Motion of A Rocket in Three-Dimension with Constant Thrust Over A Spherical Rotating Earth Holding Constant Heading and Constant Path Inclination

SN Maitra

Retired Head of Mathematics Department, National Defense Academy, Khadakwasla, Pune-411023, INDIA

Abstract— *In this paper we have determined the velocity and altitude of a spacecraft and then equation of its trajectory with constant thrust, constant heading and constant path-inclination by regulating the bank angle and angle of attack.*

***Keywords—* Path inclination, constant, relative, rotating, spherical, Bessel's equation, angular velocity, thrust, rocket.**

I. INTRODUCTION

Angelo Miele[1] derived differential equations of three- dimensional motion of a spacecraft relative to a spherical rotating Earth. In fact analytical solutions to these complicated equations are not possible.He[1] made no attempt to solve them with some simplified assumptions or introducing some constraints. In this feature are first determined the velocity and altitude of the space vehicle and the equation of its trajectory with constant thrust, constant heading and constant path-inclination by manipulating the bank angle and the angle of attack.

Three-dimensional equations of motion[1] of the space vehicle relative to the spherical rotating Earth are rewritten in a modified form considering the thrust along the flight path and neglecting the variation of gravity with altitude:

$$\dot{X} = \frac{VR\cos\gamma\cos\chi}{(h+R)\cos(\frac{Y}{R})},\ \dot{Y} = \frac{\boldsymbol{VR}\cos\gamma\,\boldsymbol{sin\chi}}{(\boldsymbol{h}+\boldsymbol{R})}$$

$$\dot{h} = V\sin\gamma,\ \dot{V} = \frac{-D}{m} + \frac{T}{m} - g\sin\gamma,\ T = \beta V_E,\ \dot{m} + \beta = 0 \tag{1}$$

$$\dot{\gamma} = (\frac{L}{mV})\cos\emptyset - \frac{g}{V}\cos\gamma + \frac{V\cos\gamma}{h+R} + 2\omega\cos\chi\cos\frac{Y}{R}$$

$$\dot{\chi} = (\frac{L}{mv\cos\gamma})\sin\emptyset - \frac{V\cos\gamma\cos\chi\tan Y/R}{h+R} + 2\omega(\tan\gamma\sin\chi\cos\frac{Y}{R} - \sin\frac{Y}{R})$$

$$D = \frac{1}{2}C_D\rho SV^2,\ L = \frac{1}{2}C_L\rho SV^2$$

II. NOMENCLATURE

D=drag ,L=lift, h=altitude, m=mass of the space craft, R= radius of the Earth, t =time, V= velocity at time t, V_E=exhaust velocity, $\emptyset$ =bank angle,X= longitudinal range over the Earth's surface, Y =latitudinal range over the Earth's surface, γ = flight − path angle, χ = heading angle, ω = Earth's rotational angular velocity, C_D=drag coefficient ,C_L=lift coefficient, ρ = atmospheric density, S = reference surface area of the spacecraft, g = acceleration due to gravity,T= thrust of the spacecraft, β = constant propellant mass flow ,ie, rate of propellant consumption, prime sign='=derivative with respect to mass m. $\frac{\frac{1}{2}C_D\rho S}{\beta}$ =K_D= drag factor.

III. FORMATION OF EQUATIONS OF MOTION

Combining two relevant equations of set(1), we obtain

$$\frac{dV}{dm} = \frac{K_D V^2}{m} - \frac{V_E}{m} + c \tag{2}$$

where $c=\frac{g\sin\gamma}{\beta}$ =constant (3)

Equation (2) can be converted into Bessel's equation by substituting,

$$V = -\frac{1}{K_D}\frac{m}{u}\frac{du}{dm} \tag{4}$$

$$m^2\frac{d^2u}{dm^2} + m\frac{du}{dm} + (cm - V_E)K_D u = 0 \tag{5}$$

Comparing equation (5) with equation (2), we get

$$x^2\frac{d^2y}{dx^2} + nx\frac{dy}{dx} + (b + c_1x^{2m_1})y = 0$$

IV. SOLUTION TO THE EQUATIONS

Solution[2] to the above equation is stated as

$$y = x^{\frac{-(n-1)}{2}}\left[Aj_\mu\left(\frac{c_1^{\frac{1}{2}}x^{m_1}}{m_1}\right) + Bj_{-\mu}\left(\frac{c_1^{\frac{1}{2}}x^{m_1}}{m_1}\right)\right]$$

where $\mu^2 m_1^2 = \frac{1}{4}(n-1)^2 - b$ and 2μ is of non integral value in general. *But* $j_{-\mu}$ is replaced by Y_μ. However in this situation with n=1, x=m, y=u, $c_1 = cK_D$, $b = -V_E K_D$ *and* $m_1 = \frac{1}{2}$, the required solution to equation (5) with $\mu = 2\sqrt{V_E K_D}$ is given by 'as usual' Bessel function[2]:

$$u = Aj_{2\sqrt{V_E K_D}}(2\sqrt{cK_D m}) - BY_{2\sqrt{V_E K_D}}(2\sqrt{cK_D m}) \tag{5.1}$$

and hence because of (4) the velocity is given by

$$V = \frac{-\sqrt{\frac{cm}{K_D}}[j'_{2\sqrt{V_E K_D}}(2\sqrt{cK_D m}) - \lambda Y'_{2\sqrt{V_E K_D}}(2\sqrt{cK_D m})]}{j_{2\sqrt{V_E K_D}}(2\sqrt{cK_D m}) - \lambda Y_{2\sqrt{V_E K_D}}(2\sqrt{cK_D m})} \tag{6}$$

where constant $\lambda = \frac{B}{A}$ is determined by use of the initial conditions at t=0,

$$m = m_0, h=0, V=0 \tag{7}$$

$$\lambda = \frac{j'_{2\sqrt{V_E K_D}}(2\sqrt{cK_D m})}{Y'_{2\sqrt{V_E K_D}}(2\sqrt{cK_D m})} \tag{8}$$

Now maneuvering of two control parameters bank angle ϕ and lift coefficient C_L is to bring about constant heading χ_0 and constant flight − path angle γ_0 from (1)

$$\text{with } \dot{\chi} = 0 \text{ and } \dot{\gamma} = 0 \Rightarrow \chi = \chi_0 \text{ and } \gamma = \gamma_0 \tag{9}$$

With this aspect two equations involving ϕ from (1) yield

$$\tan\phi = \frac{\frac{V\cos\gamma_0\cos\chi_0}{h+R}\tan\frac{Y}{R} - 2\omega(\tan\gamma_0\sin\chi_0\cos\frac{Y}{R} - \sin\frac{Y}{R})}{\frac{g}{V} - \frac{V}{h+R} - 2\omega\cos\chi_0\frac{\cos\frac{Y}{R}}{\cos\gamma_0}} \tag{10}$$

Similarly,

$$C_L=\frac{2m}{\rho SV}\left[\sqrt{\left(\frac{g}{V}-\frac{V}{h+R}-2\omega\cos\frac{Y}{R}\right)^2+\left\{\frac{V\cos\chi_0}{h+R}\tan\frac{Y}{R}-2\omega\left(\tan\gamma_0 sin\chi_0\cos\frac{Y}{R}-\sin\frac{Y}{R}\right)\right\}^2}\right]\cos\gamma_0 \quad (11)$$

In order to execute time-varying $\emptyset$ *and*

C_L, equations (9)and (11) need to be expressed as functions of time t or mass m .With the initial conditions (7),from (1)and (4) one gets

$$h(m)=\int_0^t V(t)\, sin\gamma_0 dt$$

$$=\frac{\sin\gamma_0}{\beta}\sqrt{\frac{c}{K_D}}\int_m^{m_0}\frac{[j'_{2\sqrt{V_E K_D}}(2\sqrt{cK_D m})-\lambda Y'_{2\sqrt{V_E K_D}}(2\sqrt{cK_D m})}{j_{2\sqrt{V_E K_D}}(2\sqrt{cK_D m})-\lambda Y_{2\sqrt{V_E K_D}}(2\sqrt{cK_D m})}\sqrt{m}dm \quad (12)$$

which needs to be evaluated numerically.

Considering γ *and* χ constants equal to γ_0 and χ_0 respectively as mentioned earlier by regulating / varying C_L and ϕ with time and now combining the first two of set(1),

$$\frac{dY}{dX}=\tan\chi_0\cos\frac{Y}{R} \quad (13)$$

Since initially at t=0,X=0,Y=0,h=0, (14)

the equation of the trajectory of the rocket is obtained from (13) as

$$\sec\frac{Y}{R}+\tan\frac{Y}{R}=e^{X\left(\frac{\tan\chi_0}{R}\right)} \quad (15)$$

$$\text{Now, } sec^2\frac{Y}{R}-tan^2\frac{Y}{R}=1 \quad (15.1)$$

Combining these two equations and simplifying we get

$$Y=R tan^{-1[\sin hyperbolic}\left(X\frac{tan\,\chi_0}{R}\right)] \quad (16)$$

$$\text{Similarly, } \frac{dY}{dh}=\frac{R\cot\gamma_0\sin\chi_0}{R+h} \quad (17)$$

$$\text{Or, } Y=R\cot\gamma_0\sin\chi_0\log(1+\frac{h}{R})$$

$$\text{Or, } h=R[e^{\frac{(Y\tan\gamma_0)}{R\sin\chi_0}}-1] \quad (18)$$

Owing to the initial conditions (7), from equation (1)

$$m=m_0-\beta t \quad (19)$$

V. CONCLUSION

Hence knowing the velocity- mass distribution(6),altitude-mass distribution (12), equation of the trajectory, ie, longitudinal-latitudinal range distribution (16),altitude-latitudinal range distribution (18) and finally simple mass variation law (19) with respect to time t, the bank angle (10) and the lift coefficient (11) can be made time- varying control parameters so as to accomplish such type of rocket motion.

REFERENCES

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