The Riemann Transform

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ABSTRACT

In his 1859 paper [1], Bernhard Riemann used an integral equation to develop an explicit formula for estimating the number of prime numbers less than a given quantity. It is the purpose of this present work to explore some of the properties of this equation.

Introduction

Because of my latest paper *The s-Parameter on the Transform Integrals is a Constant* [2], I've removed topics that are no longer relevant to this paper.

Consider the integral equation given below

(1) $$F(s) = \int_0^\infty f(x)\, x^{-s-1}\, dx$$

where x is the independent variable, s is a complex constant, and $F(s)$ is the trasform integral of $f(x)$. The function $f(x)$ and $\Re(s)$ are such that the integral in (1) is finite, $F(s)<\infty$. Since the integral is obtained for all x,

$$F(s) = \text{constant}$$

The derivative of $F(s)$ with respect to s is meaningless and all the integral associated with $F(s)$ will be zero

$$\int_s^s F(s)ds = 0 \quad \text{and so} \quad \frac{1}{2\pi i}\int_s^s F(s)x^s ds = 0$$

Example 1 Apply formula (1) to obtain the transform of $f(x)=e^{-x}$.
Solution.

$$F(s)=\int_0^\infty e^{-x}\, x^{-s-1}\, dx=\Gamma(-s) \quad \Re(s)<0, \quad \text{since} \quad \Gamma(s)=\int_0^\infty e^{-x} x^{s-1} dx, \quad \Re(s)>0$$

Unit Step Function (Heaviside Function)

The **unit step function** or **Heaviside function** $\mu(x-a)$ is 0 for $x < a$, has a jump size 1 at $x = a$ (where it is usually consider as undefined), and is 1 for $x > a$, in a formula:

$$\mu(x-a) = \begin{cases} 0 & \text{if } x < a \\ 1 & \text{if } x > a \end{cases} \qquad a \geq 0$$

The transform of $\mu(x-a)$ is

$$F(s) = \int_0^\infty x^{-s-1} \mu(x-a) dx = \int_a^\infty x^{-s-1} dx = \left. \frac{-x^{-s}}{s} \right|_a^\infty$$

here the integration begins at $x = a$ (>0) because $\mu(x-a)$ is 0 for $x < a$. Hence

$$F(s) = \frac{a^{-s}}{s} \qquad (a>0 \quad \text{and} \quad \Re(s)>0)$$

Example 2 The Riemann zeta sum is given by

$$\zeta(s) = 1^{-s}+2^{-s}+3^{-s}+\cdots = \sum_{n=1}^\infty n^{-s} \qquad \Re(s)>1$$

Obtain the transform of $\sum_{n=1}^\infty \mu(x-n)$, $n = 1,2,3,4,\ldots$

$$F(s)=\int_0^\infty \{\mu(x-1)+\mu(x-2)+\mu(x-3)+\cdots\}\, x^{-s-1} dx = \left.\frac{-x^{-s}}{s}\right|_1^\infty + \left.\frac{-x^{-s}}{s}\right|_2^\infty + \left.\frac{-x^{-s}}{s}\right|_3^\infty + \cdots$$

$$= \frac{1}{s}(1+2^{-s}+3^{-s}+4^{-s}+\cdots) = \frac{1}{s}\sum_{n=1}^\infty \frac{1}{n^s} = \frac{\zeta(s)}{s} \qquad \Re(s)>1$$

Example 3 Obtain the transform of $\pi(x) = \sum_p^\infty \mu(x-p)$, where p is a prime number, p = 2, 3, 5, 7, 11, ….

$$F(s) = \int_0^\infty \left\{ \sum_p^\infty \mu(x-p) x^{-s-1} dx \right\} = \int_0^\infty \{\mu(x-2)+\mu(x-3)+\mu(x-5)+\mu(x-7)+\cdots\} x^{-s-1} dx$$

$$\pi(s)=\frac{1}{s}(2^{-s}+3^{-s}+5^{-s}+7^{-s}+\cdots) = \frac{1}{s}\sum_p^\infty p^{-s} \qquad \Re(s)>1$$

Dirac's Delta Function

Consider the function

$$f_\tau(x-a) = \begin{cases} 1/\tau & \text{if } a \le x \le a+\tau \\ 0 & \text{otherwise.} \end{cases}$$

Its integral is

$$I = \int_0^\infty f_\tau(x-a)dx = \int_a^{a+\tau} \frac{1}{\tau}dx = 1$$

We let now let τ becomes smaller and smaller and take the limit as $\tau \to 0$ ($\tau > 0$). This limit is denoted by $\delta(x-a)$, that is,

$$\delta(x-a) = \lim_{\tau \to 0} f_\tau(x-a)$$

and obtain

$$\delta(x-a) = \begin{cases} \infty & \text{if } x = a \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \int_0^\infty \delta(x-a)dx = 1$$

$\delta(x-a)$ is called the **Dirac delta function** or the **unit impulse function**. For a *continuous* function $f(x)$ one uses the **sifting** property of $\delta(x-a)$,

$$\int_0^\infty f(x)\delta(x-a)dx = f(a)$$

To obtain the transform of $\delta(x-a)$, we write

$$f_\tau(x-a) = \frac{1}{\tau}\{\mu(x-a) - \mu(x-(a+\tau))\}$$

and take the transform

$$F(s) = \int_0^\infty f_\tau(x-a)x^{-s-1}dx = \frac{1}{\tau s}\left[a^{-s} - (a+\tau)^{-s}\right] = a^{-s}\frac{1-\left(1+\frac{\tau}{a}\right)^{-s}}{\tau s} \qquad a > 0 \text{ and } \Re(s) > 0$$

Take the limit as $\tau \to 0$. By l'Hopital's rule, the quotient on the right has the limit $1/a$. Hence, the right side has the limit $a^{-(s+1)}$. The transform of $\delta(x-a)$ define by this limit is

$$F(s) = \int_0^\infty \delta(x-a)x^{-s-1}dx = a^{-(s+1)} \qquad a > 0$$

Example 4 Obtain the transform of $\sum_{n=1}^{\infty} n\,\delta(x-n)$ and $\sum_{n=1}^{\infty} \delta(x-n)$.

$$\int_0^{\infty}\left\{\sum_{n=1}^{\infty} n\,\delta(x-n)\right\} x^{-s-1}dx = \sum_{n=1}^{\infty} n^{-s} = \zeta(s) \qquad \Re(s)>1$$

$$\int_0^{\infty}\left\{\sum_{n=1}^{\infty} \delta(x-n)\right\} x^{-s-1}dx = \sum_{n=1}^{\infty} n^{-(s+1)} = \zeta(s+1) \qquad \Re(s)>0$$

The Riemann Transform

Many common functions like $\sin x,\ \cos x,\ \ln x,\ etc.,$ when applied to formula (1) won't have finite integrals. But if the lower limit for (1) starts at $x = 1$, then there are suitable functions such that the integral in (1) exist.

If $f(x)$ is a function defined for all $x \geq 1$, its **Riemann transform** is the integral of $f(x)$ times x^{-s-1} for $x = 1$ to ∞. It is a function of s, say $F(s)$, and is denoted by $R\{f\}$; thus

(2)
$$F(s) = R\{f\} = \int_1^{\infty} f(x)\, x^{-s-1}\, dx$$

The given function $f(x)$ in (2) is called the **inverse transform** of $F(s)$ and is denoted by $R^{-1}\{F\}$; that is,

$$f(x) = R^{-1}\{F\}$$

Example 5 Let $f(x) = 1$, find $F(s)$.
Solution. From (2) we obtain by integration

$$R\{f\} = R\{1\} = \int_1^{\infty} x^{-s-1}dx = -\frac{1}{s}x^{-s}\Big|_1^{\infty} = \frac{1}{s} \qquad \Re(s)>0$$

Example 6 Let $f(x) = x^a$, where a is a constant. Find F(s).
Solution. From (2),

$$R\{x^a\} = \int_1^{\infty} x^a x^{-s-1}dx = -\frac{1}{s-a}x^{-(s-a)}\Big|_1^{\infty} = \frac{1}{s-a} \qquad \Re(s-a)>0$$

THEOREM 1 Linearity of the Riemann Transform

The Riemann transform is a linear operation; that is, for any functions f(x) and g(x) whose transforms exist and any constants a and b the transform of af(x) + bg(x) exists, and

$$R\{af(x)+bg(x)\} = aF(s) + bG(s)$$

Example 7 Find the transforms of cosh ($a$ln$x$) and sinh ($a$ln$x$).

Solution. Since $\cosh(a\ln x)=\frac{1}{2}(x^{a}+x^{-a})$ and $\sinh(a\ln x)=\frac{1}{2}(x^{a}-x^{-a})$, we obtain from Example 6 and Theorem 1,

$$R\{\cosh(a\ln x)\}=\frac{1}{2}(R(x^{a})+R(x^{-a}))=\frac{1}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right)=\frac{s}{s^{2}-a^{2}}$$

$$R\{\sinh(a\ln x)\}=\frac{1}{2}(R(x^{a})-R(x^{-a}))=\frac{1}{2}\left(\frac{1}{s-a}-\frac{1}{s+a}\right)=\frac{a}{s^{2}-a^{2}}$$

Example 8 Let $f(x)=x^{\alpha i}$, where i is the imaginary operator $(i=\sqrt{-1})$. Find $F(s)$.

Solution. From Example 6

$$R\{x^{\alpha i}\}=\frac{1}{s-\alpha i}=\frac{1}{s-\alpha i}\,\frac{s+\alpha i}{s+\alpha i}=\frac{s}{s^{2}+\alpha^{2}}+i\frac{\alpha}{s^{2}+\alpha^{2}}$$

Example 9 Cosine and Sine
Derive the formulas

$$R\{\cos(\alpha\ln x)\}=\frac{s}{s^{2}+\alpha^{2}}\quad\text{and}\quad R\{\sin(\alpha\ln x)\}=\frac{\alpha}{s^{2}+\alpha^{2}}$$

Solution. From Example 8 and Theorem 1

$$x^{\alpha i}=\cos(\alpha\ln x)+i\sin(\alpha\ln x)$$

$$R\{x^{\alpha i}\}=R\{\cos(\alpha\ln x)\}+iR\{\sin(\alpha\ln x)\},\quad\text{thus}$$

$$R\{\cos(\alpha\ln x)\}=\frac{s}{s^{2}+\alpha^{2}}\quad\text{and}\quad R\{\sin(\alpha\ln x)\}=\frac{\alpha}{s^{2}+\alpha^{2}}$$

THEOREM 2 *s*-Shifting Theorem

If f(x) has the transform F(s) (where s > k for some k), then x^a f(x) has the transform F(s – a) (where s – a > k). In formulas,

$$R\{x^{a}f(x)\}=F(s-a)$$

or, if we take the inverse on both sides

$$x^{a}f(x)=R^{-1}\{F(s-a)\}$$

PROOF We obtain $F(s-a)$ by replacing s with $s-a$ in the integral in (1), so that

$$F(s-a) = \int_1^\infty x^{-(s-a)-1} f(x)dx = \int_1^\infty x^{-s-1}[x^a f(x)]dx = R\{x^a f(x)\}$$

Example 10 From Example 9 and the *s*-Shifting theorem one can obtain the Riemann transform for

$$R\{x^a \cos(\alpha \ln x)\} = \frac{s-a}{(s-a)^2+\alpha^2} \quad \text{and} \quad R\{x^a \sin(\alpha \ln x)\} = \frac{\alpha}{(s-a)^2+\alpha^2}$$

Existence and Uniqueness of Riemann Transforms

A function $f(x)$ has a Riemann transform if it does not grow too fast, say, if for all $x \geq 1$ and some constants M and k it satisfies

(3) $$|f(x)| \leq Mx^k$$

THEOREM 3 Existence Theorem for Riemann Transforms

If f(x) is defined and piecewise continuous on every finite interval on $x \geq 1$ and satisfies (3) for all $x \geq 1$ and some constants M and k, then the Riemann transform R{f} exists for all $s > k$.

PROOF Since $f(x)$ is piecewise continuous, $x^{-s-1}f(x)$ is integrable over any finite interval on the x-axis,

$$|R\{f\}| = \left|\int_1^\infty f(x)x^{-s-1}\right| \leq \int_1^\infty |f(x)|x^{-s-1}dx \leq \int_1^\infty M\, x^k x^{-s-1} dx = \frac{M}{s-k}$$

Uniqueness. If the Riemann transform of a given function exists, it is uniquely determined and if two *continuous* functions have the same transform, they are completely identical.

Transforms of Derivatives and Integrals

THEOREM 4 Riemann Transform of Derivatives

The transforms of the first and second derivatives of f(x) satisfy

(4) $$R(f') = (s+1)F(s+1) - f(1)$$

(5) $$R(f'') = (s+2)(s+1)F(s+2) - (s+1)f(1) - f'(1)$$

Formula (4) holds if f(x) is continuous for all $x \geq 1$ and satisfies (3) and f'(x) is piecewise continuous on every finite interval for $x \geq 1$. Formula (5) holds if f and f' are continuous for all $x \geq 1$ and satisfy (3) and f'' is piecewise continuous on every finite interval for $x \geq 1$.

PROOF Using integration by parts on formula (4)

$$R\{f\} = \int_1^\infty f'(x)x^{-s-1}dx = [f(x)x^{-s-1}]|_1^\infty + (s+1)\int_1^\infty f(x)x^{-s-2}dx = -f(1) + (s+1)F(s+1)$$

The proof of (5) now follows by applying integration by parts twice on it, that is

$$\begin{aligned} R\{f''\} &= \int_1^\infty f''(x)x^{-s-1}dx = [f'(x)x^{-s-1}]|_1^\infty + (s+1)\int_1^\infty f'(x)x^{-s-2}dx \\ &= -f'(1) + (s+1)\left[f(x)x^{-s-2}|_1^\infty + (s+2)\int_1^\infty f(x)x^{-s-3}dx\right] \\ &= -f'(1) - (s+1)f(1) + (s+2)(s+1)F(s+2) \end{aligned}$$

Repeatedly using integration by parts as in the proof of (5) and using induction, we obtain the following Theorem.

THEOREM 5 Riemann Transform of the Derivative $f^{(n)}$ of Any Order

Let f, f', …, $f^{(n-1)}$ be continuous for all $x \geq 1$ and satisfy (2). Furthermore, let $f^{(n)}$ be piecewise continuous on every finite interval for $x \geq 1$. Then the transform of $f^{(n)}$ satisfies

$$R\{f^{(n)}\} = (s+n)(s+n-1)\cdots(s+1)F(s+n) - (s+n-1)(s+n-2)\cdots f(1) - (s+n-2)(s+n-3)\cdots f'(1) - \cdots - f^{(n-1)}(1)$$

Example 11 Let $f(x) = x^2$. Then $f(1) = 1$, $f'(x) = 2x$, $f'(1) = 2$, $f''(x) = 2$. Obtain $R\{f\}$, $R\{f'\}$, and $R\{f''\}$.

Solution. $R\{f\} = F(s) = \dfrac{1}{s-2}$, $F(s+1) = \dfrac{1}{s-1}$, $F(s+2) = \dfrac{1}{s}$. Hence, by formulas (4) and (5),

$$R(f') = (s+1)\frac{1}{s-1} - 1 = \frac{2}{s-1} \quad \text{and} \quad R(f'') = (s+2)(s+1)\frac{1}{s} - (s+1) - 2 = \frac{2}{s}$$

THEOREM 6 Riemann Transform of Integrals

Let F(s) denote the transform of a function f(x) which is piecewise continuous for $x \geq 1$ and satisfies formula (3). Then, for s > 0, s > k, and x > 1,

(6) $$R\left\{\int_1^x f(\tau)d\tau\right\} = \frac{1}{s}F(s-1), \quad \text{thus} \quad \int_1^x f(\tau)d\tau = R^{-1}\left\{\frac{1}{s}F(s-1)\right\}$$

PROOF Let the integral in (6) be g(x) then g'(x) = f(x). Since g(1) =0 (the integral from 1 to 1 is zero),

$$R\{f(x)\} = R\{g'(x)\} = (s+1)G(s+1) - g(1) = (s+1)G(s+1) = F(s)$$

replace s by $s - 1$, $([s-1] + 1)G([s-1]+1) = F(s-1) = sG(s) = F(s-1).$

Division by s and interchange of the left and right side gives the first formula in (6), from which the second follows.

Example 12 Let $f(x) = x$. Obtain $R\{g(x)\} = R\left\{\int_1^x \tau \, d\tau\right\} = G(s).$

Solution. $F(s) = R\{x\} = \dfrac{1}{s-1}$, $F(s-1) = \dfrac{1}{s-2}$, then $G(s) = \dfrac{1}{s(s-2)}$

The Riemann Transform and the Laplace Transform

The Laplace transform is the integral of $f(y)$ times e^{-sy} from $y = 0$ to ∞ where $f(y)$ is defined for all $y \geq 0$. It is denoted by $L\{f\}$,

(7) $$L\{f\} = \int_0^\infty f(y) e^{-sy} dy$$

The Riemann transform is given below

(8) $$R\{f\} = \int_1^\infty f(x) x^{-s-1} dx$$

Replace $x = e^y$ (or $y = \ln x$) in formula (8) and since $x = 1$ to ∞, $y = 0$ (ln1) to ∞ (ln∞).

$$\int_1^\infty f(x) x^{-s-1} dx = \int_0^\infty f(e^y) e^{-sy-y} d(e^y) = \int_0^\infty f(y) e^{-sy} dy$$

which is formula (7).

The Bilateral Laplace Transform

Formula (7) is usually called the **Unilateral** Laplace transform since the integral is evaluated from 0 to ∞. The integral below is known as the Bilateral Laplace transform because the integral is taken from -∞ to ∞,

(9) $$B\{f\} = \int_{-\infty}^{\infty} f(y) e^{-sy} dy$$

Now, consider the integral equation

(10) $$\int_{0}^{\infty} f(x)\, x^{-s-1}\, dx$$

Replace $x = e^y$ (or $y = \ln x$) in formula (10) and since $x = 0$ to ∞, $y = -\infty$ to ∞, thus

$$\int_{-\infty}^{\infty} f(e^y)\, e^{-ys-y}\, d(e^y) = \int_{-\infty}^{\infty} f(y)\, e^{-sy}\, dy$$

which is (9).

Riemann Transform: General Formulas

Formula	Name
$F(s) = R\{f(x)\} = \int_{1}^{\infty} f(x)\, x^{-s-1}\, dx$ $f(x) = R^{-1}(F(s))$	Definition of Transform Inverse Transform
$R\{af(x) + bg(x)\} = aR\{f(x)\} + bR\{g(x)\}$	Linearity
$R\{x^a f(x)\} = F(s-a)$ $R^{-1}\{F(s-a)\} = x^a f(x)$	s-Shifting Theorem
$R(f') = (s+1)F(s+1) - f(1)$ $R(f'') = (s+2)(s+1)F(s+2) - (s+1)f(1) - f'(1)$ $R\left\{\int_{1}^{x} f(\tau) d\tau\right\} = \frac{1}{s} F(s-1)$	Differentiation of Function Integration of Function

Table: Some Riemann Transforms

	$f(x)=R^{-1}\{F(s)\}$	$F(s)=\int_1^{\infty} f(x)x^{-s-1}dx$
1	1	$\frac{1}{s}$
2	x	$\frac{1}{s-1}$
3	x^a	$\frac{1}{s-a}$
4	$x^{\alpha i}$	$\frac{1}{s-\alpha i}$
5	$\cos(\alpha \ln x)$	$\frac{s}{s^2+\alpha^2}$
6	$\sin(\alpha \ln x)$	$\frac{\alpha}{s^2+\alpha^2}$
7	$\cosh(a \ln x)$	$\frac{s}{s^2-a^2}$
8	$\sinh(a \ln x)$	$\frac{a}{s^2-a^2}$
9	$x^b \cos(\alpha \ln x)$	$\frac{s-b}{(s-b)^2+\alpha^2}$
10	$x^b \sin(\alpha \ln x)$	$\frac{\alpha}{(s-b)^2+\alpha^2}$

REFERENCES

[1] Riemann, Bernhard (1859). *On the Number of Prime Numbers less than a Given Quantity*(pp 5-7).

[2] Evangelista, Armando M. (2019). *The s-Parameter on the Transform Integrals is a Constant.* https://zenodo.org/record/3244311#.XQpvLy17H9M