
C_{23\times23} &:= T_{23\times23} - T_{21\times21} = 4\times(S_{23\times23} - S_{21\times21}) = 27544 = 11\times8\times313\\
C_{21\times21} &:= T_{21\times21} - T_{19\times19} = 4\times(S_{21\times21} - S_{19\times19}) = 25040 = 10\times8\times313\\
C_{19\times19} &:= T_{19\times19} - T_{17\times17} = 4\times(S_{19\times19} - S_{17\times17}) = 22536 = 9\times8\times313\\
C_{17\times17} &:= T_{17\times17} - T_{15\times15} = 4\times(S_{17\times17} - S_{15\times15}) = 20032 = 8\times8\times313\\
C_{15\times15} &:= T_{15\times15} - T_{13\times13} = 4\times(S_{15\times15} - S_{13\times13}) = 17528 = 7\times8\times313\\
C_{13\times13} &:= T_{13\times13} - T_{11\times11} = 4\times(S_{11\times11} - S_{9\times9}) = 15024 = 6\times8\times313\\
C_{11\times11} &:= T_{11\times11} - T_{9\times9} = 4\times(S_{9\times9} - S_{7\times7}) = 12520 = 5\times8\times313\\
C_{9\times9} &:= T_{9\times9} - T_{7\times7} = 4\times(S_{9\times9} - S_{7\times7}) = 10016 = 4\times8\times313\\
C_{7\times7} &:= T_{7\times7} - T_{5\times5} = 4\times(S_{7\times7} - S_{5\times5}) = 7512 = 3\times8\times313\\
C_{5\times5} &:= T_{5\times5} - T_{3\times3} = 4\times(S_{5\times5} - S_{3\times3}) = 5008 = 2\times8\times313\\
C_{3\times3} &:= T_{3\times3} - T_{1\times1} = 4\times(S_{3\times3} - S_{1\times1}) = 2504 = 1\times8\times313,
\end{aligned}
$$

*where $T_{1\times1} = S_{1\times1} = 313$ is the central value. In this case, the fixed difference among the **consecutive borders** is $d_{border} := 2504 = 8\times313$.*

Finally, we get the following symmetric results

Result 2.2. *According to Result 2.1, the **nested magic square** of order 25 for the consecutive entries 1 to 625, has the following symmetric results:*

i. $S_{k\times k} := k\times T_{1\times1}$;

ii. $T_{k\times k} := k^2\times T_{1\times1}$;

iii. $C_{k\times k} := \dfrac{k-1}{2}\times8\times T_{1\times1}$.

iv. $d_{border} := 8\times T_{1\times1}$.

*where $k = 3,5,7,\ldots\ 21,23$ and 25 orders of magic squares appearing **nested magic square** of order 25, and $T_{1\times1} := 313$ is the central value of the magic square.*

2.2 Nested Magic Square of Order 24

Let's consider following two Nested Magic Square of order 24 obtained from the site by H. White [1].

Example 2.2. *Nested magic square of order 24 is given by*

																								6924
554	34	544	32	546	30	548	28	27	551	552	565	542	22	21	557	19	559	17	561	15	563	13	24	6924
575	67	519	59	517	61	515	63	513	65	511	47	520	69	507	71	505	73	503	75	501	77	509	2	6924
3	79	469	461	115	463	113	465	466	110	109	98	117	471	472	104	474	102	476	100	478	107	498	574	6924
573	497	118	433	151	427	149	429	147	431	145	442	127	435	141	437	139	439	137	441	143	459	80	4	6924
5	81	458	424	401	395	181	397	398	178	177	168	183	403	404	172	406	170	408	175	153	119	496	572	6924
571	495	120	154	184	373	209	369	207	371	205	380	191	375	201	377	199	379	203	393	423	457	82	6	6924
7	83	456	422	392	366	349	345	346	230	229	222	233	351	352	224	354	227	211	185	155	121	494	570	6924
569	493	122	156	186	212	234	329	324	254	322	256	252	242	336	240	330	343	365	391	421	455	84	8	6924
9	85	454	420	390	364	342	251	314	319	259	257	270	308	268	313	326	235	213	187	157	123	492	568	6924
567	491	124	158	188	214	236	327	311	301	277	304	271	303	275	266	250	341	363	389	419	453	86	10	6924
11	87	452	418	388	362	340	249	312	298	294	281	284	295	279	265	328	237	215	189	159	125	490	566	6924
1	489	126	160	190	216	238	334	317	280	287	292	289	286	297	260	243	339	361	387	417	451	88	576	6924
46	499	89	152	161	210	217	239	267	278	291	288	285	290	299	310	338	360	367	416	425	488	78	531	6924
532	521	97	134	167	196	221	331	262	272	282	293	296	283	305	315	246	356	381	410	443	480	56	45	6924
44	55	481	444	411	382	357	245	261	302	300	273	306	274	276	316	332	220	195	166	133	96	522	533	6924
534	523	95	132	165	194	219	333	264	258	318	320	307	269	309	263	244	358	383	412	445	482	54	43	6924
42	53	483	446	413	384	359	247	253	323	255	321	325	335	241	337	248	218	193	164	131	94	524	535	6924
536	525	93	130	163	192	350	232	231	347	348	355	344	226	225	353	223	228	385	414	447	484	52	41	6924
40	51	485	448	415	374	368	208	370	206	372	197	386	202	376	200	378	198	204	162	129	92	526	537	6924
538	527	91	128	402	182	396	180	179	399	400	409	394	174	173	405	171	407	169	176	449	486	50	39	6924
38	49	487	434	426	150	428	148	430	146	432	135	450	142	436	140	438	138	440	136	144	90	528	539	6924
540	529	470	116	462	114	464	112	111	467	468	479	460	106	105	473	103	475	101	477	99	108	48	37	6924
36	68	58	518	60	516	62	514	64	512	66	530	57	508	70	506	72	504	74	502	76	500	510	541	6924
553	543	33	545	31	547	29	549	550	26	25	12	35	555	556	20	558	18	560	16	562	14	564	23	6924
6924	6924	6924	6924	6924	6924	6924	6924	6924	6924	6924	6924	6924	6924	6924	6924	6924	6924	6924	6924	6924	6924	6924	6924	6924

Above we have a **nested magic square** of order 24. There are total 576 numbers. In this case we don't have a middle number. In order to start with magic square of order 4, let's consider 16 numbers in the middle. This means we have 280 numbers each side. Thus the magic square of order 4 starts from 281 and goes up to 296, i.e., *{281, 282, ..., 295, 296}*. Based on this philosophy the other magic squares are constructed. See below the distributions in each case:

Distribution 2.2. *In each case, the **nested magic square** of order 24 is formed by the following entries:*

$D_{4\times4} := \{281, 282, ..., 295, 296\};$ $0+16+0;$ *Total 16 entries*

$D_{6\times6} := \{271, 272, ..., 279, 280, \mathbf{D_{4\times4}}, 297, 298, ..., 305, 306\};$ $10+16+10;$ *Total 36 entries*

$$D_{8\times8} := \{257, 258, ..., 269, 270, \mathbf{D_{6\times6}}, 307, 308, ..., 319, 320\}; \quad 14+36+14; \quad \textit{Total 64 entries}$$
$$D_{10\times10} := \{239, 240, ..., 255, 256, \mathbf{D_{8\times8}}, 321, 322, ..., 317, 338\}; \quad 18+64+18; \quad \textit{Total 100 entries}$$
$$D_{12\times12} := \{217, 218, ..., 265, 266, \mathbf{D_{10\times10}}, 339, 340, ..., 359, 360\}; \quad 22+100+22; \quad \textit{Total 144 entries}$$
$$D_{14\times14} := \{191, 192, ..., 215, 216, \mathbf{D_{12\times12}}, 361, 375, ..., 385, 386\}; \quad 26+144+26; \quad \textit{Total 196 entries}$$
$$D_{16\times16} := \{161, 162, ..., 189, 190, \mathbf{D_{14\times14}}, 387, 388, ..., 415, 416\}; \quad 30+196+30; \quad \textit{Total 256 entries}$$
$$D_{18\times18} := \{127, 128, ..., 159, 160, \mathbf{D_{16\times16}}, 417, 418, ..., 449, 450\}; \quad 34+256+34; \quad \textit{Total 324 entries}$$
$$D_{20\times20} := \{89, 90, ..., 125, 126, \mathbf{D_{18\times18}}, 451, 452, ..., 487, 488\}; \quad 38+324+38; \quad \textit{Total 400 entries}$$
$$D_{22\times22} := \{47, 48, ..., 87, 88, \mathbf{D_{20\times20}}, 489, 490, ..., 531, 530\}; \quad 42+400+42; \quad \textit{Total 484 entries}$$
$$D_{24\times24} := \{1, 2, ..., 45, 46, \mathbf{D_{22\times22}}, 531, 532, ..., 575, 576\}; \quad 46+484+46; \quad \textit{Total 576 entries}$$

According to above distribution of magic squares entries, we have the following properties written as a result.

Result 2.3. *The **nested magic square** of order 24 has the following properties:*

i. The magic square sums are given by

$$S_{4\times4} := 1154 = 2\times577$$
$$S_{6\times6} := 1731 = 3\times577$$
$$S_{8\times8} := 2308 = 4\times577$$
$$S_{10\times10} := 2885 = 5\times577$$
$$S_{12\times12} := 3462 = 6\times577$$
$$S_{14\times14} := 4039 = 7\times577$$
$$S_{16\times16} := 4616 = 8\times577$$
$$S_{18\times18} := 5193 = 9\times577$$
$$S_{20\times20} := 5770 = 10\times577$$
$$S_{22\times22} := 6347 = 11\times577$$
$$S_{24\times24} := 6924 = 12\times577$$

ii. The sum of entries are given by

$$T_{4\times4} := 4\times1154 = 4616 = 2^2\times2\times577$$
$$T_{6\times6} := 6\times1731 = 10386 = 3^2\times2\times577$$
$$T_{8\times8} := 8\times2308 = 18464 = 4^2\times2\times577$$
$$T_{10\times10} := 10\times2885 = 28850 = 5^2\times2\times577$$
$$T_{12\times12} := 12\times3462 = 41544 = 6^2\times2\times577$$
$$T_{14\times14} := 14\times4039 = 56546 = 7^2\times2\times577$$
$$T_{16\times16} := 16\times4616 = 73856 = 8^2\times2\times577$$
$$T_{18\times18} := 18\times5193 = 93474 = 9^2\times2\times577$$
$$T_{20\times20} := 20\times5770 = 115400 = 10^2\times2\times577$$
$$T_{22\times22} := 22\times6347 = 139634 = 11^2\times2\times577$$
$$T_{24\times24} := 24\times6924 = 166176 = 12^2\times2\times577$$

*iii. The **borders** entries sums are given by*

$$C_{24\times24} := T_{24\times24} - T_{22\times22} = 4\times(S_{24\times24} - S_{22\times22}) = 26542 = 23\times2\times577$$

$$C_{22\times22} := T_{22\times22} - T_{20\times20} = 4\times(S_{22\times22} - S_{20\times20}) = 24234 = 21\times2\times577$$
$$C_{20\times20} := T_{20\times20} - T_{18\times18} = 4\times(S_{20\times20} - S_{18\times18}) = 21926 = 19\times2\times577$$
$$C_{18\times18} := T_{18\times18} - T_{16\times16} = 4\times(S_{18\times18} - S_{16\times16}) = 19618 = 17\times2\times577$$
$$C_{16\times16} := T_{16\times16} - T_{14\times14} = 4\times(S_{16\times16} - S_{14\times14}) = 17310 = 15\times2\times577$$
$$C_{14\times14} := T_{14\times14} - T_{12\times12} = 4\times(S_{14\times14} - S_{12\times12}) = 15002 = 13\times2\times577$$
$$C_{12\times12} := T_{12\times12} - T_{10\times10} = 4\times(S_{12\times12} - S_{10\times10}) = 12694 = 11\times2\times577$$
$$C_{10\times10} := T_{10\times10} - T_{8\times8} = 4\times(S_{10\times10} - S_{8\times8}) = 10386 = 9\times2\times577$$
$$C_{8\times8} := T_{8\times8} - T_{6\times6} = 4\times(S_{8\times8} - S_{6\times6}) = 8078 = 7\times2\times577$$
$$C_{6\times6} := T_{6\times6} - T_{4\times4} = 4\times(S_{6\times6} - S_{4\times4}) = 5770 = 5\times2\times577$$
$$C_{4\times4} := T_{4\times4} - T_{2\times2} = 3462 = 3\times2\times577,$$

where $T_{2\times2} = 285+288+289+292 = 1154 = 2\times577$ *are four central or middle values of* ***nested magic square****. In this case, the fixed difference among the* ***consecutive borders*** *is* $d_{border} := 2308 = 2\times2\times577$*. The expression* $S_{2\times2}$ *is taken out as we don't have magic square of order 2.*

Finally, we get the following symmetric results

Result 2.4. *According to Result 2.3, the* ***nested magic square*** *of order 24 for the consecutive entries 1 to 576, has the following symmetric results:*

i. $S_{k\times k} := \frac{k}{2}\times\frac{T_{2\times2}}{2}$;

ii. $T_{k\times k} := \left(\frac{k}{2}\right)^2\times T_{2\times2}$;

iii. $C_{k\times k} := (k-1)\times T_{2\times2}$.

iv. $d_{border} := 2\times T_{2\times2}$.

where $k = 4, 6,$ *and* $22, 24$ *orders of magic squares appearing* ***nested magic square*** *of order 24, and* $T_{2\times2} := 1154$ *is sum of four central values of magic square.*

The magic squares given in Examples 2.1 and 2.2 of orders 25 and 24 are of consecutive natural numbers, and are derived from H. White's site [1]. Based on above two examples 2.1 and 2.2, we shall obtain lower order **nested magic square**s up to order 4.

2.3 Nested Magic Square of Order 23

In Example 2.1 remove the external border, then we are left with **nested magic square** of order 23 for the entries 49 to 577. Subtracting 48, we get following distribution for the **nested magic square** of order 23 for the consecutive natural numbers 1 to 529.

Distribution 2.3.

$$
\begin{aligned}
D_{3\times3} &:= \{261,262,263,264,\mathbf{265},266,267,268,269\}\\
D_{5\times5} &:= \{253,254,...,259,260,\mathbf{D_{3\times3}},270,271,...,276,277\}\\
D_{7\times7} &:= \{241,242,...,251,252,\mathbf{D_{5\times5}},278,279,...,288,289\}\\
D_{9\times9} &:= \{225,226,...,239,240,\mathbf{D_{7\times7}},290,291,...,304,305\}\\
D_{11\times11} &:= \{205,206,...,223,224,\mathbf{D_{9\times9}},306,307,...,334,325\}\\
D_{13\times13} &:= \{181,182,...,203,204,\mathbf{D_{11\times11}},326,327,...,348,349\}\\
D_{15\times15} &:= \{153,154,...,179,180,\mathbf{D_{13\times13}},350,351,...,376,377\}\\
D_{17\times17} &:= \{121,122,...,151,152,\mathbf{D_{15\times15}},378,379,...,408,409\}\\
D_{19\times19} &:= \{85,86,...,119,120,\mathbf{D_{17\times17}},410,411,...,444,445\}\\
D_{21\times21} &:= \{45,46,...,83,84,\mathbf{D_{19\times19}},446,447,...,484,485\}\\
D_{23\times23} &:= \{1,2,...,43,44,\mathbf{D_{21\times21}},486,487,...,528,529\}
\end{aligned}
$$

According to above distribution, the **nested magic square** of order 23 for the consecutive natural numbers entries from 1 to 529 is given by

Example 2.3. *Nested magic square of order 23 is given by*

																							6095
508	43	41	39	37	35	33	31	29	27	25	23	511	513	515	517	519	521	523	525	527	529	24	6095
486	464	447	449	451	453	455	457	459	461	463	465	61	59	57	55	53	51	49	47	45	64	44	6095
488	46	428	119	117	115	113	111	109	107	105	103	431	433	435	437	439	441	443	445	104	484	42	6095
490	48	410	394	151	149	147	145	143	141	139	137	397	399	401	403	405	407	409	138	120	482	40	6095
492	50	412	378	364	350	352	354	356	358	360	165	164	162	160	158	156	154	362	152	118	480	38	6095
494	52	414	380	179	192	181	183	185	187	189	337	335	333	331	329	327	336	351	150	116	478	36	6095
496	54	416	382	177	348	214	205	207	209	211	315	313	311	309	307	314	182	353	148	114	476	34	6095
498	56	418	384	175	346	324	296	226	228	230	231	294	292	290	298	206	184	355	146	112	474	32	6095
500	58	420	386	173	344	322	291	282	242	244	245	280	278	284	239	208	186	357	144	110	472	30	6095
502	60	422	388	171	342	320	293	279	274	270	255	254	272	251	237	210	188	359	142	108	470	28	6095
504	62	424	390	169	340	318	295	281	259	262	267	266	271	249	235	212	190	361	140	106	468	26	6095
21	63	101	135	167	339	317	297	283	257	269	265	261	273	247	233	213	191	363	395	429	467	509	6095
20	462	100	134	367	196	218	229	243	277	264	263	268	253	287	301	312	334	163	396	430	68	510	6095
18	460	98	132	369	198	220	227	241	258	260	275	276	256	289	303	310	332	161	398	432	70	512	6095
16	458	96	130	371	200	222	225	246	288	286	285	250	252	248	305	308	330	159	400	434	72	514	6095
14	456	94	128	373	202	224	232	304	302	300	299	236	238	240	234	306	328	157	402	436	74	516	6095
12	454	92	126	375	204	216	325	323	321	319	215	217	219	221	223	316	326	155	404	438	76	518	6095
10	452	90	124	377	194	349	347	345	343	341	193	195	197	199	201	203	338	153	406	440	78	520	6095
8	450	88	122	168	180	178	176	174	172	170	365	366	368	370	372	374	376	166	408	442	80	522	6095
6	448	86	392	379	381	383	385	387	389	391	393	133	131	129	127	125	123	121	136	444	82	524	6095
4	446	426	411	413	415	417	419	421	423	425	427	99	97	95	93	91	89	87	85	102	84	526	6095
2	466	83	81	79	77	75	73	71	69	67	65	469	471	473	475	477	479	481	483	485	66	528	6095
506	487	489	491	493	495	497	499	501	503	505	507	19	17	15	13	11	9	7	5	3	1	22	6095
6095	6095	6095	6095	6095	6095	6095	6095	6095	6095	6095	6095	6095	6095	6095	6095	6095	6095	6095	6095	6095	6095	6095	6095

According to above distribution of magic squares entries, we have the following properties written as a result.

Result 2.5. *The **nested magic square** of order 23 for the entries 1 to 529 has the following properties:*

i. The magic square sums are given by

$$\begin{aligned}
S_{3\times3} &:= 795 = 3\times5\times53\\
S_{5\times5} &:= 1325 = 5\times5\times53\\
S_{7\times7} &:= 1855 = 7\times5\times53\\
S_{9\times9} &:= 2385 = 9\times5\times53\\
S_{11\times11} &:= 2915 = 11\times5\times53\\
S_{13\times13} &:= 3445 = 13\times5\times53
\end{aligned}
\qquad
\begin{aligned}
S_{15\times15} &:= 3975 = 15\times5\times53\\
S_{17\times17} &:= 4505 = 17\times5\times53\\
S_{19\times19} &:= 5035 = 19\times5\times53\\
S_{21\times21} &:= 5565 = 21\times5\times53\\
S_{23\times23} &:= 6095 = 23\times5\times53.
\end{aligned}$$

ii. *The total entries sums are given by*

$$\begin{aligned}
T_{3\times3} &:= 3\times795 = 2385 = 3^2\times5\times53\\
T_{5\times5} &:= 5\times1325 = 6625 = 5^2\times5\times53\\
T_{7\times7} &:= 7\times1855 = 12985 = 7^2\times5\times53\\
T_{9\times9} &:= 9\times2385 = 21465 = 9^2\times5\times53\\
T_{11\times11} &:= 11\times2915 = 32065 = 11^2\times5\times53\\
T_{13\times13} &:= 13\times3445 = 44785 = 13^2\times5\times53
\end{aligned}
\qquad
\begin{aligned}
T_{15\times15} &:= 15\times3975 = 50625 = 15^2\times5\times53\\
T_{17\times17} &:= 17\times4505 = 76585 = 17^2\times5\times53\\
T_{19\times19} &:= 19\times5035 = 95665 = 19^2\times5\times53\\
T_{21\times21} &:= 21\times5565 = 116865 = 21^2\times5\times53\\
T_{23\times23} &:= 23\times6095 = 140185 = 23^2\times5\times53
\end{aligned}$$

iii. *The* ***borders*** *entries sums for the* ***nested magic square*** *of order 23 are given by*

$$\begin{aligned}
C_{23\times23} &:= T_{23\times23} - T_{21\times21} = 4\times(S_{23\times23}-S_{21\times21}) = 11\times2^3\times5\times53\\
C_{21\times21} &:= T_{21\times21} - T_{19\times19} = 4\times(S_{21\times21}-S_{19\times19}) = 10\times2^3\times5\times53\\
C_{19\times19} &:= T_{19\times19} - T_{17\times17} = 4\times(S_{19\times19}-S_{17\times17}) = 9\times2^3\times5\times53\\
C_{17\times17} &:= T_{17\times17} - T_{15\times15} = 4\times(S_{17\times17}-S_{15\times15}) = 8\times2^3\times5\times53\\
C_{15\times15} &:= T_{15\times15} - T_{13\times13} = 4\times(S_{15\times15}-S_{13\times13}) = 7\times2^3\times5\times53\\
C_{13\times13} &:= T_{13\times13} - T_{11\times11} = 4\times(S_{13\times13}-S_{11\times11}) = 6\times2^3\times5\times53\\
C_{11\times11} &:= T_{11\times11} - T_{9\times9} = 4\times(S_{11\times11}-S_{9\times9}) = 5\times2^3\times5\times53\\
C_{9\times9} &:= T_{9\times9} - T_{7\times7} = 4\times(S_{9\times9}-S_{7\times7}) = 4\times2^3\times5\times53\\
C_{7\times7} &:= T_{7\times7} - T_{5\times5} = 4\times(S_{7\times7}-S_{5\times5}) = 3\times2^3\times5\times53\\
C_{5\times5} &:= T_{5\times5} - T_{3\times3} = 4\times(S_{5\times5}-S_{3\times3}) = 2\times2^3\times5\times53\\
C_{3\times3} &:= T_{3\times3} - T_{1\times1} = 4\times(S_{3\times3}-S_{1\times1}) = 1\times2^3\times5\times53
\end{aligned}$$

where $T_{1\times1} = S_{1\times1} = 265$ *is the central value. In this case, the fixed difference among the* ***consecutive borders*** *is* $d_{border} := 2120 = 2^3\times5\times53$.

Finally, we get the following symmetric results

Result 2.6. *According to Result 2.5, the **nested magic square** of order 23 for the consecutive entries 1 to 529, has the following symmetric results:*

i. $S_{k\times k} := k \times T_{1\times 1}$;

ii. $T_{k\times k} := k^2 \times T_{1\times 1}$;

iii. $C_{k\times k} := \frac{k-1}{2} \times 8 \times T_{1\times 1}$.

where $k = 3, 5, 7, \dots\ 21, 23$ *orders of squares appearing **nested magic square** of order 23, and* $T_{1\times 1} := 265$ *is the central value of the magic square.*

2.4 Nested Magic Square of Order 22

In example 2.2 if we remove the external border, we are left with **nested magic square** of order 22 for the entries 47 to 530. Subtracting 46, we get following distribution for the **nested magic square** of order 22 for the consecutive natural numbers 1 to 484.

Distribution 2.4.

$$
\begin{aligned}
D_{4\times 4} &:= \{235, 236, \dots, 249, 250\} \\
D_{6\times 6} &:= \{225, 226, \dots, 233, 234, \mathbf{D_{4\times 4}}, 251, 252, \dots, 259, 260\} \\
D_{8\times 8} &:= \{211, 212, \dots, 223, 224, \mathbf{D_{6\times 6}}, 261, 262, \dots, 273, 274\} \\
D_{10\times 10} &:= \{193, 194, \dots, 209, 210, \mathbf{D_{8\times 8}}, 275, 276, \dots, 291, 292\} \\
D_{12\times 12} &:= \{171, 172, \dots, 191, 192, \mathbf{D_{10\times 10}}, 293, 294, \dots, 313, 314\} \\
D_{14\times 14} &:= \{145, 146, \dots, 169, 170, \mathbf{D_{12\times 12}}, 315, 316, \dots, 339, 340\} \\
D_{16\times 16} &:= \{115, 116, \dots, 285, 287, \mathbf{D_{14\times 14}}, 341, 342, \dots, 369, 370\} \\
D_{18\times 18} &:= \{81, 82, \dots, 113, 114, \mathbf{D_{16\times 16}}, 371, 372, \dots, 403, 404\} \\
D_{20\times 20} &:= \{43, 44, \dots, 79, 80, \mathbf{D_{18\times 18}}, 405, 406, \dots, 441, 442\} \\
D_{22\times 22} &:= \{1, 3, \dots, 41, 42, \mathbf{D_{20\times 20}}, 443, 444, \dots, 483, 484\}
\end{aligned}
$$

According to above distribution, the **nested magic square** of order 22 for the consecutive natural numbers entries from 1 to 484 is given by

Example 2.4. *Nested magic square of order 22 is given by*

																						5335
21	473	13	471	15	469	17	467	19	465	1	474	23	461	25	459	27	457	29	455	31	463	5335
33	423	415	69	417	67	419	420	64	63	52	71	425	426	58	428	56	430	54	432	61	452	5335
451	72	387	105	381	103	383	101	385	99	396	81	389	95	391	93	393	91	395	97	413	34	5335
35	412	378	355	349	135	351	352	132	131	122	137	357	358	126	360	124	362	129	107	73	450	5335
449	74	108	138	327	163	323	161	325	159	334	145	329	155	331	153	333	157	347	377	411	36	5335
37	410	376	346	320	303	299	300	184	183	176	187	305	306	178	308	181	165	139	109	75	448	5335
447	76	110	140	166	188	283	278	208	276	210	206	196	290	194	284	297	319	345	375	409	38	5335
39	408	374	344	318	296	205	268	273	213	211	224	262	222	267	280	189	167	141	111	77	446	5335
445	78	112	142	168	190	281	265	255	231	258	225	257	229	220	204	295	317	343	373	407	40	5335
41	406	372	342	316	294	203	266	252	248	235	238	249	233	219	282	191	169	143	113	79	444	5335
443	80	114	144	170	192	288	271	234	241	246	243	240	251	214	197	293	315	341	371	405	42	5335
453	43	106	115	164	171	193	221	232	245	242	239	244	253	264	292	314	321	370	379	442	32	5335
475	51	88	121	150	175	285	216	226	236	247	250	237	259	269	200	310	335	364	397	434	10	5335
9	435	398	365	336	311	199	215	256	254	227	260	228	230	270	286	174	149	120	87	50	476	5335
477	49	86	119	148	173	287	218	212	272	274	261	223	263	217	198	312	337	366	399	436	8	5335
7	437	400	367	338	313	201	207	277	209	275	279	289	195	291	202	172	147	118	85	48	478	5335
479	47	84	117	146	304	186	185	301	302	309	298	180	179	307	177	182	339	368	401	438	6	5335
5	439	402	369	328	322	162	324	160	326	151	340	156	330	154	332	152	158	116	83	46	480	5335
481	45	82	356	136	350	134	133	353	354	363	348	128	127	359	125	361	123	130	403	440	4	5335
3	441	388	380	104	382	102	384	100	386	89	404	96	390	94	392	92	394	90	98	44	482	5335
483	424	70	416	68	418	66	65	421	422	433	414	60	59	427	57	429	55	431	53	62	2	5335
22	12	472	14	470	16	468	18	466	20	484	11	462	24	460	26	458	28	456	30	454	464	5335
5335	5335	5335	5335	5335	5335	5335	5335	5335	5335	5335	5335	5335	5335	5335	5335	5335	5335	5335	5335	5335	5335	5335

The **nested magic square** given in Example 2.4 or Distribution 2.4 satisfies some interesting properties summarized in a result below.

Result 2.7. *The **nested magic square** of order 22 has the following properties:*

i. The magic square sums are given by

$$\begin{aligned} S_{4\times4} &:= 970 = 2\times485 \\ S_{6\times6} &:= 1455 = 3\times485 \\ S_{8\times8} &:= 1940 = 4\times485 \\ S_{10\times10} &:= 2425 = 5\times485 \\ S_{12\times12} &:= 2910 = 6\times485 \end{aligned}$$

$$\begin{aligned} S_{14\times14} &:= 3395 = 7\times485 \\ S_{16\times16} &:= 3880 = 8\times485 \\ S_{18\times18} &:= 4365 = 9\times485 \\ S_{20\times20} &:= 4850 = 10\times485 \\ S_{22\times22} &:= 5335 = 11\times485 \end{aligned}$$

ii. *The sum of entries are given by*

$$\begin{aligned} T_{4\times4} &:= 4\times970 = 3880 = 2^2\times2\times485 \\ T_{6\times6} &:= 6\times1455 = 8730 = 3^2\times2\times485 \\ T_{8\times8} &:= 8\times1940 = 15520 = 4^2\times2\times485 \\ T_{10\times10} &:= 10\times2425 = 24250 = 5^2\times2\times485 \\ T_{12\times12} &:= 12\times2910 = 34920 = 6^2\times2\times485 \end{aligned}$$

$$\begin{aligned} T_{14\times14} &:= 14\times3395 = 47530 = 7^2\times2\times485 \\ T_{16\times16} &:= 16\times3880 = 62080 = 8^2\times2\times485 \\ T_{18\times18} &:= 18\times4365 = 78570 = 9^2\times2\times485 \\ T_{20\times20} &:= 20\times4850 = 97000 = 10^2\times2\times485 \\ T_{22\times22} &:= 22\times5335 = 117370 = 11^2\times2\times485 \end{aligned}$$

iii. *The **borders** entries sums are given by*

$$\begin{aligned} C_{22\times22} &:= T_{22\times22} - T_{20\times20} = 4\times(S_{22\times22} - S_{20\times20}) = 21\times2\times485 \\ C_{20\times20} &:= T_{20\times20} - T_{18\times18} = 4\times(S_{20\times20} - S_{18\times18}) = 19\times2\times485 \\ C_{18\times18} &:= T_{18\times18} - T_{16\times16} = 4\times(S_{18\times18} - S_{16\times16}) = 17\times2\times485 \\ C_{16\times16} &:= T_{16\times16} - T_{14\times14} = 4\times(S_{16\times16} - S_{14\times14}) = 15\times2\times485 \\ C_{14\times14} &:= T_{14\times14} - T_{12\times12} = 4\times(S_{14\times14} - S_{12\times12}) = 13\times2\times485 \\ C_{12\times12} &:= T_{12\times12} - T_{10\times10} = 4\times(S_{12\times12} - S_{10\times10}) = 11\times2\times485 \\ C_{10\times10} &:= T_{10\times10} - T_{8\times8} = 4\times(S_{10\times10} - S_{8\times8}) = 9\times2\times485 \\ C_{8\times8} &:= T_{8\times8} - T_{6\times6} = 4\times(S_{8\times8} - S_{6\times6}) = 7\times2\times485 \\ C_{6\times6} &:= T_{6\times6} - T_{4\times4} = 4\times(S_{6\times6} - S_{4\times4}) = 5\times2\times485 \\ C_{4\times4} &:= T_{4\times4} - T_{2\times2} = 3\times2\times485, \end{aligned}$$

where $T_{2\times2} = 241+242+243+244 = 970 = 2\times485$ *are four central or middle values of **nested magic square**. In this case, the fixed difference among the **consecutive borders** is* $d_{border} := 1940$. *The expression* $S_{2\times2}$ *is taken out as we don't have magic square of order 2.*

Finally, we get the following symmetric results

Result 2.8. *According to Result 2.7, the **nested magic square** of order 22 for the consecutive entries 1 to 484 has the following symmetric results:*

i. $S_{k\times k} := \frac{k}{2} \times \frac{T_{2\times 2}}{2}$;

ii. $T_{k\times k} := \left(\frac{k}{2}\right)^2 \times T_{2\times 2}$;

iii. $C_{k\times k} := (k-1) \times T_{2\times 2}$.

where $k = 4, 6,$ *and* $20, 22$ *orders of magic squares appearing* ***nested magic square*** *of order 22, and* $T_{2\times 2} := 970$ *is sum of four central values of magic square.*

2.5 Nested Magic Square of Order 21

In Example 2.3 remove the external border, then we are left with **nested magic square** of order 21 for the entries 45 to 485. Subtracting 44, we get following distribution for the **nested magic square** of order 21 for the consecutive natural numbers 1 to 441.

Distribution 2.5.

$$D_{3\times 3} := \{217, 218, \ldots, 224, 225\}$$
$$D_{5\times 5} := \{209, 210, \ldots, 215, 216, \mathbf{D_{3\times 3}}, 226, 227, \ldots, 232, 233\}$$
$$D_{7\times 7} := \{197, 198, \ldots, 208, 209, \mathbf{D_{5\times 5}}, 234, 235, \ldots, 244, 245\}$$
$$D_{9\times 9} := \{181, 182, \ldots, 195, 196, \mathbf{D_{7\times 7}}, 246, 247, \ldots, 260, 261\}$$
$$D_{11\times 11} := \{161, 162, \ldots, 179, 180, \mathbf{D_{9\times 9}}, 262, 263, \ldots, 280, 281\}$$
$$D_{13\times 13} := \{137, 138, \ldots, 159, 160, \mathbf{D_{11\times 11}}, 282, 283, \ldots, 304, 305\}$$
$$D_{15\times 15} := \{109, 110, \ldots, 135, 136, \mathbf{D_{13\times 13}}, 306, 307, \ldots, 332, 333\}$$
$$D_{17\times 17} := \{77, 78, \ldots, 107, 108, \mathbf{D_{15\times 15}}, 334, 335, \ldots, 364, 365\}$$
$$D_{19\times 19} := \{41, 42, \ldots, 75, 76, \mathbf{D_{17\times 17}}, 366, 367, \ldots, 400, 401\}$$
$$D_{21\times 21} := \{1, 2, \ldots, 39, 40, \mathbf{D_{19\times 19}}, 402, 403, \ldots, 440, 441\}.$$

According to above distribution, the **nested magic square** of order 21 for the consecutive natural numbers entries from 1 to 441 is given by

Example 2.5. *Nested magic square of order 21 is given by*

																					4641
420	403	405	407	409	411	413	415	417	419	421	17	15	13	11	9	7	5	3	1	20	4641
2	384	75	73	71	69	67	65	63	61	59	387	389	391	393	395	397	399	401	60	440	4641
4	366	350	107	105	103	101	99	97	95	93	353	355	357	359	361	363	365	94	76	438	4641
6	368	334	320	306	308	310	312	314	316	121	120	118	116	114	112	110	318	108	74	436	4641
8	370	336	135	148	137	139	141	143	145	293	291	289	287	285	283	292	307	106	72	434	4641
10	372	338	133	304	170	161	163	165	167	271	269	267	265	263	270	138	309	104	70	432	4641
12	374	340	131	302	280	252	182	184	186	187	250	248	246	254	162	140	311	102	68	430	4641
14	376	342	129	300	278	247	238	198	200	201	236	234	240	195	164	142	313	100	66	428	4641
16	378	344	127	298	276	249	235	230	226	211	210	228	207	193	166	144	315	98	64	426	4641
18	380	346	125	296	274	251	237	215	218	223	222	227	205	191	168	146	317	96	62	424	4641
19	57	91	123	295	273	253	239	213	225	221	217	229	203	189	169	147	319	351	385	423	4641
418	56	90	323	152	174	185	199	233	220	219	224	209	243	257	268	290	119	352	386	24	4641
416	54	88	325	154	176	183	197	214	216	231	232	212	245	259	266	288	117	354	388	26	4641
414	52	86	327	156	178	181	202	244	242	241	206	208	204	261	264	286	115	356	390	28	4641
412	50	84	329	158	180	188	260	258	256	255	192	194	196	190	262	284	113	358	392	30	4641
410	48	82	331	160	172	281	279	277	275	171	173	175	177	179	272	282	111	360	394	32	4641
408	46	80	333	150	305	303	301	299	297	149	151	153	155	157	159	294	109	362	396	34	4641
406	44	78	124	136	134	132	130	128	126	321	322	324	326	328	330	332	122	364	398	36	4641
404	42	348	335	337	339	341	343	345	347	349	89	87	85	83	81	79	77	92	400	38	4641
402	382	367	369	371	373	375	377	379	381	383	55	53	51	49	47	45	43	41	58	40	4641
422	39	37	35	33	31	29	27	25	23	21	425	427	429	431	433	435	437	439	441	22	4641
4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641

According to above distribution of magic squares entries, we have the following properties written as a result.

Result 2.9. *The* ***nested magic square*** *of order 21 for the entries 1 to 441 has the following properties:*

i. The magic square sums are given by

$$
\begin{aligned}
S_{3\times3} &:= 663 = 3\times13\times17 \\
S_{5\times5} &:= 1105 = 5\times13\times17 \\
S_{7\times7} &:= 1547 = 7\times13\times17 \\
S_{9\times9} &:= 1989 = 9\times13\times17 \\
S_{11\times11} &:= 2431 = 11\times13\times17
\end{aligned}
\qquad
\begin{aligned}
S_{13\times13} &:= 2873 = 13\times13\times17 \\
S_{15\times15} &:= 3315 = 15\times13\times17 \\
S_{17\times17} &:= 3757 = 17\times13\times17 \\
S_{19\times19} &:= 4199 = 19\times13\times17 \\
S_{21\times21} &:= 4641 = 21\times13\times17
\end{aligned}
$$

ii. *The total entries sums are given by*

$$
\begin{aligned}
T_{3\times3} &:= 3\times663 = 1989 = 3^2\times13\times17 \\
T_{5\times5} &:= 5\times1105 = 5525 = 5^2\times13\times17 \\
T_{7\times7} &:= 7\times1547 = 10829 = 7^2\times13\times17 \\
T_{9\times9} &:= 9\times1989 = 17901 = 9^2\times13\times17 \\
T_{11\times11} &:= 11\times2431 = 26741 = 11^2\times13\times17
\end{aligned}
\qquad
\begin{aligned}
T_{13\times13} &:= 13\times2873 = 37349 = 13^2\times13\times17 \\
T_{15\times15} &:= 15\times3315 = 49725 = 15^2\times13\times17 \\
T_{17\times17} &:= 17\times3757 = 63869 = 17^2\times13\times17 \\
T_{19\times19} &:= 19\times4199 = 79781 = 19^2\times13\times17 \\
T_{21\times21} &:= 21\times4641 = 97461 = 21^2\times13\times17
\end{aligned}
$$

iii. *The **borders** entries sums for the **nested magic square** of order 23 are given by*

$$
\begin{aligned}
C_{21\times21} &:= T_{21\times21} - T_{19\times19} = 4\times(S_{21\times21} - S_{19\times19}) = 10\times2^3\times13\times17 \\
C_{19\times19} &:= T_{19\times19} - T_{17\times17} = 4\times(S_{19\times19} - S_{17\times17}) = 9\times2^3\times13\times17 \\
C_{17\times17} &:= T_{17\times17} - T_{15\times15} = 4\times(S_{17\times17} - S_{15\times15}) = 8\times2^3\times13\times17 \\
C_{15\times15} &:= T_{15\times15} - T_{13\times13} = 4\times(S_{15\times15} - S_{13\times13}) = 7\times2^3\times13\times17 \\
C_{13\times13} &:= T_{13\times13} - T_{11\times11} = 4\times(S_{13\times13} - S_{11\times11}) = 6\times2^3\times13\times17 \\
C_{11\times11} &:= T_{11\times11} - T_{9\times9} = 4\times(S_{11\times11} - S_{9\times9}) = 5\times2^3\times13\times17 \\
C_{9\times9} &:= T_{9\times9} - T_{7\times7} = 4\times(S_{9\times9} - S_{7\times7}) = 4\times2^3\times13\times17 \\
C_{7\times7} &:= T_{7\times7} - T_{5\times5} = 4\times(S_{7\times7} - S_{5\times5}) = 3\times2^3\times13\times17 \\
C_{5\times5} &:= T_{5\times5} - T_{3\times3} = 4\times(S_{5\times5} - S_{3\times3}) = 2\times2^3\times13\times17 \\
C_{3\times3} &:= T_{3\times3} - T_{1\times1} = 4\times(S_{3\times3} - S_{1\times1}) = 1\times2^3\times13\times17
\end{aligned}
$$

*where $T_{1\times1} = S_{1\times1} = 221$ is the central value. In this case, the fixed difference among the **consecutive borders** is $d_{border} := 1768 = 2^3\times13\times17$.*

Finally, we get the following symmetric results

Result 2.10. *According to Result 2.9, the **nested magic square** of order 21 for the consecutive entries 1 to 441, has the following symmetric results:*

i. $S_{k\times k} := k \times T_{1\times 1}$;

ii. $T_{k\times k} := k^2 \times T_{1\times 1}$;

iii. $C_{k\times k} := \dfrac{k-1}{2} \times 8 \times T_{1\times 1}$.

where $k = 3,5,7,\ldots\ 19$ *and* 21 *orders of squares appearing* ***nested magic square*** *of order 21, and* $T_{1\times 1} := 221$ *is the central value of the magic square.*

2.6 Nested Magic Square of Order 20

In example 2.4 if we remove the external border, we are left with **nested magic square** of order 20 for the entries 43 to 443. Subtracting 42, we get following distribution for the **nested magic square** of order 20 for the consecutive natural numbers 1 to 400.

Distribution 2.6.

$$\begin{aligned}
D_{4\times 4} &:= \{193,194,\ldots,207,208\}\\
D_{6\times 6} &:= \{183,184,\ldots,191,192,\mathbf{D_{4\times 4}},209,210,\ldots,217,218\}\\
D_{8\times 8} &:= \{169,170,\ldots,181,182,\mathbf{D_{6\times 6}},219,220,\ldots,231,232\}\\
D_{10\times 10} &:= \{151,152,\ldots,167,168,\mathbf{D_{8\times 8}},233,234,\ldots,249,250\}\\
D_{12\times 12} &:= \{129,130,\ldots,149,150,\mathbf{D_{10\times 10}},251,252,\ldots,271,272\}\\
D_{14\times 14} &:= \{103,104,\ldots,127,128,\mathbf{D_{12\times 12}},273,274,\ldots,297,298\}\\
D_{16\times 16} &:= \{73,74,\ldots,101,102,\mathbf{D_{14\times 14}},299,300,\ldots,327,328\}\\
D_{18\times 18} &:= \{39,40,\ldots,71,72,\mathbf{D_{16\times 16}},329,330,\ldots,361,362\}\\
D_{20\times 20} &:= \{1,2,\ldots,37,38,\mathbf{D_{18\times 18}},363,364,\ldots,399,400\}
\end{aligned}$$

According to above distribution, the **nested magic square** of order 20 for the consecutive natural numbers entries from 1 to 400 is given by

Example 2.6. *Nested magic square of order 20 is given by*

																				4010
381	373	27	375	25	377	378	22	21	10	29	383	384	16	386	14	388	12	390	19	4010
30	345	63	339	61	341	59	343	57	354	39	347	53	349	51	351	49	353	55	371	4010
370	336	313	307	93	309	310	90	89	80	95	315	316	84	318	82	320	87	65	31	4010
32	66	96	285	121	281	119	283	117	292	103	287	113	289	111	291	115	305	335	369	4010
368	334	304	278	261	257	258	142	141	134	145	263	264	136	266	139	123	97	67	33	4010
34	68	98	124	146	241	236	166	234	168	164	154	248	152	242	255	277	303	333	367	4010
366	332	302	276	254	163	226	231	171	169	182	220	180	225	238	147	125	99	69	35	4010
36	70	100	126	148	239	223	213	189	216	183	215	187	178	162	253	275	301	331	365	4010
364	330	300	274	252	161	224	210	206	193	196	207	191	177	240	149	127	101	71	37	4010
38	72	102	128	150	246	229	192	199	204	201	198	209	172	155	251	273	299	329	363	4010
1	64	73	122	129	151	179	190	203	200	197	202	211	222	250	272	279	328	337	400	4010
9	46	79	108	133	243	174	184	194	205	208	195	217	227	158	268	293	322	355	392	4010
393	356	323	294	269	157	173	214	212	185	218	186	188	228	244	132	107	78	45	8	4010
7	44	77	106	131	245	176	170	230	232	219	181	221	175	156	270	295	324	357	394	4010
395	358	325	296	271	159	165	235	167	233	237	247	153	249	160	130	105	76	43	6	4010
5	42	75	104	262	144	143	259	260	267	256	138	137	265	135	140	297	326	359	396	4010
397	360	327	286	280	120	282	118	284	109	298	114	288	112	290	110	116	74	41	4	4010
3	40	314	94	308	92	91	311	312	321	306	86	85	317	83	319	81	88	361	398	4010
399	346	338	62	340	60	342	58	344	47	362	54	348	52	350	50	352	48	56	2	4010
382	28	374	26	376	24	23	379	380	391	372	18	17	385	15	387	13	389	11	20	4010
4010	4010	4010	4010	4010	4010	4010	4010	4010	4010	4010	4010	4010	4010	4010	4010	4010	4010	4010	4010	4010

The **nested magic square** given in Example 2.6 or Distribution **??** satisfies some interesting properties summarized in a result below.

Result 2.11. *The* ***nested magic square*** *of order 20 has the following properties:*

i. The magic square sums are given by

$$S_{4\times4} := 802 = 2\times401 \qquad S_{14\times14} := 2807 = 7\times401$$
$$S_{6\times6} := 1203 = 3\times401 \qquad S_{16\times16} := 3208 = 8\times401$$
$$S_{8\times8} := 1604 = 4\times401 \qquad S_{18\times18} := 3609 = 9\times401$$
$$S_{10\times10} := 2005 = 5\times401 \qquad S_{20\times20} := 4010 = 10\times401$$
$$S_{12\times12} := 2406 = 6\times401$$

ii. The sum of entries are given by

$$T_{4\times4} := 4\times970 = 3880 = 2^2\times2\times401 \qquad T_{14\times14} := 14\times3395 = 47530 = 7^2\times2\times401$$
$$T_{6\times6} := 6\times1455 = 8730 = 3^2\times2\times401 \qquad T_{16\times16} := 16\times3880 = 62080 = 8^2\times2\times401$$
$$T_{8\times8} := 8\times1940 = 15520 = 4^2\times2\times401 \qquad T_{18\times18} := 18\times4365 = 78570 = 9^2\times2\times401$$
$$T_{10\times10} := 10\times2425 = 24250 = 5^2\times2\times401 \qquad T_{20\times20} := 20\times4850 = 97000 = 10^2\times2\times401$$
$$T_{12\times12} := 12\times2910 = 34920 = 6^2\times2\times401$$

*iii. The **borders** entries sums are given by*

$$\begin{aligned}
C_{20\times20} &:= T_{20\times20} - T_{18\times18} = 4\times(S_{20\times20}-S_{18\times18}) = 19\times2\times401\\
C_{18\times18} &:= T_{18\times18} - T_{16\times16} = 4\times(S_{18\times18}-S_{16\times16}) = 17\times2\times401\\
C_{16\times16} &:= T_{16\times16} - T_{14\times14} = 4\times(S_{16\times16}-S_{14\times14}) = 15\times2\times401\\
C_{14\times14} &:= T_{14\times14} - T_{12\times12} = 4\times(S_{14\times14}-S_{12\times12}) = 13\times2\times401\\
C_{12\times12} &:= T_{12\times12} - T_{10\times10} = 4\times(S_{12\times12}-S_{10\times10}) = 11\times2\times401\\
C_{10\times10} &:= T_{10\times10} - T_{8\times8} = 4\times(S_{10\times10}-S_{8\times8}) = 9\times2\times401\\
C_{8\times8} &:= T_{8\times8} - T_{6\times6} = 4\times(S_{8\times8}-S_{6\times6}) = 7\times2\times401\\
C_{6\times6} &:= T_{6\times6} - T_{4\times4} = 4\times(S_{6\times6}-S_{4\times4}) = 5\times2\times401\\
C_{4\times4} &:= T_{4\times4} - T_{2\times2} = 3\times2\times401,
\end{aligned}$$

where $T_{2\times2} = 199+200+201+202 = 802$ *are four central or middle values of **nested magic square**. In this case, the fixed difference among the **consecutive borders** is* $d_{border} := 1604$*. The expression* $S_{2\times2}$ *is taken out as we don't have magic square of order 2.*

Finally, we get the following symmetric results

Result 2.12. *According to Result 2.11, the **nested magic square** of order 20 for the consecutive entries 1 to 400 has the following symmetric results:*

i. $S_{k\times k} := \frac{k}{2}\times\frac{T_{2\times2}}{2}$;

ii. $T_{k\times k} := \left(\frac{k}{2}\right)^2\times T_{2\times2}$;

iii. $C_{k\times k} := (k-1)\times T_{2\times2}$.

where $k = 4, 6,$ *and* $18, 20$ *orders of magic squares appearing **nested magic square** of order 20, and* $T_{2\times2} := 802$ *is sum of four central values of magic square.*

2.7 Nested Magic Square of Order 19

In Example 2.5 remove the external border, then we are left with **nested magic square** of order 19 for the entries 41 to 401. Subtracting 40, we get following distribution for the **nested magic square** of order 19 for the consecutive natural numbers 1 to 361.

Distribution 2.7.

$$D_{3\times 3} := \{177, 178, \ldots, 184, 185\}$$
$$D_{5\times 5} := \{169, 170, \ldots, 215, 216, \mathbf{D_{3\times 3}}, 186, 187, \ldots, 192, 193\}$$
$$D_{7\times 7} := \{157, 158, \ldots, 167, 168, \mathbf{D_{5\times 5}}, 194, 195, \ldots, 204, 205\}$$
$$D_{9\times 9} := \{141, 142, \ldots, 155, 156, \mathbf{D_{7\times 7}}, 206, 207, \ldots, 220, 221\}$$
$$D_{11\times 11} := \{121, 122, \ldots, 139, 140, \mathbf{D_{9\times 9}}, 222, 223, \ldots, 240, 241\}$$
$$D_{13\times 13} := \{97, 98, \ldots, 119, 120, \mathbf{D_{11\times 11}}, 242, 243, \ldots, 264, 265\}$$
$$D_{15\times 15} := \{69, 70, \ldots, 95, 96, \mathbf{D_{13\times 13}}, 266, 267, \ldots, 292, 293\}$$
$$D_{17\times 17} := \{37, 38, \ldots, 67, 68, \mathbf{D_{15\times 15}}, 294, 295, \ldots, 324, 325\}$$
$$D_{19\times 19} := \{1, 2, \ldots, 35, 36, \mathbf{D_{17\times 17}}, 326, 327, \ldots, 360, 361\}$$

According to above distribution, the **nested magic square** of order 19 for the consecutive natural numbers entries from 1 to 361 is given by

Example 2.7. *Nested magic square of order 19 is given by*

																			3429
344	35	33	31	29	27	25	23	21	19	347	349	351	353	355	357	359	361	20	3439
326	310	67	65	63	61	59	57	55	53	313	315	317	319	321	323	325	54	36	3439
328	294	280	266	268	270	272	274	276	81	80	78	76	74	72	70	278	68	34	3439
330	296	95	108	97	99	101	103	105	253	251	249	247	245	243	252	267	66	32	3439
332	298	93	264	130	121	123	125	127	231	229	227	225	223	230	98	269	64	30	3439
334	300	91	262	240	212	142	144	146	147	210	208	206	214	122	100	271	62	28	3439
336	302	89	260	238	207	198	158	160	161	196	194	200	155	124	102	273	60	26	3439
338	304	87	258	236	209	195	190	186	171	170	188	167	153	126	104	275	58	24	3439
340	306	85	256	234	211	197	175	178	183	182	187	165	151	128	106	277	56	22	3439
17	51	83	255	233	213	199	173	185	181	177	189	163	149	129	107	279	311	345	3439
16	50	283	112	134	145	159	193	180	179	184	169	203	217	228	250	79	312	346	3439
14	48	285	114	136	143	157	174	176	191	192	172	205	219	226	248	77	314	348	3439
12	46	287	116	138	141	162	204	202	201	166	168	164	221	224	246	75	316	350	3439
10	44	289	118	140	148	220	218	216	215	152	154	156	150	222	244	73	318	352	3439
8	42	291	120	132	241	239	237	235	131	133	135	137	139	232	242	71	320	354	3439
6	40	293	110	265	263	261	259	257	109	111	113	115	117	119	254	69	322	356	3439
4	38	84	96	94	92	90	88	86	281	282	284	286	288	290	292	82	324	358	3439
2	308	295	297	299	301	303	305	307	309	49	47	45	43	41	39	37	52	360	3439
342	327	329	331	333	335	337	339	341	343	15	13	11	9	7	5	3	1	18	3439
3439	3439	3439	3439	3439	3439	3439	3439	3439	3439	3439	3439	3439	3439	3439	3439	3439	3439	3439	3439

According to above distribution of magic squares entries, we have the following properties written as a result.

Result 2.13. *The **nested magic square** of order 19 for the entries 1 to 361 has the following properties:*

i. The magic square sums are given by

$$S_{3\times 3} := 543 = 3 \times 181 \qquad S_{13\times 13} := 2353 = 13 \times 181$$
$$S_{5\times 5} := 905 = 5 \times 181 \qquad S_{15\times 15} := 2715 = 15 \times 181$$
$$S_{7\times 7} := 1267 = 7 \times 181 \qquad S_{17\times 17} := 3077 = 17 \times 181$$
$$S_{9\times 9} := 1629 = 9 \times 181 \qquad S_{19\times 19} := 3439 = 19 \times 181$$
$$S_{11\times 11} := 1991 = 11 \times 181$$

ii. The total entries sums are given by

$$\begin{aligned}
T_{3\times3} &:= 3\times 543 = 1629 = 3^2\times 181 \\
T_{5\times5} &:= 5\times 905 = 4525 = 5^2\times 181 \\
T_{7\times7} &:= 7\times 1267 = 8869 = 7^2\times 181 \\
T_{9\times9} &:= 9\times 1629 = 14661 = 9^2\times 181 \\
T_{11\times11} &:= 11\times 1991 = 21901 = 11^2\times 181
\end{aligned}$$

$$\begin{aligned}
T_{13\times13} &:= 13\times 2353 = 30589 = 13^2\times 181 \\
T_{15\times15} &:= 15\times 2715 = 40725 = 15^2\times 181 \\
T_{17\times17} &:= 17\times 3077 = 52309 = 17^2\times 181 \\
T_{19\times19} &:= 19\times 3439 = 65341 = 19^2\times 181
\end{aligned}$$

*iii. The **borders** entries sums for the **nested magic square** of order 23 are given by*

$$\begin{aligned}
C_{19\times19} &:= T_{19\times19}-T_{17\times17} = 4\times(S_{19\times19}-S_{17\times17}) = 9\times 2^3\times 181 \\
C_{17\times17} &:= T_{17\times17}-T_{15\times15} = 4\times(S_{17\times17}-S_{15\times15}) = 8\times 2^3\times 181 \\
C_{15\times15} &:= T_{15\times15}-T_{13\times13} = 4\times(S_{15\times15}-S_{13\times13}) = 7\times 2^3\times 181 \\
C_{13\times13} &:= T_{13\times13}-T_{11\times11} = 4\times(S_{13\times13}-S_{11\times11}) = 6\times 2^3\times 181 \\
C_{11\times11} &:= T_{11\times11}-T_{9\times9} = 4\times(S_{11\times11}-S_{9\times9}) = 5\times 2^3\times 181 \\
C_{9\times9} &:= T_{9\times9}-T_{7\times7} = 4\times(S_{9\times9}-S_{7\times7}) = 4\times 2^3\times 181 \\
C_{7\times7} &:= T_{7\times7}-T_{5\times5} = 4\times(S_{7\times7}-S_{5\times5}) = 3\times 2^3\times 181 \\
C_{5\times5} &:= T_{5\times5}-T_{3\times3} = 4\times(S_{5\times5}-S_{3\times3}) = 2\times 2^3\times 181 \\
C_{3\times3} &:= T_{3\times3}-T_{1\times1} = 4\times(S_{3\times3}-S_{1\times1}) = 1\times 2^3\times 181
\end{aligned}$$

*where $T_{1\times1} = S_{1\times1} = 181$ is the central value. In this case, the fixed difference among the **consecutive borders** is $d_{border} := 1448 = 2^3 \times 181$.*

Finally, we get the following symmetric results

Result 2.14. *According to Result 2.13, the **nested magic square** of order 19 for the consecutive entries 1 to 361, has the following symmetric results:*

i. $S_{k\times k} := k \times T_{1\times1}$;

ii. $T_{k\times k} := k^2 \times T_{1\times1}$;

iii. $C_{k\times k} := \dfrac{k-1}{2} \times 8 \times T_{1\times1}$.

*where $k = 3, 5, 7, \ldots$ 17 and 19 orders of squares appearing **nested magic square** of order 19, and $T_{1\times1} := 181$ is the central value of the magic square.*

2.8 Nested Magic Square of Order 18

In Example 2.6 remove the external border, then we are left with **nested magic square** of order 18 for the entries 39 to 362. Subtracting 38, we get following distribution for the **nested magic square** of order 18 for the consecutive natural numbers 1 to 324.

Distribution 2.8.

$$
\begin{aligned}
D_{4\times4} &:= \{155, 156, \ldots, 169, 170\}\\
D_{6\times6} &:= \{145, 146, \ldots, 153, 154, \mathbf{D_{4\times4}}, 171, 172, \ldots, 179, 180\}\\
D_{8\times8} &:= \{131, 132, \ldots, 143, 144, \mathbf{D_{6\times6}}, 181, 182, \ldots, 193, 194\}\\
D_{10\times10} &:= \{113, 115, \ldots, 129, 130, \mathbf{D_{8\times8}}, 195, 196, \ldots, 211, 212\}\\
D_{12\times12} &:= \{91, 92, \ldots, 111, 112, \mathbf{D_{10\times10}}, 213, 214, \ldots, 233, 234\}\\
D_{14\times14} &:= \{65, 66, \ldots, 89, 90, \mathbf{D_{12\times12}}, 235, 236, \ldots, 259, 260\}\\
D_{16\times16} &:= \{35, 36, \ldots, 63, 64, \mathbf{D_{14\times14}}, 261, 262, \ldots, 289, 290\}\\
D_{18\times18} &:= \{1, 2, \ldots, 33, 34, \mathbf{D_{16\times16}}, 291, 292, \ldots, 323, 324\}
\end{aligned}
$$

According to above distribution, the **nested magic square** of order 18 for the consecutive natural numbers entries from 1 to 324 is given by

Example 2.8. *Nested magic square of order 18 is given by*

																		2925
307	25	301	23	303	21	305	19	316	1	309	15	311	13	313	11	315	17	2925
298	275	269	55	271	272	52	51	42	57	277	278	46	280	44	282	49	27	2925
28	58	247	83	243	81	245	79	254	65	249	75	251	73	253	77	267	297	2925
296	266	240	223	219	220	104	103	96	107	225	226	98	228	101	85	59	29	2925
30	60	86	108	203	198	128	196	130	126	116	210	114	204	217	239	265	295	2925
294	264	238	216	125	188	193	133	131	144	182	142	187	200	109	87	61	31	2925
32	62	88	110	201	185	175	151	178	145	177	149	140	124	215	237	263	293	2925
292	262	236	214	123	186	172	168	155	158	169	153	139	202	111	89	63	33	2925
34	64	90	112	208	191	154	161	166	163	160	171	134	117	213	235	261	291	2925
26	35	84	91	113	141	152	165	162	159	164	173	184	212	234	241	290	299	2925
8	41	70	95	205	136	146	156	167	170	157	179	189	120	230	255	284	317	2925
318	285	256	231	119	135	176	174	147	180	148	150	190	206	94	69	40	7	2925
6	39	68	93	207	138	132	192	194	181	143	183	137	118	232	257	286	319	2925
320	287	258	233	121	127	197	129	195	199	209	115	211	122	92	67	38	5	2925
4	37	66	224	106	105	221	222	229	218	100	99	227	97	102	259	288	321	2925
322	289	248	242	82	244	80	246	71	260	76	250	74	252	72	78	36	3	2925
2	276	56	270	54	53	273	274	283	268	48	47	279	45	281	43	50	323	2925
308	300	24	302	22	304	20	306	9	324	16	310	14	312	12	314	10	18	2925
2925	2925	2925	2925	2925	2925	2925	2925	2925	2925	2925	2925	2925	2925	2925	2925	2925	2925	2925

The **nested magic square** given in Example 2.8 or Distribution 2.8 satisfies some interesting properties summarized in a result below.

Result 2.15. *The **nested magic square** of order 18 has the following properties:*

i. *The magic square sums are given by*

$$S_{4\times 4} := 650 \quad = 2\times 325 \qquad S_{12\times 12} := 1950 = 6\times 325$$
$$S_{6\times 6} := 975 \quad = 3\times 325 \qquad S_{14\times 14} := 2275 = 7\times 325$$
$$S_{8\times 8} := 1300 = 4\times 325 \qquad S_{16\times 16} := 2600 = 8\times 325$$
$$S_{10\times 10} := 1625 = 5\times 325 \qquad S_{18\times 18} := 2925 = 9\times 325$$

ii. *The sum of entries are given by*

$$
\begin{aligned}
T_{4\times4} &:= 4\times 650 = 2600 = 2^2\times 2\times 325 \\
T_{6\times6} &:= 6\times 975 = 5850 = 3^2\times 2\times 325 \\
T_{8\times8} &:= 8\times 1300 = 10400 = 4^2\times 2\times 325 \\
T_{10\times10} &:= 10\times 1625 = 16250 = 5^2\times 2\times 325
\end{aligned}
\qquad
\begin{aligned}
T_{12\times12} &:= 12\times 1950 = 23400 = 6^2\times 2\times 325 \\
T_{14\times14} &:= 14\times 2275 = 31850 = 7^2\times 2\times 325 \\
T_{16\times16} &:= 16\times 2600 = 41600 = 8^2\times 2\times 325 \\
T_{18\times18} &:= 18\times 2925 = 52650 = 9^2\times 2\times 325
\end{aligned}
$$

iii. *The* ***borders*** *entries sums are given by*

$$
\begin{aligned}
C_{18\times18} &:= T_{18\times18} - T_{16\times16} = 4\times(S_{18\times18} - S_{16\times16}) = 17\times 2\times 325 \\
C_{16\times16} &:= T_{16\times16} - T_{14\times14} = 4\times(S_{16\times16} - S_{14\times14}) = 15\times 2\times 325 \\
C_{14\times14} &:= T_{14\times14} - T_{12\times12} = 4\times(S_{14\times14} - S_{12\times12}) = 13\times 2\times 325 \\
C_{12\times12} &:= T_{12\times12} - T_{10\times10} = 4\times(S_{12\times12} - S_{10\times10}) = 11\times 2\times 325 \\
C_{10\times10} &:= T_{10\times10} - T_{8\times8} = 4\times(S_{10\times10} - S_{8\times8}) = 9\times 2\times 325 \\
C_{8\times8} &:= T_{8\times8} - T_{6\times6} = 4\times(S_{8\times8} - S_{6\times6}) = 7\times 2\times 325 \\
C_{6\times6} &:= T_{6\times6} - T_{4\times4} = 4\times(S_{6\times6} - S_{4\times4}) = 5\times 2\times 325 \\
C_{4\times4} &:= T_{4\times4} - T_{2\times2} = 3\times 2\times 325,
\end{aligned}
$$

where $T_{2\times2} = 161+162+163+164 = 650 = 2\times 325$ *are four central or middle values of* ***nested magic square****. In this case, the fixed difference among the* ***consecutive borders*** *is* $d_{border} := 1300$. *The expression* $S_{2\times2}$ *is taken out as we don't have magic square of order 2.*

Finally, we get the following symmetric results

Result 2.16. *According to Result 2.15, the* ***nested magic square*** *of order 18 for the consecutive entries 1 to 324, has the following symmetric results:*

i. $S_{k\times k} := \frac{k}{2}\times\frac{T_{2\times2}}{2}$;

ii. $T_{k\times k} := \left(\frac{k}{2}\right)^2\times T_{2\times2}$;

iii. $C_{k\times k} := (k-1)\times T_{2\times2}$.

where $k = 4, 6,$ *and* $16, 18$ *orders of magic squares appearing* ***nested magic square*** *of order 18, and* $T_{2\times2} := 650$ *is sum of four central values of magic square.*

2.9 Nested Magic Square of Order 17

In Example 2.7 remove the external border, then we are left with **nested magic square** of order 17 for the entries 37 to 325. Subtracting 36, we get following distribution for the **nested magic square** of order 17 for the consecutive natural numbers 1 to 289.

Distribution 2.9.

$$
\begin{aligned}
D_{3\times3} &:= \{141, 142, \ldots, 148, 149\}\\
D_{5\times5} &:= \{133, 134, \ldots, 139, 140, \mathbf{D_{3\times3}}, 150, 151, \ldots, 156, 157\}\\
D_{7\times7} &:= \{121, 122, \ldots, 131, 132, \mathbf{D_{5\times5}}, 158, 159, \ldots, 168, 169\}\\
D_{9\times9} &:= \{105, 106, \ldots, 119, 120, \mathbf{D_{7\times7}}, 170, 171, \ldots, 184, 185\}\\
D_{11\times11} &:= \{85, 86, \ldots, 103, 104, \mathbf{D_{9\times9}}, 186, 187, \ldots, 204, 205\}\\
D_{13\times13} &:= \{61, 62, \ldots, 83, 84, \mathbf{D_{11\times11}}, 206, 207, \ldots, 228, 229\}\\
D_{15\times15} &:= \{33, 34, \ldots, 59, 60, \mathbf{D_{13\times13}}, 230, 231, \ldots, 256, 257\}\\
D_{17\times17} &:= \{1, 2, \ldots, 31, 32, \mathbf{D_{15\times15}}, 258, 259, \ldots, 288, 289\}
\end{aligned}
$$

According to above distribution, the **nested magic square** of order 17 for the consecutive natural numbers entries from 1 to 225 is given by

Example 2.9. *Nested magic square of order 17 is given by*

																	2565
274	31	29	27	25	23	21	19	17	277	279	281	283	285	287	289	18	2465
258	244	230	232	234	236	238	240	45	44	42	40	38	36	34	242	32	2465
260	59	72	61	63	65	67	69	217	215	213	211	209	207	216	231	30	2465
262	57	228	94	85	87	89	91	195	193	191	189	187	194	62	233	28	2465
264	55	226	204	176	106	108	110	111	174	172	170	178	86	64	235	26	2465
266	53	224	202	171	162	122	124	125	160	158	164	119	88	66	237	24	2465
268	51	222	200	173	159	154	150	135	134	152	131	117	90	68	239	22	2465
270	49	220	198	175	161	139	142	147	146	151	129	115	92	70	241	20	2465
15	47	219	197	177	163	137	149	145	141	153	127	113	93	71	243	275	2465
14	247	76	98	109	123	157	144	143	148	133	167	181	192	214	43	276	2465
12	249	78	100	107	121	138	140	155	156	136	169	183	190	212	41	278	2465
10	251	80	102	105	126	168	166	165	130	132	128	185	188	210	39	280	2465
8	253	82	104	112	184	182	180	179	116	118	120	114	186	208	37	282	2465
6	255	84	96	205	203	201	199	95	97	99	101	103	196	206	35	284	2465
4	257	74	229	227	225	223	221	73	75	77	79	81	83	218	33	286	2465
2	48	60	58	56	54	52	50	245	246	248	250	252	254	256	46	288	2465
272	259	261	263	265	267	269	271	273	13	11	9	7	5	3	1	16	2465
2465	2465	2465	2465	2465	2465	2465	2465	2465	2465	2465	2465	2465	2465	2465	2465	2465	2465

According to above distribution of magic squares entries, we have the following properties written as a result.

Result 2.17. *The **nested magic square** of order 17 for the entries 1 to 289 has the following properties:*

i. *The magic square sums are given by*

$$S_{3\times3} := 435 \quad = 3\times5\times29 \qquad S_{11\times11} := 1595 = 11\times5\times29$$
$$S_{5\times5} := 725 \quad = 5\times5\times29 \qquad S_{13\times13} := 1885 = 13\times5\times29$$
$$S_{7\times7} := 1015 = 7\times5\times29 \qquad S_{15\times15} := 2175 = 15\times5\times29$$
$$S_{9\times9} := 1305 = 9\times5\times29 \qquad S_{17\times17} := 2465 = 17\times5\times29$$

ii. *The total entries sums are given by*

$T_{3\times3} := 3\times435 = 1305 = 3^2\times5\times29$

$T_{5\times5} := 5\times725 = 3635 = 5^2\times5\times29$

$T_{7\times7} := 7\times1015 = 7105 = 7^2\times5\times29$

$T_{9\times9} := 9\times1305 = 11745 = 9^2\times5\times29$

$T_{11\times11} := 11\times1595 = 17545 = 11^2\times5\times29$

$T_{13\times13} := 13\times1885 = 24505 = 13^2\times5\times29$

$T_{15\times15} := 15\times2175 = 32625 = 15^2\times5\times29$

$T_{17\times17} := 17\times2465 = 41905 = 17^2\times5\times29$

*iii. The **borders** entries sums for the **nested magic square** of order 23 are given by*

$$\begin{aligned}
C_{17\times17} &:= T_{17\times17}-T_{15\times15} = 4\times(S_{17\times17}-S_{15\times15}) = 8\times2^3\times5\times29\\
C_{15\times15} &:= T_{15\times15}-T_{13\times13} = 4\times(S_{15\times15}-S_{13\times13}) = 7\times2^3\times5\times29\\
C_{13\times13} &:= T_{13\times13}-T_{11\times11} = 4\times(S_{13\times13}-S_{11\times11}) = 6\times2^3\times5\times29\\
C_{11\times11} &:= T_{11\times11}-T_{9\times9} = 4\times(S_{11\times11}-S_{9\times9}) = 5\times2^3\times5\times29\\
C_{9\times9} &:= T_{9\times9}-T_{7\times7} = 4\times(S_{9\times9}-S_{7\times7}) = 4\times2^3\times5\times29\\
C_{7\times7} &:= T_{7\times7}-T_{5\times5} = 4\times(S_{7\times7}-S_{5\times5}) = 3\times2^3\times5\times29\\
C_{5\times5} &:= T_{5\times5}-T_{3\times3} = 4\times(S_{5\times5}-S_{3\times3}) = 2\times2^3\times5\times29\\
C_{3\times3} &:= T_{3\times3}-T_{1\times1} = 4\times(S_{3\times3}-S_{1\times1}) = 1\times2^3\times5\times29
\end{aligned}$$

*where $T_{1\times1} = S_{1\times1} = 145$ is the central value. In this case, the fixed difference among the **consecutive borders** is $d_{border} := 1160 = 2^3\times145$.*

Finally, we get the following symmetric results

Result 2.18. *According to Result 2.17, the **nested magic square** of order 17 for the consecutive entries 1 to 289, has the following symmetric results:*

i. $S_{k\times k} := k\times T_{1\times1}$;

ii. $T_{k\times k} := k^2\times T_{1\times1}$;

iii. $C_{k\times k} := \frac{k-1}{2}\times8\times T_{1\times1}$.

*where $k = 3,5,7,\dots$ 15 and 17 orders of squares appearing **nested magic square** of order 17, and $T_{1\times1} := 145$ is the central value of the magic square.*

2.10 Nested Magic Square of Order 16

In Example 2.8 remove the external border, then we are left with **nested magic square** of order 16 for the entries 35 to 290. Subtracting 33, we get following distribution for the **nested magic square** of order 16 for the consecutive natural numbers 1 to 256.

Distribution 2.10.

$$D_{4\times4} := \{121, 122, \ldots, 135, 136\}$$
$$D_{6\times6} := \{111, 112, \ldots, 119, 120, \mathbf{D_{4\times4}}, 137, 138, \ldots, 145, 146\}$$
$$D_{8\times8} := \{97, 98, \ldots, 109, 110, \mathbf{D_{6\times6}}, 147, 148, \ldots, 159, 160\}$$
$$D_{10\times10} := \{79, 80, \ldots, 95, 96, \mathbf{D_{8\times8}}, 161, 162, \ldots, 177, 178\}$$
$$D_{12\times12} := \{57, 58, \ldots, 77, 78, \mathbf{D_{10\times10}}, 179, 180, \ldots, 199, 200\}$$
$$D_{14\times14} := \{31, 32, \ldots, 55, 56, \mathbf{D_{12\times12}}, 201, 202, \ldots, 225, 226\}$$
$$D_{16\times16} := \{1, 2, \ldots, 29, 30, \mathbf{D_{14\times14}}, 227, 228, \ldots, 255, 256\}$$

According to above distribution, the **nested magic square** of order 16 for the consecutive natural numbers entries from 1 to 256 is given by

Example 2.10. *Nested magic square of order 16 is given by*

																2056
241	235	21	237	238	18	17	8	23	243	244	12	246	10	248	15	2056
24	213	49	209	47	211	45	220	31	215	41	217	39	219	43	233	2056
232	206	189	185	186	70	69	62	73	191	192	64	194	67	51	25	2056
26	52	74	169	164	94	162	96	92	82	176	80	170	183	205	231	2056
230	204	182	91	154	159	99	97	110	148	108	153	166	75	53	27	2056
28	54	76	167	151	141	117	144	111	143	115	106	90	181	203	229	2056
228	202	180	89	152	138	134	121	124	135	119	105	168	77	55	29	2056
30	56	78	174	157	120	127	132	129	126	137	100	83	179	201	227	2056
1	50	57	79	107	118	131	128	125	130	139	150	178	200	207	256	2056
7	36	61	171	102	112	122	133	136	123	145	155	86	196	221	250	2056
251	222	197	85	101	142	140	113	146	114	116	156	172	60	35	6	2056
5	34	59	173	104	98	158	160	147	109	149	103	84	198	223	252	2056
253	224	199	87	93	163	95	161	165	175	81	177	88	58	33	4	2056
3	32	190	72	71	187	188	195	184	66	65	193	63	68	225	254	2056
255	214	208	48	210	46	212	37	226	42	216	40	218	38	44	2	2056
242	22	236	20	19	239	240	249	234	14	13	245	11	247	9	16	2056
2056	2056	2056	2056	2056	2056	2056	2056	2056	2056	2056	2056	2056	2056	2056	2056	2056

The **nested magic square** given in Example 2.10 or Distribution 2.10 satisfies some interesting properties summarized in a result below.

Result 2.19. *The **nested magic square** of order 16 has the following properties:*

i. The magic square sums are given by

$$\begin{aligned} S_{4\times 4} &:= 514 = 2\times 257 \\ S_{6\times 6} &:= 771 = 3\times 257 \\ S_{8\times 8} &:= 1028 = 4\times 257 \\ S_{10\times 10} &:= 1285 = 5\times 257 \end{aligned} \qquad \begin{aligned} S_{12\times 12} &:= 1542 = 6\times 257 \\ S_{14\times 14} &:= 1799 = 7\times 257 \\ S_{16\times 16} &:= 2056 = 8\times 257 \end{aligned}$$

ii. The sum of entries are given by

$$\begin{aligned} T_{4\times 4} &:= 4\times 514 = 2056 = 2^2\times 2\times 257 \\ T_{6\times 6} &:= 6\times 771 = 4626 = 3^2\times 2\times 257 \\ T_{8\times 8} &:= 8\times 1028 = 8224 = 4^2\times 2\times 257 \\ T_{10\times 10} &:= 10\times 1285 = 12850 = 5^2\times 2\times 257 \end{aligned} \qquad \begin{aligned} T_{12\times 12} &:= 12\times 1542 = 18504 = 6^2\times 2\times 257 \\ T_{14\times 14} &:= 14\times 1799 = 25186 = 7^2\times 2\times 257 \\ T_{16\times 16} &:= 16\times 2056 = 32896 = 8^2\times 2\times 257 \end{aligned}$$

*iii. The **borders** entries sums are given by*

$$\begin{aligned} C_{16\times 16} &:= T_{16\times 16} - T_{14\times 14} = 4\times (S_{16\times 16} - S_{14\times 14}) = 15\times 2\times 257 \\ C_{14\times 14} &:= T_{14\times 14} - T_{12\times 12} = 4\times (S_{14\times 14} - S_{12\times 12}) = 13\times 2\times 257 \\ C_{12\times 12} &:= T_{12\times 12} - T_{10\times 10} = 4\times (S_{12\times 12} - S_{10\times 10}) = 11\times 2\times 257 \\ C_{10\times 10} &:= T_{10\times 10} - T_{8\times 8} = 4\times (S_{10\times 10} - S_{8\times 8}) = 9\times 2\times 257 \\ C_{8\times 8} &:= T_{8\times 8} - T_{6\times 6} = 4\times (S_{8\times 8} - S_{6\times 6}) = 7\times 2\times 257 \\ C_{6\times 6} &:= T_{6\times 6} - T_{4\times 4} = 4\times (S_{6\times 6} - S_{4\times 4}) = 5\times 2\times 257 \\ C_{4\times 4} &:= T_{4\times 4} - T_{2\times 2} = 3\times 2\times 257, \end{aligned}$$

*where $T_{2\times 2} = 127+128+129+130 = 514$ are four central or middle values of **nested magic square**. In this case, the fixed difference among the **consecutive borders** is $d_{border} := 1028$. The expression $S_{2\times 2}$ is taken out as we don't have magic square of order 2.*

Finally, we get the following symmetric results

Result 2.20. *According to Result 2.19, the **nested magic square** of order 16 for the consecutive entries 1 to 256, has the following symmetric results:*

i. $S_{k\times k} := \dfrac{k}{2}\times \dfrac{T_{2\times 2}}{2}$;

ii. $T_{k\times k} := \left(\dfrac{k}{2}\right)^2 \times T_{2\times 2}$;

iii. $C_{k\times k} := (k-1)\times T_{2\times 2}$.

*where $k = 4, 6, 8, 10, 12, 14$ and 16 orders of magic squares appearing **nested magic square** of order 16, and $T_{2\times 2} := 514$ is sum of four central values of magic square.*

2.11 Nested Magic Square of Order 15

In Example 2.9 remove the external border, then we are left with **nested magic square** of order 15 for the entries 33 to 257. Subtracting 32, we get following distribution for the **nested magic square** of order 15 for the consecutive natural numbers 1 to 225.

Distribution 2.11.

$$D_{3\times3} := \{109, 110, \ldots, 116, 117\}$$
$$D_{5\times5} := \{101, 102, \ldots, 107, 108, \mathbf{D_{3\times3}}, 118, 119, \ldots, 124, 125\}$$
$$D_{7\times7} := \{89, 90, \ldots, 99, 100, \mathbf{D_{5\times5}}, 126, 127, \ldots, 136, 137\}$$
$$D_{9\times9} := \{73, 74, \ldots, 87, 88, \mathbf{D_{7\times7}}, 138, 139, \ldots, 152, 153\}$$
$$D_{11\times11} := \{53, 54, \ldots, 71, 72, \mathbf{D_{9\times9}}, 154, 155, \ldots, 172, 173\}$$
$$D_{13\times13} := \{29, 30, \ldots, 51, 52, \mathbf{D_{11\times11}}, 174, 175, \ldots, 196, 197\}$$
$$D_{15\times15} := \{1, 2, \ldots, 27, 28, \mathbf{D_{13\times13}}, 198, 199, \ldots, 224, 225\}$$

According to above distribution, the **nested magic square** of order 15 for the consecutive natural numbers entries from 1 to 225 is given by

Example 2.11. *Nested magic square of order 15 is given by*

															1695
212	198	200	202	204	206	208	13	12	10	8	6	4	2	210	1695
27	40	29	31	33	35	37	185	183	181	179	177	175	184	199	1695
25	196	62	53	55	57	59	163	161	159	157	155	162	30	201	1695
23	194	172	144	74	76	78	79	142	140	138	146	54	32	203	1695
21	192	170	139	130	90	92	93	128	126	132	87	56	34	205	1695
19	190	168	141	127	122	118	103	102	120	99	85	58	36	207	1695
17	188	166	143	129	107	110	115	114	119	97	83	60	38	209	1695
15	187	165	145	131	105	117	113	109	121	95	81	61	39	211	1695
215	44	66	77	91	125	112	111	116	101	135	149	160	182	11	1695
217	46	68	75	89	106	108	123	124	104	137	151	158	180	9	1695
219	48	70	73	94	136	134	133	98	100	96	153	156	178	7	1695
221	50	72	80	152	150	148	147	84	86	88	82	154	176	5	1695
223	52	64	173	171	169	167	63	65	67	69	71	164	174	3	1695
225	42	197	195	193	191	189	41	43	45	47	49	51	186	1	1695
16	28	26	24	22	20	18	213	214	216	218	220	222	224	14	1695
1695	1695	1695	1695	1695	1695	1695	1695	1695	1695	1695	1695	1695	1695	1695	1695

According to above distribution of magic squares entries, we have the following properties written as a result.

Result 2.21. *The **nested magic square** of order 15 for the entries 1 to 225 has the following properties:*

i. *The magic square sums are given by*

$$S_{3\times3} := 339 \quad = 3\times113 \qquad S_{11\times11} := 1243 = 11\times113$$
$$S_{5\times5} := 565 \quad = 5\times113 \qquad S_{13\times13} := 1469 = 13\times113$$
$$S_{7\times7} := 791 \quad = 7\times113 \qquad S_{15\times15} := 1695 = 15\times113$$
$$S_{9\times9} := 1071 = 9\times113$$

ii. *The total entries sums are given by*

$$T_{3\times3} := 3\times339 \quad = 1017 = 3^2\times113 \qquad T_{11\times11} := 11\times1243 = 13673 = 11^2\times113$$
$$T_{5\times5} := 5\times565 \quad = 2825 = 5^2\times113 \qquad T_{13\times13} := 13\times1469 = 19097 = 13^2\times113$$
$$T_{7\times7} := 7\times791 \quad = 5537 = 7^2\times113 \qquad T_{15\times15} := 15\times1695 = 25425 = 15^2\times113$$
$$T_{9\times9} := 9\times1071 = 9153 = 9^2\times113$$

iii. *The **borders** entries sums for the **nested magic square** of order 23 are given by*

$$C_{15\times15} := T_{15\times15} - T_{13\times13} = 4\times(S_{15\times15} - S_{13\times13}) = 7\times2^3\times113$$
$$C_{13\times13} := T_{13\times13} - T_{11\times11} = 4\times(S_{13\times13} - S_{11\times11}) = 6\times2^3\times113$$
$$C_{11\times11} := T_{11\times11} - T_{9\times9} \quad = 4\times(S_{11\times11} - S_{9\times9}) \quad = 5\times2^3\times113$$
$$C_{9\times9} := T_{9\times9} - T_{7\times7} \quad = 4\times(S_{9\times9} - S_{7\times7}) \quad = 4\times2^3\times113$$
$$C_{7\times7} := T_{7\times7} - T_{5\times5} \quad = 4\times(S_{7\times7} - S_{5\times5}) \quad = 3\times2^3\times113$$
$$C_{5\times5} := T_{5\times5} - T_{3\times3} \quad = 4\times(S_{5\times5} - S_{3\times3}) \quad = 2\times2^3\times113$$

*where $T_{1\times1} = S_{1\times1} = 113$ is the central value. In this case, the fixed difference among the **consecutive borders** is $d_{border} := 904 = 2^3\times113$.*

Finally, we get the following symmetric results

Result 2.22. *According to Result 2.21, the **nested magic square** of order 15 for the consecutive entries 1 to 221, has the following symmetric results:*

i. $S_{k\times k} := k\times T_{1\times1}$;

ii. $T_{k\times k} := k^2 \times T_{1\times 1}$;

iii. $C_{k\times k} := \dfrac{k-1}{2} \times 8 \times T_{1\times 1}$.

where $k = 3, 5, \ldots$ 13 *and* 15 *orders of squares appearing* ***nested magic square*** *of order 15, and* $T_{1\times 1} := 113$ *is the central value of the magic square.*

2.12 Nested Magic Square of Order 14

In Example 2.10 remove the external border, then we are left with **nested magic square** of order 14 for the entries 31 to 226. Subtracting 30, we get following distribution for the **nested magic square** of order 14 for the consecutive natural numbers 1 to 196.

Distribution 2.12.

$$\begin{aligned}
D_{4\times 4} &:= \{91, 92, \ldots, 105, 106\}\\
D_{6\times 6} &:= \{81, 82, \ldots, 89, 90, \mathbf{D_{4\times 4}}, 107, 108, \ldots, 115, 116\}\\
D_{8\times 8} &:= \{67, 68, \ldots, 79, 80, \mathbf{D_{6\times 6}}, 117, 118, \ldots, 129, 130\}\\
D_{10\times 10} &:= \{49, 50, \ldots, 65, 66, \mathbf{D_{8\times 8}}, 131, 132, \ldots, 147, 148\}\\
D_{12\times 12} &:= \{27, 28, \ldots, 47, 48, \mathbf{D_{10\times 10}}, 149, 150, \ldots, 169, 170\}\\
D_{14\times 14} &:= \{1, 2, \ldots, 25, 26, \mathbf{D_{12\times 12}}, 171, 172, \ldots, 195, 196\}
\end{aligned}$$

According to above distribution, the **nested magic square** of order 14 for the consecutive natural numbers entries from 1 to 196 is given by

Example 2.12. *Nested magic square of order 14 is given by*

														1379
183	19	179	17	181	15	190	1	185	11	187	9	189	13	1379
176	159	155	156	40	39	32	43	161	162	34	164	37	21	1379
22	44	139	134	64	132	66	62	52	146	50	140	153	175	1379
174	152	61	124	129	69	67	80	118	78	123	136	45	23	1379
24	46	137	121	111	87	114	81	113	85	76	60	151	173	1379
172	150	59	122	108	104	91	94	105	89	75	138	47	25	1379
26	48	144	127	90	97	102	99	96	107	70	53	149	171	1379
20	27	49	77	88	101	98	95	100	109	120	148	170	177	1379
6	31	141	72	82	92	103	106	93	115	125	56	166	191	1379
192	167	55	71	112	110	83	116	84	86	126	142	30	5	1379
4	29	143	74	68	128	130	117	79	119	73	54	168	193	1379
194	169	57	63	133	65	131	135	145	51	147	58	28	3	1379
2	160	42	41	157	158	165	154	36	35	163	33	38	195	1379
184	178	18	180	16	182	7	196	12	186	10	188	8	14	1379
1379	1379	1379	1379	1379	1379	1379	1379	1379	1379	1379	1379	1379	1379	1379

According to above distribution of magic squares entries, we have the following properties written as a result.

Result 2.23. *The* ***nested magic square*** *of order 14 has the following properties:*

i. *The magic square sums are given by*

$$S_{4\times4} := 394 = 2\times197 \qquad S_{10\times10} := 985 = 5\times197$$
$$S_{6\times6} := 591 = 3\times197 \qquad S_{12\times12} := 1182 = 6\times197$$
$$S_{8\times8} := 788 = 4\times197 \qquad S_{14\times14} := 1379 = 7\times197$$

ii. *The sum of entries are given by*

$$T_{4\times4} := 4\times394 = 1576 = 2^2\times2\times197 \qquad T_{10\times10} := 10\times985 = 9850 = 5^2\times2\times197$$
$$T_{6\times6} := 6\times591 = 3546 = 3^2\times2\times197 \qquad T_{12\times12} := 12\times1182 = 14184 = 6^2\times2\times197$$
$$T_{8\times8} := 8\times788 = 6304 = 4^2\times2\times197 \qquad T_{14\times14} := 14\times1379 = 19306 = 7^2\times2\times197$$

iii. *The **borders** entries sums are given by*

$$\begin{aligned}
C_{14\times14} &:= T_{14\times14} - T_{12\times12} = 4\times(S_{14\times14} - S_{12\times12}) = 13\times2\times197\\
C_{12\times12} &:= T_{12\times12} - T_{10\times10} = 4\times(S_{12\times12} - S_{10\times10}) = 11\times2\times197\\
C_{10\times10} &:= T_{10\times10} - T_{8\times8} = 4\times(S_{10\times10} - S_{8\times8}) = 9\times2\times197\\
C_{8\times8} &:= T_{8\times8} - T_{6\times6} = 4\times(S_{8\times8} - S_{6\times6}) = 7\times2\times197\\
C_{6\times6} &:= T_{6\times6} - T_{4\times4} = 4\times(S_{6\times6} - S_{4\times4}) = 5\times2\times197\\
C_{4\times4} &:= T_{4\times4} - T_{2\times2} = 3\times2\times197,
\end{aligned}$$

*where $T_{2\times2} = 97+98+99+100 = 394$ are four central or middle values of **nested magic square**. In this case, the fixed difference among the **consecutive borders** is $d_{border} := 788$. The expression $S_{2\times2}$ is taken out as we don't have magic square of order 2.*

Finally, we get the following symmetric results

Result 2.24. *According to Result 2.23, the **nested magic square** of order 14 for the consecutive entries 1 to 196, has the following symmetric results:*

i. $S_{k\times k} := \frac{k}{2} \times \frac{T_{2\times2}}{2}$;

ii. $T_{k\times k} := \left(\frac{k}{2}\right)^2 \times T_{2\times2}$;

iii. $C_{k\times k} := (k-1)\times T_{2\times2}$.

*where $k = 4, 6, 8, 10, 12$ and 14 orders of magic squares appearing **nested magic square** of order 14, and $T_{2\times2} := 394$ is sum of four central values of magic square.*

2.13 Nested Magic Square of Order 13

In Example 2.11 remove the external border, then we are left with **nested magic square** of order 13 for the entries 29 to 197. Subtracting 28, we get following distribution for the **nested magic square** of order 13 for the consecutive natural numbers 1 to 169.

Distribution 2.13.

$$\begin{aligned}
D_{3\times3} &:= \{81, 82, \ldots, 88, 89\}\\
D_{5\times5} &:= \{73, 74, \ldots, 79, 80, \mathbf{D_{3\times3}}, 90, 91, \ldots, 96, 97\}\\
D_{7\times7} &:= \{61, 62, \ldots, 71, 72, \mathbf{D_{5\times5}}, 98, 99, \ldots, 108, 109\}\\
D_{9\times9} &:= \{45, 46, \ldots, 59, 60, \mathbf{D_{7\times7}}, 110, 111, \ldots, 124, 125\}\\
D_{11\times11} &:= \{25, 26, \ldots, 43, 44, \mathbf{D_{9\times9}}, 126, 127, \ldots, 144, 145\}\\
D_{13\times13} &:= \{1, 2, \ldots, 23, 24, \mathbf{D_{11\times11}}, 146, 147, \ldots, 168, 169\}
\end{aligned}$$

According to above distribution, the **nested magic square** of order 13 for the consecutive natural numbers entries from 1 to 169 is given by

Example 2.13. *Nested magic square of order 13 is given by*

													1105
12	1	3	5	7	9	157	155	153	151	149	147	156	1105
168	34	25	27	29	31	135	133	131	129	127	134	2	1105
166	144	116	46	48	50	51	114	112	110	118	26	4	1105
164	142	111	102	62	64	65	100	98	104	59	28	6	1105
162	140	113	99	94	90	75	74	92	71	57	30	8	1105
160	138	115	101	79	82	87	86	91	69	55	32	10	1105
159	137	117	103	77	89	85	81	93	67	53	33	11	1105
16	38	49	63	97	84	83	88	73	107	121	132	154	1105
18	40	47	61	78	80	95	96	76	109	123	130	152	1105
20	42	45	66	108	106	105	70	72	68	125	128	150	1105
22	44	52	124	122	120	119	56	58	60	54	126	148	1105
24	36	145	143	141	139	35	37	39	41	43	136	146	1105
14	169	167	165	163	161	13	15	17	19	21	23	158	1105
1105	1105	1105	1105	1105	1105	1105	1105	1105	1105	1105	1105	1105	1105

According to above distribution of magic squares entries, we have the following properties written as a result.

Result 2.25. *The **nested magic square** of order 13 for the entries 1 to 169 has the following properties:*

i. *The magic square sums are given by*

$$S_{3\times3} := 255 = 3\times5\times17 \qquad S_{9\times9} := 765 = 9\times5\times17$$
$$S_{5\times5} := 425 = 5\times5\times17 \qquad S_{11\times11} := 935 = 11\times5\times17$$
$$S_{7\times7} := 595 = 7\times5\times17 \qquad S_{13\times13} := 1105 = 13\times5\times17$$

ii. *The total entries sums are given by*

$$T_{3\times3} := 3\times255 = 765 = 3^2\times5\times17 \qquad T_{9\times9} := 9\times765 = 6885 = 9^2\times5\times17$$
$$T_{5\times5} := 5\times425 = 2125 = 5^2\times5\times17 \qquad T_{11\times11} := 11\times935 = 10285 = 11^2\times5\times17$$
$$T_{7\times7} := 7\times595 = 4165 = 7^2\times5\times17 \qquad T_{13\times13} := 13\times1105 = 14365 = 13^2\times5\times17$$

iii. *The* ***borders*** *entries sums for the* ***nested magic square*** *of order 23 are given by*

$$\begin{aligned}
C_{13\times13} &:= T_{13\times13} - T_{11\times11} = 4\times(S_{13\times13} - S_{11\times11}) = 6\times 2^3\times 5\times 17\\
C_{11\times11} &:= T_{11\times11} - T_{9\times9} \quad = 4\times(S_{11\times11} - S_{9\times9}) \quad = 5\times 2^3\times 5\times 17\\
C_{9\times9} &:= T_{9\times9} - T_{7\times7} \quad = 4\times(S_{9\times9} - S_{7\times7}) \quad = 4\times 2^3\times 5\times 17\\
C_{7\times7} &:= T_{7\times7} - T_{5\times5} \quad = 4\times(S_{7\times7} - S_{5\times5}) \quad = 3\times 2^3\times 5\times 17\\
C_{5\times5} &:= T_{5\times5} - T_{3\times3} \quad = 4\times(S_{5\times5} - S_{3\times3}) \quad = 2\times 2^3\times 5\times 17
\end{aligned}$$

where $T_{1\times1} = S_{1\times1} = 85$ *is the central value. In this case, the fixed difference among the* ***consecutive borders*** *is* $d_{border} := 680 = 2^3\times 5\times 17$.

Finally, we get the following symmetric results

Result 2.26. *According to Result 2.25, the* ***nested magic square*** *of order 13 for the consecutive entries 1 to 169, has the following symmetric results:*

i. $S_{k\times k} := k\times T_{1\times1}$;

ii. $T_{k\times k} := k^2\times T_{1\times1}$;

iii. $C_{k\times k} := \dfrac{k-1}{2}\times 8\times T_{1\times1}$.

where $k = 3, 5, \ldots\ 11$ *and* 13 *orders of squares appearing* ***nested magic square*** *of order 13, and* $T_{1\times1} := 85$ *is the central value of the magic square.*

2.14 Nested Magic Square of Order 12

In Example 2.12 remove the external border, then we are left with **nested magic square** of order 12 for the entries 27 to 170. Subtracting 26, we get following distribution for the **nested magic square** of order 12 for the consecutive natural numbers 1 to 144.

Distribution 2.14.

$$\begin{aligned}
D_{4\times4} &:= \{65, 66, \ldots, 79, 80\}\\
D_{6\times6} &:= \{55, 56, \ldots, 63, 64, \mathbf{D_{4\times4}}, 81, 82, \ldots, 89, 90\}\\
D_{8\times8} &:= \{41, 42, \ldots, 53, 54, \mathbf{D_{6\times6}}, 91, 92, \ldots, 103, 104\}\\
D_{10\times10} &:= \{23, 24, \ldots, 39, 40, \mathbf{D_{8\times8}}, 105, 106, \ldots, 121, 122\}\\
D_{12\times12} &:= \{1, 2, \ldots, 21, 22, \mathbf{D_{10\times10}}, 123, 124, \ldots, 143, 144\}
\end{aligned}$$

According to above distribution, the **nested magic square** of order 12 for the consecutive natural numbers entries from 1 to 144 is given by

Example 2.14. *Nested magic square of order 12 is given by*

												870
133	129	130	14	13	6	17	135	136	8	138	11	870
18	113	108	38	106	40	36	26	120	24	114	127	870
126	35	98	103	43	41	54	92	52	97	110	19	870
20	111	95	85	61	88	55	87	59	50	34	125	870
124	33	96	82	78	65	68	79	63	49	112	21	870
22	118	101	64	71	76	73	70	81	44	27	123	870
1	23	51	62	75	72	69	74	83	94	122	144	870
5	115	46	56	66	77	80	67	89	99	30	140	870
141	29	45	86	84	57	90	58	60	100	116	4	870
3	117	48	42	102	104	91	53	93	47	28	142	870
143	31	37	107	39	105	109	119	25	121	32	2	870
134	16	15	131	132	139	128	10	9	137	7	12	870
870	870	870	870	870	870	870	870	870	870	870	870	870

According to above distribution of magic squares entries, we have the following properties written as a result.

Result 2.27. *The **nested magic square** of order 12 has the following properties:*

i. *The magic square sums are given by*

$$S_{4\times4} := 290 = 2\times5\times29$$
$$S_{6\times6} := 435 = 3\times5\times29$$
$$S_{8\times8} := 580 = 4\times5\times29$$
$$S_{10\times10} := 725 = 5\times5\times29$$
$$S_{12\times12} := 870 = 6\times5\times29$$

ii. *The sum of entries are given by*

$$T_{4\times4} := 4\times290 = 1160 = 2^2\times2\times5\times29$$
$$T_{6\times6} := 6\times435 = 2610 = 3^2\times2\times5\times29$$
$$T_{8\times8} := 8\times580 = 4640 = 4^2\times2\times5\times29$$
$$T_{10\times10} := 10\times725 = 7250 = 5^2\times2\times5\times29$$
$$T_{12\times12} := 12\times870 = 10440 = 6^2\times2\times5\times29$$

iii. *The **borders** entries sums are given by*

$$C_{12\times12} := T_{12\times12} - T_{10\times10} = 4\times(S_{12\times12} - S_{10\times10}) = 11\times2\times5\times29$$

$$\begin{aligned}
C_{10\times 10} &:= T_{10\times 10} - T_{8\times 8} &&= 4\times(S_{10\times 10} - S_{8\times 8}) &&= 9\times 5\times 29\\
C_{8\times 8} &:= T_{8\times 8} - T_{6\times 6} &&= 4\times(S_{8\times 8} - S_{6\times 6}) &&= 7\times 2\times 5\times 29\\
C_{6\times 6} &:= T_{6\times 6} - T_{4\times 4} &&= 4\times(S_{6\times 6} - S_{4\times 4}) &&= 5\times 2\times 5\times 29\\
C_{4\times 4} &:= T_{4\times 4} - T_{2\times 2} && &&= 3\times 2\times 5\times 29,
\end{aligned}$$

where $T_{2\times 2} = 71+72+73+74 = 290$ *are four central or middle values of* ***nested magic square****. In this case, the fixed difference among the* ***consecutive borders*** *is* $d_{border} := 580$. *The expression* $S_{2\times 2}$ *is taken out as we don't have magic square of order 2.*

Finally, we get the following symmetric results

Result 2.28. *According to Result 2.27, the* ***nested magic square*** *of order 12 for the consecutive entries 1 to 144, has the following symmetric results:*

i. $S_{k\times k} := \dfrac{k}{2} \times \dfrac{T_{2\times 2}}{2}$;

ii. $T_{k\times k} := \left(\dfrac{k}{2}\right)^2 \times T_{2\times 2}$;

iii. $C_{k\times k} := (k-1) \times T_{2\times 2}$.

where $k = 4, 6, 8, 10$ *and* 12 *orders of magic squares appearing* ***nested magic square*** *of order 12, and* $T_{2\times 2} := 290$ *is sum of four central values of magic square.*

2.15 Nested Magic Square of Order 11

In Example 2.13 remove the external border, then we are left with **nested magic square** of order 11 for the entries 25 to 145. Subtracting 24, we get following distribution for the **nested magic square** of order 11 for the consecutive natural numbers 1 to 121.

Distribution 2.15.

$$\begin{aligned}
D_{3\times 3} &:= \{57, 58, \ldots, 64, 65\}\\
D_{5\times 5} &:= \{49, 50, \ldots, 55, 56, \mathbf{D_{3\times 3}}, 66, 67, \ldots, 72, 73\}\\
D_{7\times 7} &:= \{37, 38, \ldots, 47, 48, \mathbf{D_{5\times 5}}, 74, 75, \ldots, 84, 85\}\\
D_{9\times 9} &:= \{21, 22, \ldots, 35, 36, \mathbf{D_{7\times 7}}, 86, 87, \ldots, 100, 101\}\\
D_{11\times 11} &:= \{1, 2, \ldots, 19, 20, \mathbf{D_{9\times 9}}, 102, 103, \ldots, 121, 121\}
\end{aligned}$$

According to above distribution, the **nested magic square** of order 11 for the consecutive natural numbers entries from 1 to 121 is given by

Example 2.15. *Nested magic square of order 11 is given by*

											671
10	1	3	5	7	111	109	107	105	103	110	671
120	92	22	24	26	27	90	88	86	94	2	671
118	87	78	38	40	41	76	74	80	35	4	671
116	89	75	70	66	51	50	68	47	33	6	671
114	91	77	55	58	63	62	67	45	31	8	671
113	93	79	53	65	61	57	69	43	29	9	671
14	25	39	73	60	59	64	49	83	97	108	671
16	23	37	54	56	71	72	52	85	99	106	671
18	21	42	84	82	81	46	48	44	101	104	671
20	28	100	98	96	95	32	34	36	30	102	671
12	121	119	117	115	11	13	15	17	19	112	671
671	671	671	671	671	671	671	671	671	671	671	671

According to above distribution of magic squares entries, we have the following properties written as a result.

Result 2.29. *The* ***nested magic square*** *of order 11 for the entries 1 to 121 has the following properties:*

i. *The magic square sums are given by*

$$S_{3\times3} := 183 = 3\times61 \qquad S_{9\times9} := 549 = 9\times61$$
$$S_{5\times5} := 305 = 5\times61 \qquad S_{11\times11} := 671 = 11\times61$$
$$S_{7\times7} := 427 = 7\times61$$

ii. *The total entries sums are given by*

$$T_{3\times3} := 3\times183 = 549 = 3^2\times61 \qquad T_{9\times9} := 9\times549 = 4941 = 9^2\times61$$
$$T_{5\times5} := 5\times305 = 1525 = 5^2\times61 \qquad T_{11\times11} := 11\times671 = 7381 = 11^2\times61$$
$$T_{7\times7} := 7\times427 = 2989 = 7^2\times61$$

iii. *The* ***borders*** *entries sums for the* ***nested magic square*** *of order 23 are given by*

$$C_{11\times11} := T_{11\times11} - T_{9\times9} = 4\times(S_{11\times11} - S_{9\times9}) = 5\times2^3\times61$$
$$C_{9\times9} := T_{9\times9} - T_{7\times7} = 4\times(S_{9\times9} - S_{7\times7}) = 4\times2^3\times61$$

$$C_{7\times7} := T_{7\times7} - T_{5\times5} \quad = 4\times(S_{7\times7} - S_{5\times5}) \quad = 3\times2^3\times61$$
$$C_{5\times5} := T_{5\times5} - T_{3\times3} \quad = 4\times(S_{5\times5} - S_{3\times3}) \quad = 2\times2^3\times61$$

*where $T_{1\times1} = S_{1\times1} = 61$ is the central value. In this case, the fixed difference among the **consecutive borders** is $d_{border} := 488 = 2^3\times61$.*

Finally, we get the following symmetric results

Result 2.30. *According to Result 2.29, the **nested magic square** of order 11 for the consecutive entries 1 to 121, has the following symmetric results:*

i. $S_{k\times k} := k\times T_{1\times1}$;

ii. $T_{k\times k} := k^2\times T_{1\times1}$;

iii. $C_{k\times k} := \dfrac{k-1}{2}\times8\times T_{1\times1}$.

*where $k = 3,5,7,9$ and 11 orders of squares appearing **nested magic square** of order 11, and $T_{1\times1} := 61$ is the central value of the magic square.*

2.16 Nested Magic Square of Order 10

In Example 2.14 remove the external border, then we are left with **nested magic square** of order 10 for the entries 23 to 122. Subtracting 22, we get following distribution for the **nested magic square** of order 10 for the consecutive natural numbers 1 to 100.

Distribution 2.16.

$$D_{4\times4} := \{43,44,\ldots,57,58\}$$
$$D_{6\times6} := \{33,34,\ldots,41,42,\mathbf{D_{4\times4}},59,60,\ldots,67,68\}$$
$$D_{8\times8} := \{19,20,\ldots,31,32,\mathbf{D_{6\times6}},69,70,\ldots,81,82\}$$
$$D_{10\times10} := \{1,2,\ldots,17,18,\mathbf{D_{8\times8}},83,84,\ldots,99,100\}$$

According to above distribution, the **nested magic square** of order 10 for the consecutive natural numbers entries from 1 to 100 is given by

Example 2.16. *Nested magic square of order 12 is given by*

										505
91	86	16	84	18	14	4	98	2	92	505
13	76	81	21	19	32	70	30	75	88	505
89	73	63	39	66	33	65	37	28	12	505
11	74	60	56	43	46	57	41	27	90	505
96	79	42	49	54	51	48	59	22	5	505
1	29	40	53	50	47	52	61	72	100	505
93	24	34	44	55	58	45	67	77	8	505
7	23	64	62	35	68	36	38	78	94	505
95	26	20	80	82	69	31	71	25	6	505
9	15	85	17	83	87	97	3	99	10	505
505	505	505	505	505	505	505	505	505	505	505

According to above distribution of magic squares entries, we have the following properties written as a result.

Result 2.31. *The* ***nested magic square*** *of order 10 has the following properties:*

i. *The magic square sums are given by*

$$S_{4\times4} := 202 = 2\times101 \qquad S_{8\times8} := 404 = 4\times101$$
$$S_{6\times6} := 303 = 3\times101 \qquad S_{10\times10} := 505 = 5\times101$$

ii. *The sum of entries are given by*

$$T_{4\times4} := 4\times202 = 808 = 2^2\times2\times101 \qquad T_{8\times8} := 8\times404 = 3232 = 4^2\times2\times101$$
$$T_{6\times6} := 6\times303 = 1818 = 3^2\times2\times101 \qquad T_{10\times10} := 10\times505 = 5050 = 5^2\times2\times101$$

iii. *The* ***borders*** *entries sums are given by*

$$\begin{aligned}
C_{10\times10} &:= T_{10\times10} - T_{8\times8} = 4\times(S_{10\times10} - S_{8\times8}) = 9\times101\\
C_{8\times8} &:= T_{8\times8} - T_{6\times6} = 4\times(S_{8\times8} - S_{6\times6}) = 7\times2\times101\\
C_{6\times6} &:= T_{6\times6} - T_{4\times4} = 4\times(S_{6\times6} - S_{4\times4}) = 5\times2\times101\\
C_{4\times4} &:= T_{4\times4} - T_{2\times2} = 3\times2\times101,
\end{aligned}$$

where $T_{2\times2} = 49+50+51+52 = 202$ *are four central or middle values of* ***nested magic square****. In this case, the fixed difference among the* ***consecutive borders*** *is* $d_{border} := 404$*. The expression* $S_{2\times2}$ *is taken out as we don't have magic square of order 2.*

Finally, we get the following symmetric results

Result 2.32. *According to Result 2.31, the **nested magic square** of order 10 for the consecutive entries 1 to 100, has the following symmetric results:*

i. $S_{k\times k} := \frac{k}{2} \times \frac{T_{2\times 2}}{2}$;

ii. $T_{k\times k} := \left(\frac{k}{2}\right)^2 \times T_{2\times 2}$;

iii. $C_{k\times k} := (k-1) \times T_{2\times 2}$.

where $k = 4, 6, 8$ *and* 10 *orders of magic squares appearing **nested magic square** of order 10, and* $T_{2\times 2} := 202$ *is sum of four central values of magic square.*

2.17 Nested Magic Square of Order 9

In Example 2.15 remove the external border, then we are left with **nested magic square** of order 9 for the entries 21 to 101. Subtracting 20, we get following distribution for the **nested magic square** of order 9 for the consecutive natural numbers 1 to 91.

Distribution 2.17.

$$D_{3\times 3} := \{37, 38, \ldots, 44, 45\}$$
$$D_{5\times 5} := \{29, 30, \ldots, 35, 36, \mathbf{D_{3\times 3}}, 46, 47, \ldots, 52, 53\}$$
$$D_{7\times 7} := \{17, 18, \ldots, 27, 28, \mathbf{D_{5\times 5}}, 54, 55, \ldots, 64, 65\}$$
$$D_{9\times 9} := \{1, 2, \ldots, 15, 16, \mathbf{D_{7\times 7}}, 66, 67, \ldots, 80, 81\}$$

According to above distribution, the **nested magic square** of order 9 for the consecutive natural numbers entries from 1 to 81 is given by

Example 2.17. *Nested magic square of order 9 is given by*

									369
72	2	4	6	7	70	68	66	74	369
67	58	18	20	21	56	54	60	15	369
69	55	50	46	31	30	48	27	13	369
71	57	35	38	43	42	47	25	11	369
73	59	33	45	41	37	49	23	9	369
5	19	53	40	39	44	29	63	77	369
3	17	34	36	51	52	32	65	79	369
1	22	64	62	61	26	28	24	81	369
8	80	78	76	75	12	14	16	10	369
369	369	369	369	369	369	369	369	369	369

According to above distribution of magic squares entries, we have the following properties written as a result.

Result 2.33. *The **nested magic square** of order 9 for the entries 1 to 81 has the following properties:*

i. *The magic square sums are given by*

$$S_{3\times3} := 123 = 3\times 41 \qquad S_{7\times7} := 287 = 7\times 41$$
$$S_{5\times5} := 205 = 5\times 41 \qquad S_{9\times9} := 369 = 9\times 41$$

ii. *The total entries sums are given by*

$$T_{3\times3} := 3\times 123 = 369 = 3^2\times 41 \qquad T_{7\times7} := 7\times 287 = 2009 = 7^2\times 41$$
$$T_{5\times5} := 5\times 205 = 1025 = 5^2\times 41 \qquad T_{9\times9} := 9\times 369 = 3321 = 9^2\times 41$$

iii. *The **borders** entries sums for the **nested magic square** of order 23 are given by*

$$C_{9\times9} := T_{9\times9} - T_{7\times7} = 4\times(S_{9\times9} - S_{7\times7}) = 4\times 2^3\times 41$$
$$C_{7\times7} := T_{7\times7} - T_{5\times5} = 4\times(S_{7\times7} - S_{5\times5}) = 3\times 2^3\times 41$$
$$C_{5\times5} := T_{5\times5} - T_{3\times3} = 4\times(S_{5\times5} - S_{3\times3}) = 2\times 2^3\times 41$$
$$C_{3\times3} := T_{3\times3} - T_{1\times1} = 4\times(S_{3\times3} - S_{1\times1}) = 1\times 2^3\times 41$$

where $T_{1\times1} = S_{1\times1} = 61$ *is the central value. In this case, the fixed difference among the **consecutive borders** is* $d_{border} := 328 = 2^3\times 41$.

Finally, we get the following symmetric results

Result 2.34. *According to Result 2.33, the **nested magic square** of order 9 for the consecutive entries 1 to 81, has the following symmetric results:*

i. $S_{k\times k} := k\times T_{1\times1}$;

ii. $T_{k\times k} := k^2\times T_{1\times1}$;

iii. $C_{k\times k} := \frac{k-1}{2}\times 8\times T_{1\times1}$.

where $k = 3,5,7$ *and* 9 *orders of squares appearing **nested magic square** of order 9, and* $T_{1\times1} := 41$ *is the central value of the magic square.*

2.18 Nested Magic Square of Order 8

In Example 2.16 remove the external border, then we are left with **nested magic square** of order 8 for the entries 19 to 82. Subtracting 18, we get following distribution for the **nested magic square** of order 8 for the consecutive natural numbers 1 to 64.

Distribution 2.18.

$$D_{4\times 4} := \{25, 26, \ldots, 39, 40\}$$
$$D_{6\times 6} := \{15, 16, \ldots, 23, 24, \mathbf{D_{4\times 4}}, 41, 42, \ldots, 49, 50\}$$
$$D_{8\times 8} := \{1, 2, \ldots, 13, 14, \mathbf{D_{6\times 6}}, 51, 52, \ldots, 63, 64\}$$

According to above distribution, the **nested magic square** of order 8 for the consecutive natural numbers entries from 1 to 64 is given by

Example 2.18. *Nested magic square of order 8 is given by*

								260
58	63	3	1	14	52	12	57	260
55	45	21	48	15	47	19	10	260
56	42	38	25	28	39	23	9	260
61	24	31	36	33	30	41	4	260
11	22	35	32	29	34	43	54	260
6	16	26	37	40	27	49	59	260
5	46	44	17	50	18	20	60	260
8	2	62	64	51	13	53	7	260
260	260	260	260	260	260	260	260	260

According to above distribution of magic squares entries, we have the following properties written as a result.

Result 2.35. *The* ***nested magic square*** *of order 8 has the following properties:*

i. *The magic square sums are given by*

$$S_{4\times 4} := 130 = 2 \times 5 \times 13$$
$$S_{6\times 6} := 195 = 3 \times 5 \times 13$$
$$S_{8\times 8} := 260 = 4 \times 5 \times 13$$

ii. The sum of entries are given by

$$T_{4\times4} := 4\times202 = 808 \quad = 2^2\times2\times5\times13$$
$$T_{6\times6} := 6\times303 = 1818 = 3^2\times2\times5\times13$$
$$T_{8\times8} := 8\times404 = 3232 = 4^2\times2\times5\times13$$

*iii. The **borders** entries sums are given by*

$$C_{8\times8} := T_{8\times8} - T_{6\times6} = 4\times(S_{8\times8}-S_{6\times6}) = 7\times2\times5\times13$$
$$C_{6\times6} := T_{6\times6} - T_{4\times4} = 4\times(S_{6\times6}-S_{4\times4}) = 5\times2\times5\times13$$
$$C_{4\times4} := T_{4\times4} - T_{2\times2} \qquad = 3\times2\times5\times13,$$

*where $T_{2\times2} = 31+32+33+34 = 130$ are four central or middle values of **nested magic square**. In this case, the fixed difference among the **consecutive borders** is $d_{border} := 260$. The expression $S_{2\times2}$ is taken out as we don't have magic square of order 2.*

Finally, we get the following symmetric results

Result 2.36. *According to Result 2.35, the **nested magic square** of order 8 for the consecutive entries 1 to 64, has the following symmetric results:*

i. $S_{k\times k} := \frac{k}{2}\times\frac{T_{2\times2}}{2}$;

ii. $T_{k\times k} := \left(\frac{k}{2}\right)^2\times T_{2\times2}$;

iii. $C_{k\times k} := (k-1)\times T_{2\times2}$.

*where $k = 4, 6$ and 8 orders of magic squares appearing **nested magic square** of order 8, and $T_{2\times2} := 130$ is sum of four central values of magic square.*

2.19 Nested Magic Square of Order 7

In Example 2.17 remove the external border, then we are left with **nested magic square** of order 7 for the entries 17 to 65. Subtracting 16, we get following distribution for the **nested magic square** of order 7 for the consecutive natural numbers 1 to 49.

Distribution 2.19.

$$D_{3\times3} := \{21, 22, \ldots, 28, 29\}$$
$$D_{5\times5} := \{13, 14, \ldots, 19, 20, \mathbf{D_{3\times3}}, 30, 31, \ldots, 36, 37\}$$
$$D_{7\times7} := \{1, 2, \ldots, 11, 12, \mathbf{D_{5\times5}}, 38, 39, \ldots, 48, 49\}$$

According to above distribution, the **nested magic square** of order 7 for the consecutive natural numbers entries from 1 to 49 is given by

Example 2.19. *Nested magic square of order 7 is given by*

							175
42	2	4	5	40	38	44	175
39	34	30	15	14	32	11	175
41	19	22	27	26	31	9	175
43	17	29	25	21	33	7	175
3	37	24	23	28	13	47	175
1	18	20	35	36	16	49	175
6	48	46	45	10	12	8	175
175	175	175	175	175	175	175	175

According to above distribution of magic squares entries, we have the following properties written as a result.

Result 2.37. *The **nested magic square** of order 7 for the entries 1 to 49 has the following properties:*

i. The magic square sums are given by

$$S_{3\times 3} := 75 \quad = 3 \times 25$$
$$S_{5\times 5} := 125 = 5 \times 25$$
$$S_{7\times 7} := 175 = 7 \times 25$$

ii. The total entries sums are given by

$$T_{3\times 3} := 3 \times 75 \quad = 225 \quad = 3^2 \times 25$$
$$T_{5\times 5} := 5 \times 125 = 625 \quad = 5^2 \times 25$$
$$T_{7\times 7} := 7 \times 175 = 1225 = 7^2 \times 25$$

*iii. The **borders** entries sums for the **nested magic square** of order 23 are given by*

$$C_{7\times 7} := T_{7\times 7} - T_{5\times 5} = 4 \times (S_{7\times 7} - S_{5\times 5}) = 3 \times 2^3 \times 25$$
$$C_{5\times 5} := T_{5\times 5} - T_{3\times 3} = 4 \times (S_{5\times 5} - S_{3\times 3}) = 2 \times 2^3 \times 25$$
$$C_{3\times 3} := T_{3\times 3} - T_{1\times 1} = 4 \times (S_{3\times 3} - S_{1\times 1}) = 1 \times 2^3 \times 25$$

*where $T_{1\times 1} = S_{1\times 1} = 25$ is the central value. In this case, the fixed difference among the **consecutive borders** is $d_{border} := 200 = 2^3 \times 25$.*

Finally, we get the following symmetric results

Result 2.38. *According to Result 2.37, the **nested magic square** of order 7 for the consecutive entries 1 to 49, has the following symmetric results:*

i. $S_{k\times k} := k \times T_{1\times 1}$;

ii. $T_{k\times k} := k^2 \times T_{1\times 1}$;

iii. $C_{k\times k} := \dfrac{k-1}{2} \times 8 \times T_{1\times 1}$.

where $k = 3, 5$ *and* 7 *orders of squares appearing **nested magic square** of order 7, and* $T_{1\times 1} := 25$ *is the central value of the magic square.*

2.20 Nested Magic Square of Order 6

In Example 2.18 remove the external border, then we are left with **nested magic square** of order 6 for the entries 15 to 50. Subtracting 14, we get following distribution for the **nested magic square** of order 6 for the consecutive natural numbers 1 to 36.

Distribution 2.20.

$$D_{4\times 4} := \{11, 12, \ldots, 25, 26\}$$
$$D_{6\times 6} := \{1, 2, \ldots, 9, 10, \mathbf{D_{4\times 4}}, 27, 28, \ldots, 35, 36\}$$

According to above distribution, the **nested magic square** of order 6 for the consecutive natural numbers entries from 1 to 36 is given by

Example 2.20. *Nested magic square of order 6 is given by*

						111
31	7	34	1	33	5	111
28	24	11	14	25	9	111
10	17	22	19	16	27	111
8	21	18	15	20	29	111
2	12	23	26	13	35	111
32	30	3	36	4	6	111
111	111	111	111	111	111	111

According to above distribution of magic squares entries, we have the following properties written as a result.

Result 2.39. *The **nested magic square** of order 6 has the following properties:*

i. The magic square sums are given by

$$S_{4\times4} := 74 \quad = 2\times37$$
$$S_{6\times6} := 111 = 3\times37$$

ii. The sum of entries are given by

$$T_{4\times4} := 4\times74 \quad = 296 = 2^2\times2\times37$$
$$T_{6\times6} := 6\times111 = 666 = 3^2\times2\times37$$

*iii. The **borders** entries sums are given by*

$$C_{6\times6} := T_{6\times6} - T_{4\times4} = 4\times(S_{6\times6} - S_{4\times4}) = 5\times2\times37$$
$$C_{4\times4} := T_{4\times4} - T_{2\times2} = 3\times2\times37,$$

*where $T_{2\times2} = 17+18+19+20 = 74$ are four central or middle values of **nested magic square**. In this case, the fixed difference among the **consecutive borders** is $d_{border} := 148$. The expression $S_{2\times2}$ is taken out as we don't have magic square of order 2.*

Finally, we get the following symmetric results

Result 2.40. *According to Result 2.39, the **nested magic square** of order 6 for the consecutive entries 1 to 36, has the following symmetric results:*

i. $S_{k\times k} := \frac{k}{2}\times\frac{T_{2\times2}}{2}$;

ii. $T_{k\times k} := \left(\frac{k}{2}\right)^2\times T_{2\times2}$;

iii. $C_{k\times k} := (k-1)\times T_{2\times2}$.

*where $k = 4$ and 6 orders of magic squares appearing **nested magic square** of order 6, and $T_{2\times2} := 74$ is sum of four central values of magic square.*

2.21 Nested Magic Square of Order 5

In Example 2.19 remove the external border, then we are left with **nested magic square** of order 5 for the entries 13 to 37. Subtracting 12, we get following distribution for the **nested magic square** of order 5 for the consecutive natural numbers 1 to 25.

Distribution 2.21.

$$D_{3\times 3} := \{9, 10, \ldots, 16, 17\}$$
$$D_{5\times 5} := \{1, 2, \ldots, 7, 8, \mathbf{D_{3\times 3}}, 18, 19, \ldots, 24, 25\}$$

According to above distribution, the **nested magic square** of order 5 for the consecutive natural numbers entries from 1 to 25 is given by

Example 2.21. *Nested magic square of order 7 is given by*

					65
22	18	3	2	20	65
7	10	15	14	19	65
5	17	13	9	21	65
25	12	11	16	1	65
6	8	23	24	4	65
65	65	65	65	65	65

According to above distribution of magic squares entries, we have the following properties written as a result.

Result 2.41. *The **nested magic square** of order 9 for the entries 1 to 81 has the following properties:*

i. *The magic square sums are given by*

$$S_{3\times 3} := 39 = 3 \times 13$$
$$S_{5\times 5} := 65 = 5 \times 13$$

ii. *The total entries sums are given by*

$$T_{3\times 3} := 3 \times 39 = 117 = 3^2 \times 13$$
$$T_{5\times 5} := 5 \times 65 = 325 = 5^2 \times 13$$

iii. *The **borders** entries sums for the **nested magic square** of order 23 are given by*

$$C_{5\times 5} := T_{5\times 5} - T_{3\times 3} = 4 \times (S_{5\times 5} - S_{3\times 3}) = 2 \times 2^3 \times 13$$
$$C_{3\times 3} := T_{3\times 3} - T_{1\times 1} = 4 \times (S_{3\times 3} - S_{1\times 1}) = 1 \times 2^3 \times 13$$

*where $T_{1\times 1} = S_{1\times 1} = 13$ is the central value. In this case, the fixed difference among the **consecutive borders** is $d_{border} := 114 = 2^3 \times 13$.*

Finally, we get the following symmetric results

Result 2.42. *According to Result 2.41, the* ***nested magic square*** *of order 5 for the consecutive entries 1 to 25, has the following symmetric results:*

i. $S_{k\times k} := k \times T_{1\times 1}$;

ii. $T_{k\times k} := k^2 \times T_{1\times 1}$;

iii. $C_{k\times k} := \frac{k-1}{2} \times 8 \times T_{1\times 1}$.

where $k = 3$ *and* 5 *orders of squares appearing* ***nested magic square*** *of order 5, and* $T_{1\times 1} := 13$ *is the central value of the magic square.*

3 Final Results

In view of nested magic squares studied above, we have two main results summarized below. One is for even orders and another is for odd orders nested magic squares. Let's consider the following notations.

$$\begin{aligned} S_{k\times k} &\longrightarrow \text{magic square sums;} \\ T_{k\times k} &\longrightarrow \text{total entries sums;} \\ C_{k\times k} &\longrightarrow \text{borders entries sums;} \\ d_{border} &\longrightarrow \text{difference among borders value.} \end{aligned}$$

We have following two results.

Result 3.1. *The* ***nested magic squares of even orders*** *constructed according to Section 2, for consecutive numbers starting from 1 satisfy the following properties:*

i. $S_{k\times k} := k \times \frac{T_{2\times 2}}{4}$;

ii. $T_{k\times k} := k^2 \times \frac{T_{2\times 2}}{4}$;

iii. $C_{k\times k} := (k-1) \times 4 \times \frac{T_{2\times 2}}{4}$;

iv. $d_{border} := 8 \times \frac{T_{2\times 2}}{4}$.

where k *is the order of nested magic square and* $T_{2\times 2}$ *is the sum of four central or middle values.*

Result 3.2. *The* ***nested magic squares of odd orders*** *constructed according to Section 2, for consecutive numbers starting from 1 satisfy the following properties:*

i. $S_{k\times k} := k \times T_{1\times 1}$;

ii. $T_{k\times k} := k^2 \times T_{1\times 1}$;

iii. $C_{k\times k} := (k-1)\times 4 \times T_{1\times 1}$;

iv. $d_{border} := 8 \times T_{1\times 1}$.

where k *is the order of nested magic square and* $T_{1\times 1}$ *is the central value.*

Remark 1. *$T_{2\times 2}$ is the sum of for values and $T_{1\times 1}$ is single value. If we understand that $T_{1\times 1}$ is same as $\frac{T_{2\times 2}}{4}$, then both the Results 3.1 and 3.2 turns to be same. In view of this we can unify both the Results 3.1 and 3.2*

Result 3.3. *The **nested magic squares** constructed according to Section 2, for consecutive numbers starting from 1 satisfy the following properties:*

i. $S_{k\times k} := k \times L$;

ii. $T_{k\times k} := k^2 \times L$;

iii. $C_{k\times k} := (k-1)\times 4 \times L$;

iv. $d_{border} := 8 \times m$.

where k *is the order of nested magic square and*

$$L := T_{1\times 1}, \text{ odd order magic squares}$$
$$L := \frac{T_{2\times 2}}{4}, \text{ even order magic squares}$$

Study on **nested magic squares** for the **consecutive odd numbers** can be seen in author's another work [12]. Its summary is given in result below.

Result 3.4. *[12] For **nested magic squares** for **consecutive odd numbers**, the total entries sums are given by*

$$T_{k\times m} := k^2 \times m^2,$$

*where k is the order of **nested magic squares**, and m is the **order of each nested sub-magic square**.*

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