Mathematical Models and Methods in Applied Sciences

TIME-DEPENDENT RESCALINGS AND LYAPUNOV FUNCTIONALS FOR THE VLASOV-POISSON AND EULER-POISSON SYSTEMS, AND FOR RELATED MODELS OF KINETIC EQUATIONS, FLUID DYNAMICS AND QUANTUM PHYSICS

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We investigate rescaling transformations for the Vlasov-Poisson and Euler-Poisson systems and derive in the plasma physics case Lyapunov functionals which can be used to analyze dispersion effects. The method is also used for studying the long time behaviour of the solutions and can be applied to other models in kinetic theory (2-dimensional symmetric Vlasov-Poisson system with an external magnetic field), in fluid dynamics (Euler system for gases) and in quantum physics (Schrödinger-Poisson system, nonlinear Schrödinger equation).

Keywords: scalings – Lyapunov functional – intermediate asymptotics – Strichartz estimate – dispersion – kinetic equations – Vlasov-Poisson system – Euler-Poisson system – fluid dynamics – Wigner equation – Schrödinger equation

1. Introduction

Consider the Vlasov-Poisson system (VP)

$$\left\{ \begin{array}{l} \partial_t f + v\cdot\partial_x f - \partial_x U\cdot\partial_v f = 0, \\ \triangle U = \varepsilon\rho,\ \ \rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,dv \end{array} \right.$$

and the pressureless Euler-Poisson system (EP)

$$\left\{ \begin{array}{l} \partial_t \rho + \mathrm{div}(\rho\, u) = 0, \\ \partial_t u + (u\cdot\partial_x)u = -\partial_x U, \\ \triangle U = \varepsilon\rho. \end{array} \right.$$

Here $t\geq 0$, x, v, $u = u(t,x)\in\mathbb{R}^d$, $d\geq 1$ is the dimension of the physical space, $\varepsilon = +1$ corresponds to the stellar dynamics and $\varepsilon = -1$ to the plasma physics case. Throughout this paper, we shall assume that f is a nonnegative function in $L^\infty(\mathbb{R}^+,\, L^1(\mathbb{R}^d \times \mathbb{R}^d))$. Formally, we have the following relation between these two systems: a pair (ρ, u) is a solution of (EP) if and only if

$$f(t,x,v) = \rho(t,x)\delta\Big(v - u(t,x)\Big)$$

is a solution of (VP) where δ denotes the Dirac delta distribution. In this situation u can be recovered from f via the identity

$$\rho(t,x)\; u(t,x) = \int_{\mathbb{R}^d} v\, f(t,x,v)\, dv. \tag{1.1}$$

In this sense (EP) is a special case of (VP), and we will see later that the asymptotic behaviour of (VP) for large times is connected with a special solution of (EP). On a rigorous level the relation of (VP) with (EP) is investigated in [10].

Throughout this paper, we will also assume for simplicity that the solutions of (VP) are of class C^1 with compact support with respect to x and v, which allows us to perform any integration by parts without further justifications (except maybe in dimension 2). The results then pass to less smooth classes of solutions, assuming for instance that f belongs to $C^0(\mathbb{R}^+, L^1(\mathbb{R}^d \times \mathbb{R}^d))$ (see for instance [25] or [29]) and is a global in time solution to the Cauchy problem corresponding to an initial data f_0 satisfying for instance:

$(d = 2)$ $f_0 \in L^1 \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ is such that for some $\epsilon > 0$ the quantity

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} f_0(x,v)\; (|x|^{2+\epsilon} + |v|^{2+\epsilon} + |U_0(x)|)\; dxdv$$

(with $U_0(x) = -\frac{1}{2}\log|x| * \int f_0\, dv$) is bounded (see [11]).

$(d = 3)$ $f_0 \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ is such that for some $\epsilon > 0$ and $p > 3$ the quantity

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} f_0(x,v)(|v|^{2+\epsilon} + |x|^p)\; dxdv$$

is bounded (see [20,27,28,29] and [5,6] for the propagation of moments).

For weak solutions obtained as a limit of an approximating sequence (for instance, if we assume no moments higher than 2), the equalities have to be replaced by inequalities.

For the Euler-Poisson system, we shall consider only C^1 solutions. The results presented in this paper have to be understood as either a general method on how to obtain dispersion effects without taking care of the existence or the regularity of the solutions, or as a method to derive *a priori* estimates for less regular solutions (by passing to the limit with smooth approximating solutions).

Our paper is organized as follows. In Section 2, we introduce linear scalings and explain why they give rise to singular self-similar problems. How to remedy this pathology with time-dependent scalings is explained in Section 3. In the one-dimensional case, the information on the solution is sufficient to provide the convergence of the rescaled solution to the asymptotic measure. Section 4 is concerned with the Lyapunov functionals and constitutes the heart of this paper: the energy of the rescaled system turns out to be a Lyapunov functional for the initial problem. A more straightforward (than the full time-dependent scaling method) approach to the Lyapunov functionals is also given. In Section 5, we use the Lyapunov functionals to describe the asymptotic behaviour (dispersion rate) of the solutions in the plasma physics case.

Rescalings for the study of large time behaviour have been widely used in various fields of applied mathematics but appear to be rather new in the context of kinetic equations: in that direction we may mention the studies made by J. R. Burgan, M. R. Feix, E. Fijalkow, and A. Munier (see [2]) and J. Batt, M. Kunze, G. Rein in [1]. Our main point is to make the link between rescalings preserving the L^1-norm and Lyapunov functionals (or pseudo-conformal laws) and to explain on various examples of conservative systems why it actually provides a general method for the study of large time asymptotics.

While Sections 1–5 are exclusively devoted to the Vlasov-Poisson and Euler-Poisson systems, Sections 6–8 are concerned with other problems of kinetic theory, fluid mechanics and quantum physics. The relation between these various domains has been noticed for a long time (see for instance [26]), but it is surprising that the estimates given in [17] have been adapted to kinetic models only recently. Here we proceed in the reverse historical order, from kinetic equations to fluids and quantum physics, and this approach actually seems to be very powerful.

To conclude with the introduction, it is worth mentioning that many of the estimates we are giving in this paper were already at least partially known. The point is that we present a systematic and elementary method which takes the nonlinearity of the model very well into account (this was not necessarily the case in the preceding papers) and gives rise to a more precise form of the Lyapunov functionals (in the sense that these Lyapunov functionals also include second-moments in the x-variable) which are natural for the problems we consider.

2. Linear Scalings

Let $f = f(t,x,v)$ be a solution of (VP). Then for any $\lambda, \mu > 0$

$$f_{\lambda,\mu}(t,x,v) = \lambda^{2-d}\mu^d f(\lambda t, \mu x, \lambda^{-1}\mu v)$$

is again a solution of (VP), as can be checked by direct computation using for instance the following integral representation of $\partial_x U$. Similarly, if $(\rho, u) = (\rho, u)(t,x)$ is a solution of (EP) then for any $\lambda, \mu > 0$

$$\rho_{\lambda,\mu}(t,x) = \lambda^2 \rho(\lambda t, \mu x),\ u_{\lambda,\mu}(t,x) = \lambda\mu^{-1}u(\lambda t, \mu x)$$

is again a solution of (EP), and the potential is transformed as for (VP). This also follows from Relation (1.1) between (VP) and (EP).

If we require that the L^1-norm of $\rho(t)$, which is a conserved quantity for (EP) as well as for (VP), is preserved by the scaling, λ and μ must satisfy

$$\lambda^2 \mu^{-d} = 1$$

and the rescaled distribution function is $f_{\lambda,\lambda^{2/d}}$. A standard way of studying the asymptotic behaviour of f would then be to consider a self-similar solution, *i.e.* a solution which satisfies $f(t,x,v) = f_{\lambda,\lambda^{2/d}}(t,x,v) = \lambda^{4-d} f(\lambda t, \lambda^{2/d} x, \lambda^{2/d-1} v)$ for any $\lambda > 0$. This solution would then be given by its self-similar profile $\tilde{f}(\xi,\eta) = f(1,\xi,\eta)$ (choose λ to be $\frac{1}{t}$). Then

$$f(t,x,v) = f_{t^{-1},t^{-2/d}}(t,x,v) = t^{d-4} \tilde{f}(t^{-\frac{2}{d}} x, t^{1-\frac{2}{d}} v) \tag{2.1}$$

is a solution of (VP) if, at least formally, $\tilde{f}$ is a solution of

$$\begin{gathered}(d-4)\tilde{f} + \eta \cdot \partial_\xi \tilde{f} - \tfrac{2}{d}\xi \cdot \partial_\xi \tilde{f} + (1-\tfrac{2}{d})\eta \cdot \partial_\eta \tilde{f} - \partial_\xi \tilde{U} \cdot \partial_\eta \tilde{f} = 0, \\ \triangle_\xi \tilde{U} = \varepsilon \tilde{\rho}, \quad \tilde{\rho}(\xi) = \textstyle\int_{\mathbb{R}^d} \tilde{f}(\xi,\eta)\, d\eta\end{gathered}$$

in the new variables $\xi = t^{-\frac{2}{d}} x$, $\eta = t^{1-\frac{2}{d}} v$. However, it is clear that as $t \to 0+$, $g(t,x,v)$ does not converge to a well defined measure for which one might establish an existence result, except for $d = 1$. This difficulty is completely removed by considering general, non-singular, time-dependent scalings.

3. Time-dependent Scalings

Consider the following transformation of variables in (VP), where the positive functions $A(t)$, $R(t)$, $G(t)$ will be determined later:

$$dt = A^2(t) d\tau, \quad x = R(t)\xi.$$

Thus, assuming that $t \mapsto x(t)$ and $\tau \mapsto \xi(\tau)$ satisfy $\frac{dx}{dt} = v$ and $\frac{d\xi}{d\tau} = \eta$ respectively, the new velocity variable η has to satisfy

$$v = \frac{dx}{dt} = \dot{R}(t)\xi + R(t)\frac{d\xi}{d\tau}\frac{d\tau}{dt} = \dot{R}(t)\xi + \frac{R(t)}{A^2(t)}\eta.$$

Let F be the rescaled distribution function:

$$f(t,x,v) = G(t) F(\tau,\xi,\eta).$$

The aim is to choose this transformation in such a way that the transformed Vlasov equation is still a transport equation on phase space and contains a given, external force and a friction term. The inverse transformation is

$$d\tau = A^{-2}(t) dt, \ \xi = R^{-1}(t) x, \ \eta = \frac{A^2(t)}{R(t)}\left(v - \frac{\dot{R}(t)}{R(t)} x\right).$$

Here $\dot{}$ always denotes derivative with respect to t. If ν and W are defined as the rescaled spatial density and the rescaled potential respectively, then

$$\nu(\tau,\xi) = \int_{\mathbb{R}^d} F(\tau,\xi,\eta)\, d\eta = \tfrac{A^{2d}}{R^d G}\rho(t,x),$$

$$W(\tau,\xi) = \tfrac{A^{2d}}{R^{d+2}G}U(t,x), \quad \partial_\xi W(\tau,\xi) = \tfrac{A^{2d}}{R^{d+1}G}\partial_x U(t,x),$$

and the Vlasov equation transforms into

$$\begin{aligned}\partial_\tau F + \eta\cdot\partial_\xi F \quad &+ \quad 2A^2\left(\tfrac{\dot A}{A} - \tfrac{\dot R}{R}\right)\eta\cdot\partial_\eta F \\ &- \quad \ddot R \tfrac{A^4}{R}\xi\cdot\partial_\eta F - R^d G A^{4-2d}\partial_\xi W\cdot\partial_\eta F + A^2\tfrac{\dot G}{G}F = 0.\end{aligned}$$

We want this to be a conservation law on (ξ,η)-space, so we require

$$\frac{\dot A}{A} - \frac{\dot R}{R} = \frac{1}{2d}\frac{\dot G}{G} \tag{3.1}$$

which holds if and only if

$$G = \left(\frac{A}{R}\right)^{2d} \tag{3.2}$$

up to a multiplicative constant. Recall that G should be positive. Next we require that the external force in the above Vlasov equation becomes time-independent and that there is no time-dependent factor in front of the nonlinear term, *i.e.* $\ddot R\frac{A^4}{R} = -\varepsilon c_0$, $R^d G A^{4-2d} = 1$, where $c_0 > 0$ is an arbitrary constant. In view of (3.2) we get $A = R^{d/4}$, $G = R^{\frac{d-4}{2}d}$ and R has to solve

$$\ddot R + \varepsilon c_0 R^{1-d} = 0. \tag{3.3}$$

Remark 1 *Every solution of Eq. (3.3) has the following properties:*

(i) For any $\lambda > 0$, $t \mapsto R_\lambda(t) = c_0^{-\frac{1}{d}}\lambda^{-\frac{2}{d}}R(\lambda t)$ is a solution of

$$\ddot R + \varepsilon R^{1-d} = 0\ . \tag{3.4}$$

Without loss of generality we therefore assume that $c_0 = 1$ in what follows.

(ii) With $R_0 = R(0)$ and $\dot R_0 = \dot R(0)$ we get, for $d = 1$, $R(t) = -\frac{\varepsilon}{2}t^2 + \dot R_0 t + R_0$. If $d \geq 2$, it is easy to carry out one integration of Eq. (3.4):

$$\begin{aligned}&\tfrac12\dot R^2(t) + \varepsilon\log R(t) = \tfrac12\dot R_0^2 + \varepsilon\log R_0 \quad \textit{for } d = 2,\\ &\tfrac12\dot R^2(t) - \tfrac{\varepsilon}{d-2}R^{2-d}(t) = \tfrac12\dot R_0^2 - \tfrac{\varepsilon}{d-2}R_0^{2-d} \quad \textit{for } d \geq 3.\end{aligned}$$

In the plasma physics case, $R(t)$ cannot change sign and is well defined for any $t \in \mathbb{R}$. Moreover,

$$\begin{aligned}&\log R(t) = \tfrac12\dot R^2(t) - \tfrac12\dot R_0^2 + \log R_0 \geq -\tfrac12\dot R_0^2 + \log R_0 \quad \textit{for } d = 2,\\ &0 \leq R^{2-d}(t) = \tfrac{d-2}{2}\left(\dot R_0^2 - \dot R^2\right) + R_0^{2-d} \leq \tfrac{d-2}{2}\dot R_0^2 + R_0^{2-d} \quad \textit{for } d \geq 3.\end{aligned}$$

Together with Eq. (3.4) this proves that there exists a unique $t_0 \in \mathbb{R}$ such that $R(t_0) > 0$ and $\dot R(t_0) = 0$, and $R(t) > R_0$ for any $t \neq t_0$, provided $\varepsilon = -1$.

(iii) If $t \mapsto R(t)$ is a solution of Eq. (3.4) with $\varepsilon = -1$, $t \mapsto R(t+a)$ is a solution too for any given $a \in \mathbb{R}$. Combining this with the invariance through the rescaling $\lambda \mapsto R_\lambda(t)$ with $R_\lambda(t) = \lambda^{-\frac{2}{d}} R(\lambda t)$, we may always require $R_0 = 1$ and $\dot R_0 = 0$ without loss of generality as long as we are interested in the asymptotic behaviour of f when $t \to +\infty$. Note that with this special choice for R_0 and $\dot R_0$, at $t = 0$, $G(0) = A(0) = 1$, and if we assume $\tau(0) = 0$, then $\xi(\tau = 0, x) = x$, $\eta(\tau = 0, x, v) = v$ and $f(t = 0, x, v) = F(\tau = 0, x, v)$. The time-dependent rescaling has the interesting property that it does not introduce any singularity at $t = 0$, and with $R_0 = 1$ and $\dot R_0 = 0$, the initial data for f and F are the same.

(iv) The singular self-similar solution (2.1) corresponding to the linear scalings of Section 2 is — when it exists — the solution one expects to get in the limit case $R_0 = 0$. Formally, this solution also corresponds to the limit of $R_\lambda(t)$ as $\lambda \to +\infty$.

(v) For $\varepsilon = -1$ and $d \geq 1$, $\frac{\dot R}{R} \sim \frac{1}{t}$ as $t \to +\infty$ and

$$\begin{array}{rl} R(t) \sim t^2 & \text{for } d = 1\,, \\ R(t) \sim t\sqrt{\log t} & \text{for } d = 2\,, \\ R(t) \sim t & \text{for } d \geq 3\,. \end{array} \tag{3.5}$$

With R solving Eq. (3.4), we obtain the following rescaled Vlasov-Poisson system (RVP):

$$\begin{array}{c} \partial_\tau F + \eta \cdot \partial_\xi F + \mathrm{div}_\eta \left[\left(\varepsilon \xi - \partial_\xi W + \frac{d-4}{2} R^{\frac{d}{2}-1} \dot R \eta \right) F \right] = 0, \\ \triangle W = \varepsilon \nu(\tau, \xi) = \varepsilon \int_{\mathbb{R}^d} F(\tau, \xi, \eta)\, d\eta. \end{array}$$

The relation between the old and the new variables is

$$\begin{array}{c} dt = R^{d/2} d\tau, \quad d\tau = R^{-d/2} dt, \\ x = R\,\xi, \quad \xi = R^{-1} x, \\ v = \dot R \xi + R^{1-\frac{d}{2}} \eta, \quad \eta = R^{\frac{d}{2}-1} \left(v - \frac{\dot R}{R} x \right), \end{array}$$

and the rescaled functions are given by

$$\begin{array}{c} F(\tau, \xi, \eta) = R^{\frac{4-d}{2} d} f(t, x, v), \quad \nu(\tau, \xi) = R^d \rho(t, x), \\ W(\tau, \xi) = R^{d-2} U(t, x), \quad \partial_\xi W(\tau, \xi) = R^{d-1} \partial_x U(t, x). \end{array}$$

If we consider (EP) we find that with

$$\eta(\tau, \xi) = R^{\frac{d}{2}-1} \left(u(t, x) - \frac{\dot R}{R} x \right)$$

the rescaled Euler-Poisson system (REP) is

$$\begin{array}{l} \partial_\tau \nu + \mathrm{div}(\nu\, \eta) = 0, \\ \partial_\tau \eta + (\eta \cdot \partial_\xi)\, \eta = \varepsilon \xi - \partial_\xi W + \frac{d-4}{2} R^{\frac{d}{2}-1} \dot R \eta, \\ \triangle W = \varepsilon \nu. \end{array}$$

Note that this rescaling as the one for the Vlasov-Poisson system introduces a harmonic force term $\varepsilon\xi$ and a friction term which is proportional to the velocity.

There exists a unique steady state with a given L^1-norm M for which the particles are at rest and uniformly distributed in the unit ball centered at 0, and the self-consistent force is exactly balanced by the external force. Define $F_\infty^M(\xi,\eta) = \nu_\infty^M(\xi)\delta(\eta)$ where δ is the usual Dirac distribution. If F_∞^M and $(\nu_\infty^M, \eta_\infty^M = 0)$ are the stationary solutions of (RVP) and (REP) respectively such that $\|F_\infty^M\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)} = \|\nu_\infty^M\|_{L^1(\mathbb{R}^d)} = M$, then

$$\partial_\xi W_\infty^M(\xi) = \varepsilon \left\{ \begin{array}{ccl} \xi & , & |\xi| \le (M/|S^{d-1}|)^{1/d}, \\ \xi/|\xi|^d & , & |\xi| > (M/|S^{d-1}|)^{1/d}, \end{array} \right. \tag{3.6}$$

and

$$\nu_\infty^M(\xi) = d \, \cdot \, \mathbb{1}_{B^d((M/|S^{d-1}|)^{1/d})}. \tag{3.7}$$

Here $B^d(r)$ denotes the ball with radius r centered at $0 \in \mathbb{R}^d$, and $\mathbb{1}_\omega$ denotes the characteristic function of the set ω. The inverse rescaling transformation takes this steady state into

$$f_\infty^M(t,x,v) = \frac{d}{R(t)^d} \mathbb{1}_{B^d(R(t)(M/|S^{d-1}|)^{1/d})}(x)\, \delta\left(v - \frac{\dot R(t)}{R(t)} x\right), \tag{3.8}$$

and $\rho_\infty^M(t,x) = \frac{d}{R(t)^d}\mathbb{1}_{B^d(R(t)(M/|S^{d-1}|)^{1/d})}(x)$, $\quad u_\infty^M(t,x) = \frac{\dot R(t)}{R(t)} x$. It is easy to see that this defines a weak solution of (VP) or (EP) respectively.

In the plasma physics case we have $\dot R(t) > 0$ for $t > 0$ provided $\dot R(0) \ge 0$ so that for $d \le 3$ the particles are slowed down by a friction force, and on physical grounds one would expect that the steady state written above is a global attractor for (RVP) or (REP) respectively. In [1] this was carried out rigorously for the case $d = 1$. We will see in the next section that this is not true in general, at least in dimension $d = 3$ for (EP). However, the rescaling still provides informations on the asymptotic behaviour of the original system for large times: the energy for the rescaled system gives rise to a Lyapunov functional for the original system by which dispersion effects and the asymptotic behaviour can be analyzed.

4. Lyapunov Functionals

In this section, we investigate the behaviour of the total energy of (RVP) and (REP). Let us consider first the case $d \ge 3$. The potential energy term is the same for both systems, namely $E_\mathrm{p}(\tau) = \int (W(\tau,\xi) - \varepsilon|\xi|^2)\nu(\tau,\xi)\,d\xi$. For (RVP) the kinetic energy reads $E_\mathrm{k}(\tau) = \int |\eta|^2 F(\tau,\xi,\eta)\,d\eta\,d\xi$, while for (REP) it reads $E_\mathrm{k}(\tau) = \int |\eta|^2(\tau,\xi)\,\nu(\tau,\xi)\,d\xi$. Recalling the remark on the relation between (VP) and (EP) from the introduction, the second formula can be viewed as a special case of the first one, and for both systems we find after a standard computation:

$$\frac{d}{d\tau}\Big(E_\mathrm{k}(\tau) + E_\mathrm{p}(\tau)\Big) = (d-4)\, R^{\frac{d}{2}-1} \dot R\, E_\mathrm{k}(\tau) \quad \text{for} \quad d \ge 3\,. \tag{4.1}$$

Note that for this computation one has to check that no boundary term appears. This is true for $d \geq 3$, but not for $d = 2$ as we shall see below.

Recall also that $R^{d/2}\dot{R} = R^{d/2}dR/dt = d[R(t(\tau))]/d\tau$. Let us rewrite the energy for the rescaled systems in terms of the original variables: if we define P and K by $P(t) = E_{\rm p}(\tau(t)) = R^{d-2}(t)\int \left(U(t,x) - \varepsilon\frac{|x|^2}{R^2(t)}\right)\,\rho(t,x)\,dx$,
and $K(t) = E_{\rm k}(\tau(t)) = R^{d-2}(t)\int |v - \frac{\dot{R}}{R}x|^2 f(t,x,v)\,dx\,dv$ for (VP) or, for (EP), $K(t) = E_{\rm k}(\tau(t)) = R^{d-2}(t)\int\ |u(t,x) - \frac{\dot{R}}{R}x|^2\rho(t,x)\,dx$, then because of (4.1)

$$L(t) = K(t) + P(t) \tag{4.2}$$

is a non-increasing quantity with respect to t for $d = 3$, 4:

$$\frac{dL}{dt} = (d-4)\frac{\dot{R}}{R}K \leq 0\ . \tag{4.3}$$

Because of the integrations by parts in the intermediate computations, the above formulas are true only for $d \geq 3$. We will now consider the cases $d = 1$ and $d = 2$.

In dimension $d = 1$ with $\varepsilon = -1$ (plasma physics case), direct computations involving the kinetic energy and integral quantities related to the force field have been used in [1] to prove the exponential convergence (in the rescaled time variable τ) of $F(\tau,\cdot,\cdot)$ towards F_∞ in $(W^{1,\infty}(\mathbb{R}^2))'$ and of $\partial_\xi W(\tau,\cdot)$ towards $\partial_\xi W_\infty$ in $L^2(\mathbb{R})$. The same computation also holds true for the solution of (EP) if it exists globally in time:

Proposition 1 *Assume that $d = 1$, $\varepsilon = -1$ and consider a global solution $(t,x) \mapsto (\rho(t,x), u(t,x))$ of (EP) in $C^1(\mathbb{R}^+\times\mathbb{R})$ such that for any $t > 0$, $\rho(t,\cdot)$ has a compact support. Then*

$$\nu(\tau(t),\xi) = R(t)\rho(t,R(t)\xi)\ ,\quad \eta(\tau(t),\xi) = \frac{1}{\sqrt{R(t)}}\Big(u(t,R(t)\xi) - \dot{R}(t)\xi\Big)$$

with $\tau(t) = 2\log(1+t)$ and $R(t) = (1+t)^2$ is a solution of (REP) and converges to $(\nu_\infty^M, 0)$ where ν_∞^M is given by Eq. (3.7), with $M = \|\rho(t,\cdot)\|_{L^1}$: there exists a positive constant C such that

$$\|\nu(\tau,\cdot) - \nu_\infty^M\|_{(W^{1,\infty})'} \leq C\cdot e^{-\tau}\ ,$$

while the electric field $-\partial_\xi W(\tau,\cdot) = \int_{-\infty}^{\xi}\nu(\tau,\zeta)\,d\zeta - \frac{1}{2}\|\nu(\tau,\cdot)\|_{L^1(\mathbb{R})}$ converges in $L^2(\mathbb{R})$ to $-\partial_\xi W_\infty^M$ which is given by Eq. (3.6):

$$\|\partial_\xi W(\tau,\cdot) - \partial_\xi W_\infty^M\|_{L^2(\mathbb{R})} \leq C\cdot e^{-\tau}\ .$$

In terms of the original variables and with the notation of Section 3, this means that $\|(1+t)^2\rho(t,(1+t)^2\cdot) - \nu_\infty^M\|_{(W^{1,\infty})'}$ and $\|\partial_x U(t,(1+t)^2\cdot) - \partial_\xi W_\infty^M\|_{L^2(\mathbb{R})}$ are bounded by $\frac{C}{(1+t)}$ for some $C > 0$.

Note that in Proposition 1, we made for $R(t)$ the same choice as in [1], which means that with the notation of Remark 1 we consider the solution of Eq. (3.3) corresponding to $R_0 = 1$ and $\dot{R}_0 = 2$.

The proof follows the same arguments as in [1].

In the case $\varepsilon = +1$ (gravitational case), essentially nothing is known concerning the asymptotic behavior of the solution. If $d = 2, 3, 4$ and $\varepsilon = -1$, the question of identifying the limit of $F(\tau, \cdot, \cdot)$ or $\nu(\tau, \cdot)$ in the sense of measures as $\tau \to \tau_\infty = \int_0^{+\infty} R^{-d/2}(t)\, dt$ (which is finite as soon as $d \geq 3$) is an open question. As already noted, a natural conjecture would be to identify this limit with F_∞^M for the solution of (RVP) and ν_∞^M for the solution of (REP) as in dimension $d = 1$. In other terms, the stationary state of the rescaled equation would be an attractor for the solutions of the rescaled system in dimension $d > 1$. If $d \geq 3$, this is not true in general.

Counter-examples. *Consider a solution for which it is the case and shift the initial data by a constant velocity. Since asymptotically the support of the unscaled solution grows linearly in time, after rescaling, the shifted solution cannot converge to the stationary profile. One may then ask the same question in the reference frame of the center of mass. The following counter-example for (EP) again shows that for $d \geq 3$, $\varepsilon = -1$, the answer is negative.*

Consider in $\mathbb{R}^3$ the solution corresponding to the following initial data:

$$\begin{aligned}
&\rho(t=0,x) = 3\mathbb{1}_{B^3(1)}(x) + \mathbb{1}_{B^3(3)\setminus B^3(2)}(x)\ , \\
&u(t=0,x) = 0 \quad \textit{if} \quad |x| < 1\ , \quad u(t=0,x) = x \quad \textit{if} \quad 2 < |x| < 3\ .
\end{aligned}$$

For any $t > 0$, the solution is supported in the union of a centered ball of radius $R(t)$ (which obeys to Eq. (3.3)) and of a centered annulus of inner radius $R_1(t)$. A straightforward computation shows that R and R_1 satisfy

$$\ddot{R} = \frac{1}{R^2},\ R(0) = 1,\ \dot{R}(0) = 0, \quad \ddot{R}_1 = \frac{1}{R_1^2},\ R_1(0) = 2,\ \dot{R}_1(0) = 2\ ,$$

respectively, and an integration with respect to t gives

$$\dot{R}^2(t) = 2 - \frac{2}{R(t)} < 2 < 4 < 5 - \frac{2}{R_1(t)} = \dot{R}_1^2(t)$$

for any $t > 0$. As $t \to +\infty$, $\sqrt{2} = \lim_{\to +\infty} \frac{R(t)}{t} < \lim_{\to +\infty} \frac{R_1(t)}{t} = \sqrt{5}$ which again forbids the convergence to the stationary solution after rescaling.

In dimension $d = 2$ for $\varepsilon = -1$, the situation is different: $t \mapsto R(t)$ grows superlinearly (as for $d = 1$: see Remark 1, *(v)*), and the question is still open.

Consider now the case $d = 2$ for (VP). The main difficulty comes from the integration by parts, and one has to be very careful with the terms involving the self-consistent potential U since ∇U essentially decays like $1/|x|$. Let $(\rho, \rho u) = \int f(t,x,v)(1,v)\,dv$, $(\nu, \nu\eta) = \int F(\tau,\xi,\eta)(1,\eta)\,d\eta$ and $M = \|f(t,\cdot,\cdot)\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)}$.

$$\int \frac{\xi}{|\xi|^2}\cdot(\nu\eta)\,d\xi = \int \frac{Rx}{|x|^2}\cdot R^2\rho(u - \frac{\dot R}{R}x)\,\frac{dx}{R^2} = -M\dot R - R\int \frac{1}{|x|}\partial_t\rho\,dx$$

using the local conservation of mass $\partial_t\rho + \partial_x(\rho u) = 0$. Similarly,

$$\int \partial_\xi W\cdot(\int \eta F(\tau,\xi,\eta)\,d\eta)\,d\xi = -\frac{M^2}{2\pi}\frac{\dot R}{R} - \frac{1}{2}\frac{d}{d\tau}\int W\nu(\tau,\xi)\,d\xi$$

cf. [11] for more details. Of course the computations are exactly the same for (EP). Thus in dimension $d=2$, the definition (4.2) has to be replaced by

$$L(t) = K(t) + P(t) + \frac{M^2}{2\pi}\log R(t)\ , \tag{4.4}$$

so that Eq. (4.3) still holds, see also Remark 3.

Eq. (4.3) provides an identity which is a sharpened form of the Lyapunov functional (also called pseudo-conformal law: see Section 10 for the relation with the Schrödinger equation). A simple form of this identity had been discovered independently by R. Illner and G. Rein, and by B. Perthame, *cf.* [16,25]. The improved Lyapunov functional has the striking property that it easily provides all the terms that one has to take into account in the case $d=2$ (see [11] for (VP)) in a quite straightforward manner, while a direct approach was far from being obvious.

Theorem 4.1 *Assume that f is a solution of (VP) with $M = \|f(t,\cdot,\cdot)\|$ and that $t \mapsto R(t)$ is the solution of Eq. (3.4) with $R(0)=1$, $\dot R(0)=0$. The function $t \mapsto L(t)$ given by*

$$L(t) = R^{d-2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left|v - \frac{\dot R}{R}x\right|^2 f\,dv\,dx + R^{d-2}\int_{\mathbb{R}^d}\left(U(t,x) - \varepsilon\frac{|x|^2}{R^d}\right)\rho\,dx$$

for $d \geq 3$ and

$$L(t) = \iint_{\mathbb{R}^2\times\mathbb{R}^2}\left|v - \frac{\dot R}{R}x\right|^2 f\,dv\,dx + \int_{\mathbb{R}^2}\left(U(t,x) - \varepsilon\frac{|x|^2}{R^2}\right)\rho\,dx + \frac{M^2}{2\pi}\log R$$

if $d=2$ is decreasing for $d=2,3$, constant for $d=4$, and for any $d\geq 2$ satisfies

$$\frac{dL}{dt} = (d-4)\,\dot R R^{d-3}\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left|v - \frac{\dot R}{R}x\right|^2 f\,dv\,dx\ .$$

Moreover in the plasma physics case $\varepsilon = -1$, L is bounded from below, and for $d = 2,3$,

$$\int_0^{+\infty}\dot R(s)R^{d-3}(s)\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left|v - \frac{\dot R(s)}{R(s)}x\right|^2 f(s,x,v)\,dv\,dx\right)ds < +\infty\ .$$

Proof. dL/dt has already been computed above. For $\varepsilon = -1$, the proof of the existence of a lower bound is straightforward except maybe for $d = 2$. In that case, the Lyapunov functional is decreasing but might *a priori* be unbounded from below, and we have to estimate it. This can be done with Jensen's inequality using the fact that $(-\log)$ is a convex function:

$$
\begin{aligned}
-\frac{1}{2\pi} &\iint_{\mathbb{R}^2\times\mathbb{R}^2} \log|x-y|\,\rho(t,x)\rho(t,y)\;dxdy \\
&= \frac{M^2}{4\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\Big(-\log|x-y|^2\Big)\cdot\rho(t,x)\rho(t,y)\;\frac{dxdy}{M^2} \\
&\geq -\frac{M^2}{4\pi}\log\Big(\iint_{\mathbb{R}^2\times\mathbb{R}^2}|x-y|^2\cdot\rho(t,x)\rho(t,y)\;\frac{dxdy}{M^2}\Big) - \frac{M^2}{4\pi}\log(2I/M)\;,
\end{aligned}
$$

where $I = \int |x|^2\rho\;dx$, and an optimization on $I > 0$ gives

$$
\frac{M^2}{2\pi}\log R + \frac{I}{R^2} - \frac{M^2}{4\pi}\log(2I/M) \geq \frac{M^2}{4\pi}[1-\log(M/2\pi)]\;,
$$

which proves the result. $\square$

Remark 2 *For $\varepsilon = +1$ and $d = 3$ or 4, using the Hardy-Littlewood-Sobolev inequality and classical interpolation identities, one proves that the self-consistent potential energy term $\|\nabla U\|^2_{L^2}$ is bounded in terms of K by*

$$
\|\nabla U\|^2_{L^2(\mathbb{R}^d)} \leq C\|f\|^{2(1-(d^2-4)/4d)}_{L^1(\mathbb{R}^d\times\mathbb{R}^d)}\|f\|^{(d-2)/d}_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}K(t)^{(d-2)/2}\;,
$$

see Section 5 for more details on interpolations.

Remark 3 *In dimension $d = 2$, for $\varepsilon = -1$, it is probably easier to compute dL/dt and prove (4.3) directly from (VP) using the identity*

$$
\iint_{\mathbb{R}^2\times\mathbb{R}^2}(x\cdot v)\;(\partial_x U\cdot\partial_v f)\;dv\,dx = -\int_{\mathbb{R}^2}(x\cdot\partial_x U)\rho\;dx = \frac{M^2}{4\pi}
$$

once the equation for R is known, cf. [11].

Note that with the help of (3.5) and the results of Theorem 4.1, we recover the results of [16,25] in dimension $d = 3$ as well as the results of [11] in dimension $d = 2$. Very similar results of course hold for (EP) since the estimates on the Lyapunov functional in dimension $d = 2, 3, 4$ are the same.

Theorem 4.2 *Assume that (ρ, u) is a global strong solution of (EP) with $M = \|\rho(t,\cdot)\|_{L^1(\mathbb{R}^d)}$ and that $t \mapsto R(t)$ is the solution of Eq. (3.4) with $R(0) = 1$ and $\dot R(0) = 0$. The function $t \mapsto L(t)$ given by*

$$
L(t) = R^{d-2}\int_{\mathbb{R}^d}\Big|u - \frac{\dot R}{R}x\Big|^2\rho\,dx + R^{d-2}\int_{\mathbb{R}^d}\Big(U - \varepsilon\frac{|x|^2}{R^d}\Big)\;\rho\,dx
$$

for $d \geq 3$, with the additional term $\frac{M^2}{2\pi} \log R$ for $d = 2$, is decreasing for $d = 2, 3$, constant for $d = 4$, and for any $d \geq 2$ satisfies Eq. (4.3). Moreover in the plasma physics case $\varepsilon = -1$, L is bounded from below, and for $d = 2, 3$

$$\int_0^{+\infty} \dot{R}(s) R^{d-3}(s) \left(\int_{\mathbb{R}^d} \left| u(s,x) - \frac{\dot{R}(s)}{R(s)} x \right|^2 \rho(s,x)\, dx \right) ds < +\infty \, .$$

The case $d = 4$ appears to be the limit case to which the above method for finding Lyapunov functionals in the plasma physics case applies since for $d \geq 5$, $t \mapsto L(t)$ is increasing. However for $d \geq 4$, we may write

$$\frac{d}{dt} \left(R^2 \frac{L}{R^{d-2}} \right) \leq 0 \, ,$$

and thus obtain $\int_{\mathbb{R}^d \times \mathbb{R}^d} |v - \frac{\dot{R}}{R} x|^2 f(t,x,v)\, dv\, dx = O\left(R^{-2}\right) = O\left(t^{-2}\right)$ since for $d \geq 3$ all the quantities involved in $L(t)$ are nonnegative and $R(t) \sim t$ as $t \to +\infty$.

In this last part of Section 4, we will derive the Lyapunov functionals in another way, not because of the case $d > 4$ (which is of minor interest for (EP) or (VP) in itself), but because the method is simpler and will be applied to other systems in Sections 6–8. We assume that $\varepsilon = -1$ in the rest of this section.

We may indeed notice that all the quantities we have been taking into account are integrated in the x variable, so that the change of variable $\xi(t,x) = x/R(t)$ does not play any role in the estimates. Let us first consider the Vlasov-Poisson system (VP). According to the above remark, we may use the change of variables

$$\eta(t,x,v) = v - \frac{\dot{R}}{R} x \, , \quad f(t,x,v) = F(t,x,\eta)$$

so that F solves the rescaled system (R$'$VP):

$$\partial_t F + \eta \cdot \partial_x F - \tfrac{\ddot{R}}{R} x \cdot \partial_\eta F - \partial_x U(t,x) \cdot \partial_\eta F + \tfrac{\dot{R}}{R} \big[\partial_x (xF) - \partial_\eta (\eta F) \big] = 0 \, ,$$

$$-\partial_x U(t,x) = \tfrac{x}{|S^{d-1}|\, |x|^d} * \int_{R^d} F(t,x,\eta)\, d\eta \, .$$

As for (RVP), we may compute the energy:

$$E(t) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(|\eta|^2 + \frac{\ddot{R}}{R} |x|^2 + U \right) F \; dx d\eta$$

if $d \geq 3$, with the additional term $\frac{M^2}{2\pi} \log R$ for $d = 2$. This energy is a decaying function of t : for any $d \geq 2$,

$$\begin{aligned} \frac{dE}{dt}(t) \;&=\; -(d-2) \frac{\dot{R}}{R} \iint_{\mathbb{R}^d \times \mathbb{R}^d} FU \; dx d\eta \\ &\quad + \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left[\left(\frac{d}{dt} (\frac{\ddot{R}}{R}) + 2 \frac{\ddot{R}}{R} \frac{\dot{R}}{R} \right) |x|^2 - 2 \frac{\dot{R}}{R} |\eta|^2 \right] F \; dx d\eta \, . \end{aligned}$$

We may now define $L(t) = B(t)E(t)$. For any $d \geq 3$,

$$\begin{aligned}\frac{dL}{dt} &= \left(\dot{B} - (d-2)\frac{\dot{R}}{R}B\right)\int_{\mathbb{R}^d} |\nabla U|^2 \, dx \\ &\quad + \left(\dot{B} - 2\frac{\dot{R}}{R}B\right)\iint_{\mathbb{R}^d\times\mathbb{R}^d} F(t,x,\eta)|\eta|^2 \, dxd\eta \\ &\quad + \left(\dot{B}\frac{\ddot{R}}{R} + \left(\frac{d}{dt}(\frac{\ddot{R}}{R}) + 2\frac{\ddot{R}}{R}\frac{\dot{R}}{R}\right)B\right)\iint_{\mathbb{R}^d\times\mathbb{R}^d} F(t,x,\eta)|x|^2 \, dxd\eta \ ,\end{aligned} \tag{4.5}$$

while for $d = 2$, there is an additional term: $(\dot{B}\log R - B\frac{\dot{R}}{R})M^2$. For $d \geq 3$, the following conditions are sufficient for L to be nonincreasing:

1) $B(t) = R(t)^{d-2}$, which implies $\dot{B} - (d-2)\, B\, \dot{R}/R \leq 0$,

2) $d \leq 4$, which implies $\dot{B} - B\,\dot{R}/R = -(4-d)\, B\, \dot{R}/R \leq 0$,

3) $\ddot{R} = R^p$, $\quad R(0) = 1$, $\quad \dot{R}(0) = 0$ with $p \leq -(d-1)$, which implies $(\frac{d}{dt}(\frac{\ddot{R}}{R}) + 2\frac{\ddot{R}}{R}\frac{\dot{R}}{R})B + \dot{B}\frac{\ddot{R}}{R} \leq 0$,

and we recover the results of Theorem (4.1) for $d = 3, 4$; for $d = 2$ take $B = 1$.

Remark 4 *If $d \geq 2$ (including the case $d \geq 4$), we may choose $B = R^{d-2-\theta}$, $\theta \geq \max(0, d-4)$, and R solving the equation $\ddot{R} = R^p$, $R(0) = 1$, $\dot{R}(0) = 0$ for some $p \leq \theta - (d-1)$ without any further restriction on d. Note that for $d \geq 4$, $p < -1$ and $\theta < d-2$, one recovers the estimate one would have for the free transport $\partial_t f + v \cdot \partial_x f = 0$, since in that case $f(t,x,v) = f_0(x - vt, v)$ and $\int\int f(t,x,v)|x - vt|^2 \, dxdv = \int\int f_0(x,v)|x|^2 \, dxdv$. For the consequences on the dispersion rate, see Section 5.*

An analogous method also works for the Euler-Poisson system (EP). If we shift the velocity $u(t,x)$ by an unknown "bulk" velocity $\frac{\dot{R}}{R}x$, so that $\eta(t,x) = u(t,x) - \frac{\dot{R}}{R}x$, then $(\rho(t,x), \eta(t,x))$ solves the system (R$'$EP):

$$\begin{gathered}\partial_t\rho + \partial_x(\rho(\eta + \tfrac{\dot{R}}{R}x)) = 0 \ , \\ \partial_t\eta + \tfrac{d}{dt}(\tfrac{\dot{R}}{R})x + ((\eta + \tfrac{\dot{R}}{R}x)\cdot\partial_x)\eta + \tfrac{\dot{R}}{R}(\eta + \tfrac{\dot{R}}{R}x) = -\partial_x U(t,x) \ , \\ -\partial_x U(t,x) = \tfrac{x}{|S^{d-1}|\,|x|^d} * \rho \ .\end{gathered}$$

As for (RVP), we may consider the energy:

$$E(t) = \int_{\mathbb{R}^d}\left(|\eta(t,x)|^2 + \frac{\ddot{R}}{R}|x|^2 + U(t,x)\right)\rho(t,x)\, dx$$

for $d \geq 3$ (if $d = 2$, one has to add the term $\frac{M^2}{2\pi}\frac{\dot{R}}{R}$). Again $t \mapsto E(t)$ is decaying: for any $d \geq 2$, as for (VP), we may also define $L(t) = B(t)E(t)$, and the rest of the discussion is exactly the same.

This method for finding a Lyapunov functional can be summarized as follows: first change the velocity variable by subtracting a velocity $\frac{\dot{R}}{R}x$ for some increasing function R, then compute the energy associated to the new equation and finally choose the Lyapunov functional to be $L(t) = B(t)E(t)$ where $B(t)$ is the function of t which has the maximal growth in order that $L(t)$ is still a decaying function of t and corresponds to a function $t \mapsto R(t)$ solving an adequate ordinary differential equation which takes the nonlinearity into account and has to be chosen well. Of course, one way to find an equation for R is to apply the method of the time-dependent rescalings of the beginning of this section. This method is sufficient to extract the asymptotic rate of decay of the relevant quantities, as we shall see later in several other cases, *cf.* Sections 6–8.

5. Asymptotic Behaviour, Dispersion

An estimate of the rate of dispersion of a solution f of the Vlasov-Poisson system (VP) in the plasma physics case $\varepsilon = -1$ is given by the interpolation of $\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\, dv$ between the L^∞-norm of f, which is preserved for strong solutions, and the momentum $\int_{\mathbb{R}^d\times\mathbb{R}^d} f|v-\frac{x}{t}|^2\, dxdv \sim \int_{\mathbb{R}^d\times\mathbb{R}^d} f|v-\frac{\dot R}{R}x|^2\, dxdv$ as $t\to+\infty$: there exists a constant $C = C(d) > 0$ such that

$$\left\|\int_{\mathbb{R}^d} f\, dv\right\|_{L^{\frac{d}{d+2}}(\mathbb{R}^d)} \le C\cdot \|f\|^{\frac{2}{d+2}}_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\cdot\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} f\left|v-\frac{\dot R}{R}x\right|^2 dxdv\right)^{\frac{d}{d+2}} \quad (5.1)$$

(for a systematic study of these interpolation inequalities see [11] and references therein).

The asymptotic form of the Lyapunov functional was given in [16,25] for the case $d = 3$ and in [11] for the case $d = 2$. Using $R(t)$, we remove the difficulty due to the singularity at $t = 0$ and recover the known results. The use of the decay term of the Lyapunov functional allows us to prove that the decay is not optimal.

Proposition 2 *Assume that f is a strong solution of (VP) in the plasma physics case $\varepsilon = -1$ and $t\mapsto R(t)$ is the solution of $\ddot R = R^{1-d}$ with $R(0) = 1$ and $\dot R(0) = 0$. Then f obeys the following Strichartz type estimate: if $d = 2$ or 3,*

$$\int_0^{+\infty} R^{d-3}(t)\dot R(t)\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)\left|v-\frac{\dot R(t)}{R(t)}x\right|^2 dxdv\right) dt \le C\,, \quad (5.2)$$

and for $d = 3, 4$, we have the following dispersion estimate

$$\|\rho(t,\cdot)\|_{L^{\frac{d+2}{d}}(\mathbb{R}^d)} \le C\, R(t)^{-d\frac{d-2}{d+2}} \sim t^{-d\frac{d-2}{d+2}}\,. \quad (5.3)$$

Here C denotes various positive constants which depend only on d and f_0, and L is the Lyapunov functional of Theorem 4.1. If $d = 2$,

$$\liminf_{t\to+\infty} \|\rho(t)\|_{L^2(\mathbb{R}^2)} = 0\,. \quad (5.4)$$

The proof follows from Theorem 4.1 and the interpolation identity (5.1) given above. The decay of $\rho(t,\cdot)$ in $L^2(\mathbb{R}^2)$ is given by the decay term of the Lyapunov functional; see Remark 5 below. Estimate (5.2) for $d = 2$ has been improved compared to [11].

Remark 5 *The decay given in Proposition 2 is not optimal. Consider indeed a function $t \mapsto h(t)$ such that $h \geq 1$, $\lim_{t\to+\infty} h(t) = +\infty$ and*

$$\int_0^{+\infty} \frac{ds}{s\,h(s)} = +\infty \quad \textit{if } d = 2\;, \quad \int_0^{+\infty} \frac{ds}{h(s)} = +\infty \quad \textit{if } d = 3\;.$$

For instance, one may take for $t > 0$, $h(t) = \log(t+2)$ if $d = 2$ and $h(t) = t\,\log(t+2)$ if $d = 3$. The bounds (5.2) immediately provide for $d = 2, 3$

$$\liminf_{t\to+\infty}\, h(t) \iint_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)\left| v - \frac{\dot R(t)}{R(t)}x \right|^2 dxdv = 0,$$

and as a consequence, the decays in (5.3) and (5.4) are not optimal.

Similar results can of course be obtained for any $d \geq 5$, using Remark 4. We may notice that the decay in (5.3) for $d \geq 4$ is the one which is obtained for the free transport equation when considering the second moment in $x - vt$.

Remark 6 *The Lyapunov functionals given in [11,16,25] correspond to the asymptotic form of $R(t)$ as $t \to +\infty$. The fact that this asymptotic form also gives a Lyapunov functional is easily explained by the scaling invariance of the equation (see Remark 1): if one replaces $R(t)$ by t for $d = 3, 4$ or $t\sqrt{\log t}$ for $d = 2$ in the expression of the Lyapunov functional $L(t)$ of Theorem 4.1, $L(t)$ would still be a Lyapunov functional.*

Similar results for the pressureless Euler-Poisson system (EP) also hold except that no direct interpolation can be used. The decay only holds in a weak norm defined as follows (assume here that $d \geq 3$): let us consider the space $\mathcal{D}^{1,2}(\mathbb{R}^d) = \left\{\phi \in L^{\frac{2d}{d-2}}(\mathbb{R}^d)\ :\ \nabla\phi \in L^2(\mathbb{R}^d)\right\}$ and define on its dual space the norm

$$|||\rho||| = \|\rho\|_{(\mathcal{D}^{1,2}(\mathbb{R}^d))'} = \sup\left\{ \int \rho\phi\, dx \mid \phi \in \mathcal{D}^{1,2}(\mathbb{R}^d),\ \|\nabla\phi\|_{L^2(\mathbb{R}^d)} \leq 1 \right\}.$$

If $U \in \mathcal{D}^{1,2}(\mathbb{R}^d)$ is such that $-\Delta U = \rho$, then $|||\rho||| \leq \|\nabla U\|_{L^2(\mathbb{R}^d)}$. Using the same notation as in Section 4, if $d = 3, 4$, there exists a positive constant C such that $R(t)^{d-2}\int_{\mathbb{R}^d} \rho(t,x)|u(t,x) - \frac{\dot R}{R}x|^2\, dx \leq C$ and $R(t)^{d-2}\int_{\mathbb{R}^d} |\nabla U(t,x)|^2\, dx \leq C$. The last inequality can be reinterpreted as an estimate on $|||\rho(t,\cdot)|||$.

Proposition 3 *Assume that (ρ, u) is a C^1 solution on $\mathbb{R}^+ \times \mathbb{R}^d$ of (EP) in the plasma physics case $\varepsilon = -1$ and $t \mapsto R(t)$ is the solution of $\ddot R + \varepsilon R^{1-d} = 0$ with $R(0) = 1$ and $\dot R(0) = 0$. Then (ρ, u) obeys to the following Strichartz type estimate for $d = 2$, 3:*

$$\int_0^{+\infty} R^{d-3}(t)\dot R(t)\left(\int_{\mathbb{R}^d} \rho(t,x)|u(t,x) - \frac{\dot R(t)}{R(t)}x|^2\, dx\right) dt \ <\ +\infty\ .$$

Moreover, if $d = 3, 4$, then

$$\limsup_{t\to+\infty}\, t^{\frac{d}{2}-1}|||\rho(t,\cdot)||| = \limsup_{t\to+\infty}\, R(t)^{\frac{d}{2}-1}\|\nabla U(t,\cdot)\|_{L^2(\mathbb{R}^d)} < +\infty\ .$$

Remark 7 *If $d = 2$, we cannot use the $(\mathcal{D}^{1,2}(\mathbb{R}^d))'$-norm as in the case $d \geq 3$, but the following estimates for the solutions in the plasma physics case $\varepsilon = -1$ of the pressureless Euler-Poisson system (EP) hold:*

$$\lim_{t\to+\infty} \tfrac{1}{\log R(t)} \int_{\mathbb{R}^2} \rho U \, dx = -\tfrac{M^2}{2\pi} \; ,$$

$$\lim_{t\to+\infty} \tfrac{1}{\log R(t)} \int_{\mathbb{R}^2} \rho |u|^2 \, dx = \tfrac{M^2}{2\pi} \; , \quad \textit{and} \quad \lim_{t\to+\infty} \tfrac{1}{t^2} \int_{\mathbb{R}^2} \rho |x|^2 \, dx = \tfrac{M^2}{2\pi} \; .$$

These estimates are easily deduced from the conservation of the energy, the expression of the Lyapunov functional $L(t)$ and the estimate given in the proof of Theorem 4.1.

Maybe more interesting is the observation (see [11]) that for $d = 2$, which is the limit case for dispersion results, the dispersion estimate gives a lower bound for the growth of the support of a solution corresponding to a compactly supported initial datum:

Corollary 1 *Consider for $d = 2$ solutions of (VP) or (EP) corresponding to compactly supported initial data. Assume that $r(t)$ is the minimal radius of the balls containing the support of $\rho(t, \cdot)$. Then there exists a constant $C > 0$ such that $r(t) \geq C\, R(t) \quad$ as $\quad t \to +\infty$.*

Proof. As in [11] one may simply notice that

$\frac{M^2}{2\pi} \Big(\log R(t) - \log(2r(t)) \Big) \leq L(t) \leq L(0).$ □

6. The 2-dimensional Symmetric Vlasov-Poisson System with an External Magnetic Field

In dimension $d = 2$, we may consider the following system (VPM)

$$\begin{aligned} &\partial_t f + v \cdot \partial_x f + \Big(-\partial_x U(t,x) + B_0 v^\perp \Big) \cdot \partial_v f = 0 \\ &-\partial_x U(t,x) = \tfrac{x}{2\pi\, |x|^2} * \textstyle\int_{\mathbb{R}^2} f(t,x,v) \, dv \end{aligned}$$

corresponding to a system of particles with a self-interaction through electrostatic forces, in the presence of an external constant magnetic field B_0. Here we use the notation $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}^\perp = \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix}$. For the linear system without self-consistent electrostatic forces, all the characteristics are circles and a solution with an initially compact support will remain supported in a fixed compact set for all time. With a self-consistent Poisson term, the situation is radically different since we get the same estimates as for the Vlasov-Poisson system without a magnetic field.

We may indeed shift the velocity variable $\eta(t,x,v) = v - \frac{\dot R}{R} x$, and the new distribution function $f(t,x,v) = F(t,x,\eta)$ obeys to the system

$$\partial_t F + \eta \cdot \partial_x F - \frac{\ddot R}{R} x \cdot \partial_\eta F + \Big(B_0 (\eta^\perp + \frac{\dot R}{R} x^\perp) - \partial_x U \Big) \cdot \partial_\eta F + \frac{\dot R}{R} \Big(\partial_x (xF) - \partial_\eta (\eta F) \Big) = 0 \; ,$$

Th energy is the same as for the Vlasov-Poisson system (VP) (see Section 4) and decays according to

$$\frac{dE}{dt} = -\iint F(t,x,\eta)\left[2\frac{\dot R}{R}|\eta|^2 + \left(\frac{d}{dt}(\frac{\ddot R}{R}) + 2\frac{\ddot R}{R}\frac{\dot R}{R}\right)|x|^2 + 2\frac{\ddot R}{R}(x\cdot\eta^\perp)\right] dx d\eta\ .$$

If f is radially symmetric, *i.e.* depends only on t, $|x|$, $(x\cdot v)$ and $|x|^2|v|^2-(x\cdot v)^2$, then the analogous property holds for F: F only depends on t, $|x|$, $(x\cdot\eta)$ and $|x|^2|\eta|^2-(x\cdot\eta)^2$, and $\int_{\mathbb{R}^2}\left(x^\perp\cdot\int_{\mathbb{R}^2}\eta F(t,x,\eta)\,d\eta\right)dx = \int_{\mathbb{R}^2\times\mathbb{R}^2}(x\cdot\eta^\perp)F(t,x,\eta)=0$. The system has the same Lyapunov functional as (VP), and we obtain the same dispersion results as for the Vlasov-Poisson system:

Proposition 4 *Let $d=2$. Assume that f is a solution of (VPM) and that $t\mapsto R(t)$ is the solution of $\ddot R=\frac{1}{R}$ with $R(0)=1$, $\dot R(0)=0$. The function $L(t) = \frac{M^2}{2\pi}\log R + \int_{\mathbb{R}^2\times\mathbb{R}^2}|v-\frac{\dot R}{R}x|^2 f(t,x,v)\,dv\,dx + \int_{\mathbb{R}^2}(U+\frac{|x|^2}{R^2})\,\rho\,dx$ is decreasing, bounded from below and satisfies: $\frac{dL}{dt} = -2\frac{\dot R}{R}\int_{\mathbb{R}^2\times\mathbb{R}^2}|v-\frac{\dot R}{R}x|^2 f(t,x,v)\ dv\,dx$. Moreover $\int_0^{+\infty}\frac{\dot R(s)}{R(s)}(\int_{\mathbb{R}^2\times\mathbb{R}^2}|v-\frac{\dot R(s)}{R(s)}x|^2 f(s,x,v)\,dv\,dx)\,ds < +\infty$ and $\liminf_{t\to+\infty}\ \|\rho(t)\|_{L^2(\mathbb{R}^2)}=0$.*

7. The Isentropic Euler System for Perfect Gases

As another example, which does not belong to the field of kinetic equations, we consider the isentropic Euler system (IE) for perfect gases (for $\gamma>1$)

$$\begin{aligned}\partial_t\rho + \partial_x(\rho u) = 0\ ,\\ \partial_t u + (u\cdot\partial_x)u = -\partial_x p\ ,\\ p = \rho^{\gamma-1}\ .\end{aligned}$$

The method goes exactly as for the pressureless Euler-Poisson system (here we use the second method of Section 4): the rescaled system (R′IE) given by $\eta(t,x) = u(t,x) - \frac{\dot R}{R}x$ is

$$\begin{aligned}\partial_t\rho + \partial_x(\rho(\eta + \tfrac{\dot R}{R}x)) = 0\ ,\\ \partial_t\eta + \eta\cdot\partial_x\eta + \tfrac{\dot R}{R}x\cdot\partial_x\eta + \tfrac{\ddot R}{R}x + \tfrac{\dot R}{R}\eta = -\partial_x\rho^{\gamma-1}\ .\end{aligned}$$

If we define the energy by

$$E(t) = \int_{\mathbb{R}^d}\rho(t,x)|\eta(t,x)|^2\ dx + \frac{\ddot R}{R}\int_{\mathbb{R}^d}\rho(t,x)|x|^2\ dx + \frac{2}{\gamma}\int_{\mathbb{R}^d}\rho^\gamma(t,x)\ dx\ ,$$

a Lyapunov functional is easily exhibited by considering $L(t) = B(t)E(t)$. The energy is indeed decreasing:

$$\frac{dE}{dt} = \int_{\mathbb{R}^d}\rho\left[-2\frac{\dot R}{R}|\eta|^2 + \left(\frac{d}{dt}(\frac{\ddot R}{R}) + 2\frac{\ddot R}{R}\frac{\dot R}{R}\right)|x|^2 - 2d\frac{\gamma-1}{\gamma}\frac{\dot R}{R}\rho^{\gamma-1}\right]dx$$

so that

$$\begin{aligned}\frac{dL}{dt} &= \left(\dot{B}-2\frac{\dot{R}}{R}B\right)\int_{\mathbb{R}^d}\rho(t,x)|\eta(t,x)|^2\,dx\\ &\quad+\left[\dot{B}\frac{\ddot{R}}{R}+\left(\frac{d}{dt}(\frac{\ddot{R}}{R})+2\frac{\ddot{R}}{R}\frac{\dot{R}}{R}\right)B\right]\int_{\mathbb{R}^d}\rho(t,x)|x|^2\,dx\\ &\quad+\frac{2}{\gamma}\left(\dot{B}-(\gamma-1)\,d\,B\,\frac{\dot{R}}{R}\right)\int_{\mathbb{R}^d}\rho^\gamma(t,x)\,dx,\end{aligned}$$

and sufficient conditions for L to be decreasing are therefore given by:

1) $B = R^q$ with $q \le \min(2,(\gamma-1)d)$, which implies $\dot{B}-2\frac{\dot{R}}{R}B \le 0$ and $\dot{B}-d\,(\gamma-1)\,B\,\dot{R}/R \le 0$.

2) $\ddot{R} = R^p$ with $p \le -(q+1)$, which implies $\dot{B}\ddot{R}/R + (\frac{d}{dt}(\frac{\ddot{R}}{R})+2\frac{\ddot{R}}{R}\frac{\dot{R}}{R})B \le 0$.

It turns out that these dispersion relations (or at least their asymptotic form) are already known and have been used for the Navier-Stokes equation by J.-Y. Chemin in [7], and by D. Serre in [14,30] and B. Perthame in [26]. One of the interests of these estimates is that one may use them as an *a priori* estimate to control the behaviour for large times and build a global (in time) solution to the Cauchy problem. An equivalent remark (see [30]) is that it is possible to build a solution by a fixed-point method for a finite time (this is not in contradiction with T. Sideris' results [31] on non-existence, if the initial data is small in the correct sense) and that one may choose the rescaling $t \mapsto R(t)$ such that (for the complete rescaling as defined in Section 3 of course) the evolution with respect to the rescaled time holds only on a finite time interval $0 \le \tau < \tau_\infty = \int_0^{+\infty} A^{-2}(t)\,dt$. However, we are here interested only in the dispersion relations which were easily obtained by the mean of the second method of Section 4. These dispersion relations can be summarized as follows:

Proposition 5 *If (ρ,u) is a global classical solution of (IE) with $\gamma > 1$, then it satisfies the following dispersion relation*

$$\frac{d}{dt}\left(R^q\int_{\mathbb{R}^d}\rho\left|u-\frac{\dot{R}}{R}x\right|^2dx+\frac{1}{R^2}\int_{\mathbb{R}^d}\rho|x|^2\,dx+\frac{2}{\gamma}R^q\int_{\mathbb{R}^d}\rho^\gamma\,dx\right)\le 0$$

with $q=\min(2,(\gamma-1)d)$ and $t\mapsto R(t)$ such that $\ddot{R}=R^{-(q+1)}$, $R(0)=1$, $\dot{R}(0)=0$.

8. Wigner and Schrödinger Equations

The relation between the Schrödinger equation, the Wigner equation and the Vlasov equation is now quite well understood. It has been the subject of a considerable number of papers in the recent years: we mention [13,15] as some of the most recent ones, and also [19,23] for the limit of the Schrödinger-Poisson to the Vlasov-Poisson system. Historically, the dispersion relations have been studied for the Schrödinger equation first and then adapted to the corresponding kinetic equation [8,9,17]. The analysis of the dispersion relations in the kinetic framework came only after, but now seems to provide powerful tools to build new dispersion identities, *cf.* [12].

Consider the Schrödinger equation

$$i\hbar\partial_t\psi = -\frac{1}{2}\hbar^2\Delta\psi + V\psi \ .$$

If $w(t,x,v) = \int_{\mathbb{R}^d} e^{-ivy}\ \overline{\psi}(t,x+\frac{\hbar}{2}y)\psi(t,x-\frac{\hbar}{2}y)\ dy$ is the Wigner transform of ψ, it has to satisfy the Wigner equation

$$\partial_t w + v\cdot\partial_x w - \frac{i}{\hbar}\Theta(V)w = 0$$

where the pseudo-differential operator $\Theta(V)$ is defined by

$$\Theta(V)f(x,v) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-ivy}\left[V(x+\tfrac{\hbar}{2}y) - V(x-\tfrac{\hbar}{2}y)\right]\cdot\left(\int_{\mathbb{R}^d} e^{+iy\xi}f(x,\xi)\ d\xi\right)dy \ .$$

In the semi-classical limit $\hbar \to 0+$, the operator $\Theta(V)$ is formally expected to converge to $-\partial_x V\cdot\partial_v$, and it is the purpose of many papers to justify this limit, *cf.* [13,15,19,23].

In this section we will only derive some dispersion identities according to the technique developed at the end of Section 4 and give some easy consequences of these estimates.

We shall consider three cases:

The linear case **(L)**: V is a given fixed nonnegative potential which does not depend on t and decays as $|x| \to +\infty$. We will not go further into this case since the dispersion properties would depend on the local properties of V and $x\cdot\partial_x V$, but the computations are essentially the same as for the other cases up to Eq. (8.1).

The Poisson case **(P)**: V is given by $-\Delta V = |\psi|^2 = \int_{\mathbb{R}^d} w(t,x,v)\ dv$ (we consider only the electrostatic case). We shall state a result on the Wigner and the Schrödinger formulations of the problem, which clearly proves that this case can be handled in full generality with our methods. The estimates are slightly improved in dimension $d = 3$ and can obviously be generalized to any dimension $d \geq 4$. The results are new for $d = 2$.

The nonlinear case **(NL)**: $V = |\psi|^{p-1}$ and ψ is a solution of the nonlinear Schrödinger equation (NLS)

$$i\hbar\partial_t\psi = -\frac{1}{2}\hbar^2\Delta\psi - \varepsilon|\psi|^{p-1}\psi$$

(in the following, we shall only study the defocusing case $\varepsilon = -1$). This case is mentioned here to make the link with the pseudo-conformal methods and to recover the pseudo-conformal law, which has been studied extensively.

8.1. *Wigner equation*

For the Wigner equation, we introduce as for the Vlasov-Poisson system the new velocity variable $\eta(t,x,v) = v - \frac{\dot R}{R}x$ and exactly as for the Vlasov-Poisson system, $F(t,x,\eta) = w(t,x,v)$ solves the rescaled Wigner equation (R′W):

$$\partial_t F + \eta \cdot \partial_x F - \frac{\ddot R}{R} x \cdot \partial_\eta F - \frac{i}{\hbar}\Theta(V)F + \frac{\dot R}{R}\Big(\partial_x(xF) - \partial_\eta(\eta F)\Big) = 0\ .$$

Again as for (R′VP), we compute the energy

$$E(t) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} F\left(|\eta|^2 + \frac{\ddot R}{R}|x|^2 + \alpha V\right)\, dx d\eta$$

if $d \geq 3$, with an additional term $\frac{M^2}{2\pi}\log R$ if $d = 2$ in case (P). Here α is a coefficient which takes different values according to the case we consider: $\alpha = 2$, 1, and $\frac{2}{p+1}$ in case (L), (P) and (NL) respectively. The same computation as before provides

$$\begin{aligned}\frac{dE}{dt} \quad &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} F(t,x,\eta)\left[-2\frac{\dot R}{R}|\eta|^2 + \left(\frac{d}{dt}(\frac{\ddot R}{R}) + 2\frac{\ddot R}{R}\frac{\dot R}{R}\right)|x|^2\right]\, dx d\eta \\ &\quad - \alpha\frac{\dot R}{R}\iint_{\mathbb{R}^d\times\mathbb{R}^d} V(t,x)\partial_x(xF(t,x,\eta))\, dx d\eta\end{aligned}$$

(for $d \geq 3$ in case (P) — the case (P), $d = 2$ is similar up to the integrations by parts that are to be done with care) and we may define $L(t) = B(t)\,E(t)$ and, as for the Vlasov-Poisson system,

$$\begin{aligned}\frac{dL}{dt} \quad &= \alpha \iint_{\mathbb{R}^d\times\mathbb{R}^d} V(t,x)\left(\dot B F - \frac{\dot R}{R}B\partial_x(xF)\right)\, dx d\eta \\ &\quad + \left(\dot B - 2\frac{\dot R}{R}B\right)\iint_{\mathbb{R}^d\times\mathbb{R}^d} F(t,x,\eta)|\eta|^2\, dx d\eta \\ &\quad + \left[\dot B\frac{\ddot R}{R} + \left(\frac{d}{dt}(\frac{\ddot R}{R}) + 2\frac{\ddot R}{R}\frac{\dot R}{R}\right)B\right]\iint_{\mathbb{R}^d\times\mathbb{R}^d} F(t,x,\eta)|x|^2\, dx d\eta\ .\end{aligned}$$

In the case of the coupling with the Poisson equation ($d \geq 2$) , the conditions on L that are sufficient for it to be nonincreasing are exactly the same as for (R′VP) in the Poisson case (P): see Section 4. The detailed justifications of the computations for initial data $|\psi(t=0,\cdot)|^2 = \int_{\mathbb{R}^d} w(t=0,\cdot,v)\, dv$ in $L^1(\mathbb{R}^d)$ are not given here, and we shall refer to [3] for a proof if $d \geq 3$ in the context of the Schrödinger-Poisson system.

Theorem 8.3 *Assume that w is a solution of (WP) with $M = \|w(t,\cdot,\cdot)\|$ and that $t \mapsto R(t)$ is the solution of Eq. (3.4): $\ddot R + \varepsilon R^{1-d} = 0$, $R(0) = 1$, $\dot R(0) = 0$. The function $t \mapsto L(t)$ defined above for $d \geq 2$ (with $B = R^{d-2}$) is decreasing for $d = 2, 3$, constant for $d = 4$, and satisfies for any $d \geq 2$*

$$\frac{dL}{dt} = (d-4)\,\dot R R^{d-3}\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left|v - \frac{\dot R}{R}x\right|^2 w\, dv\, dx\ .$$

In the plasma physics case $\varepsilon = -1$, L is bounded from below and for $d = 2, 3$,

$$\int_0^{+\infty} \dot{R}(s) R^{d-3}(s) \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} \left| v - \frac{\dot{R}(s)}{R(s)} x \right|^2 w(s,x,v)\, dv\, dx \right) ds < +\infty\ .$$

However, the results on the dispersion for the Vlasov-Poisson system cannot be transposed straightforwardly because of the lack of positivity of w and one has to be very careful to recover the estimates given in [17] for $d = 3$. In dimension $d = 2$, the situation is even worse because the boundedness of L from below is not obvious at all. In that sense, the Schrödinger formulation of the problem is more suitable.

8.2. *Schrödinger equation*

The Lyapunov function for the Schrödinger equation is easily found by simply considering the Wigner transform. However, it is interesting to realize how the method of Section 4 applies directly. According to the Weyl quantification and the Wigner transform, the operator $i\hbar\partial_x$ corresponds to the variable v: the change of variables $\eta = v - \frac{\dot{R}}{R} x$ therefore means that instead of $i\hbar\partial_x$ we consider the new operator $i\hbar\partial_x - \frac{\dot{R}}{R} x$:

$$\phi \mapsto (i\hbar\partial_x - \frac{\dot{R}}{R} x)\phi = e^{-i\frac{\dot{R}}{R}\frac{|x|^2}{2\hbar}} i\hbar\partial_x \left(e^{i\frac{\dot{R}}{R}\frac{|x|^2}{2\hbar}} \phi \right)\ .$$

For that purpose, we may consider the new wave function $\phi(t,x) = e^{-i\frac{\dot{R}}{R}\frac{|x|^2}{2\hbar}} \psi(t,x)$ which solves the rescaled Schrödinger equation (R′S)

$$i\hbar\partial_t\phi = -\frac{1}{2}\hbar^2\Delta\phi + (V + \frac{\ddot{R}}{2R}|x|^2)\phi - \frac{i\hbar\dot{R}}{2R}(d\phi + 2x \cdot \partial_x\phi)\ .$$

If we define the potential energy term by $W[\phi] = 2V|\phi|^2$, $W[\phi] = V|\phi|^2$, or $W[\phi] = \frac{2}{p+1}|\phi|^{p+1}$ in case (L), (P), or (NL) respectively, the corresponding energy is given by

$$E(t) = \int_{\mathbb{R}^d} \left(\hbar^2|\nabla\phi|^2 + W[\phi] + \frac{\ddot{R}}{R}|x|^2|\phi|^2 \right) dx$$

if $d \geq 3$, with the additional term $\frac{1}{2\pi}\log R(t)\|\phi(t,.)\|^2_{L^2(\mathbb{R}^2)}$ if $d = 2$ in case (P). We may then build the Lyapunov functional in the same way as for the solution of the Wigner equation. Going back to the original variables, we have to replace $|\nabla\phi|^2$ by $|(\nabla - i\frac{\dot{R}}{\hbar R}x)\psi|^2$.

The Schrödinger-Poisson system and its asymptotics has been studied in [8,9,17]. More recently, a theory for L^2 solutions corresponding to mixed quantum states has been established by F. Castella (see [3,4]). In the case of a pure quantum state, J. L. Lopez and J. Soler in [21,22] also gave detailed results on the asymptotic behaviour using a linear scaling approach in the continuation of the method developed S. Kamin and J. L. Vázquez. The main interest of our approach is that it gives a refined estimate for $d = 3$ and is adapted to the limit case $d = 2$ as well.

Concerning the notion of solution we may assume that it is as smooth as desired and refer to [3,4] for minimal requirements (estimates for weak solutions are built using approximating smooth solutions).

Theorem 8.4 *Assume that $d \geq 2$ and consider a solution of the Schrödinger-Poisson system. With the above notation*

$$L(t)=R^{d-2}(t)\int\left|(\nabla - i\frac{\dot{R}}{\hbar R}x)\psi\right|^2 dx + R^{d-2}(t)\int V\ |\psi|^2\ dx + \frac{1}{R^2(t)}\int |x|^2\ |\psi|^2\ dx$$

for $d = 3, 4$, with the additional term $\frac{1}{2\pi}\log R(t)\|\psi(t,.)\|^2_{L^2(\mathbb{R}^2)}$ for $d = 2$, is decreasing for $d = 2$, 3 and constant for $d = 4$ if $t \mapsto R(t)$ is a solution of $\ddot{R} = R^{1-d}$, $R(0) = 1$, $\dot{R}(0) = 0$. As a consequence $n(t,x) = |\psi(t,x)|^2$ is decreasing: there exists a constant $C > 0$ such that

$$\|n(t,\cdot)\|_{L^p(\mathbb{R}^d)} \leq C \cdot \dot{R}^{d(\frac{2}{p}-1)} \cdot R^{d(\frac{1}{p}-\frac{1}{2})(\frac{d}{2}-1)} \tag{8.1}$$

for any $p \in [2, \frac{2d}{d-2}]$ if $d = 3$, 4 and $\liminf_{t\to+\infty} \|n(t,\cdot)\|_{L^p(\mathbb{R}^2)} = 0$ for $p \in]2, +\infty[$ if $d = 2$.

Note that for $d = 3$, $p = 10/3$, we recover the same exponents as for the Vlasov-Poisson system. For $d = 2$, exactly the same estimate as in the proof of Theorem 4.1 holds: $\liminf_{t\to+\infty} \int_{\mathbb{R}^2} |(\nabla - i\frac{\dot{R}}{\hbar R}x)\psi|^2\ dx = 0$. The crucial ingredient in the proof of this theorem is the following interpolation lemma (see [8,9] and [Cor.5.5][17]) which plays a role similar to the one of Eq. (5.1) for the Vlasov-Poisson system:

Lemma 8.1 *Assume that $d \geq 3$. There exists a constant $C > 0$ depending only on d such that, for any $u \in H^1(\mathbb{R}^d)$ such that $x \mapsto xu(x)$ belongs to $L^2(\mathbb{R}^d)$,*

$$\|u\|_{L^p(\mathbb{R}^d)} \leq C\|u\|^a_{L^2(\mathbb{R}^d)}\|(x + it\nabla)u\|^{1-a}_{L^2(\mathbb{R}^d)} \cdot t^{-(1-a)}$$

for any $p \in [2, \frac{2d}{d-2}]$, $a = \frac{d}{2}(\frac{2}{p} - \frac{d-2}{d})$.

The proof of Lemma 8.1 is easily established using the Gidas-Nirenberg inequality

$$\|u\|_{L^p(\mathbb{R}^d)} \leq [C(d)]^{1-a}\|u\|^a_{L^2(\mathbb{R}^d)}\|\nabla u\|^{1-a}_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}$$

where $C(d)$ is the Sobolev constant corresponding to the injection of $H^1(\mathbb{R}^d)$ into $L^{\frac{2d}{d-2}}(\mathbb{R}^d)$, and the decomposition $u = \rho e^{i\varphi}$ which holds at least for smooth enough functions (the conclusion holds by a density argument). We may then write

$$\|(x + it\nabla)u\|^2_{L^2} = t^2\int |\nabla\rho|^2\ dx + \int |x\rho + t\rho\nabla\varphi|^2\ dx \geq t^2\|\ \nabla|u|\ \|^2_{L^2}\ ,$$

which proves the interpolation results.

Proof of Theorem 8.4. One has to replace $1/t$ by $\dot{R}/R$ in Lemma 8.1: for $d = 3$ or 4, we may refer to [17] for the proof of Eq. (8.1), where it is done in the case $R = t$. For $d = 2$, the argument is similar, the main step being the proof of the boundedness of L which goes exactly as in the Vlasov-Poisson case. □

We conclude this section by considering the case of the Nonlinear Schrödinger equation which allows us to make an explicit link with the pseudo-conformal law. If $W[\phi] = \frac{2}{p+1}|\phi|^{p+1}$, a direct computation gives

$$\frac{dE}{dt} = -\frac{d(p-1)\dot{R}}{2R}\int W[\phi]\,dx - 2\frac{\dot{R}}{R}\hbar^2\int|\nabla\phi|^2 dx + \left(\frac{d}{dt}(\frac{\ddot{R}}{R}) + 2\frac{\ddot{R}}{R}\frac{\dot{R}}{R}\right)\int|x|^2|\phi|^2 dx$$

and $L(t) = B(t)\,E(t)$ is decreasing if $B(t) = R^q(t)$, $q = \min((p-1)d/2, 2)$, $\ddot{R} = 1/R^{q+1}$, $R(0) = 1$, $\dot{R}(0) = 0$. In the next result we are again not interested in the weakest notion of solution and assume that the solution is global in t and as smooth and sufficiently decreasing at spatial infinity as necessary to justify any integration by parts in the computations.

Theorem 8.5 *Assume that $d \geq 2$ and consider a global solution of the Nonlinear Schrödinger equation (NLS) in the defocusing case. Then with the above notation*

$$L(t) = R^q(t)\int\left(|(\nabla - i\frac{\dot{R}}{\hbar R}x)\psi|^2 + \frac{2}{p+1}|\psi|^{p+1}\right)dx + \frac{1}{R^2(t)}\int|x|^2\,|\psi|^2\,dx \quad (8.2)$$

is decreasing.

Decay estimates can of course be deduced from Lemma 8.1 as for the Schrödinger-Poisson system. The details of the computations for the proof of Theorem 8.5 are left to the reader.

A simple method to understand the pseudo-conformal law is simply to look for a pseudo-conformal invariance of the equation, *i.e.* a transformation which leaves the equation invariant. Let $u(t,x)$ be a solution of (NLS) in the focusing or in the defocusing case ($\varepsilon = -1$). A function $t \mapsto (R(t), \tau(t), \omega(t))$, $(\tau, \xi) \mapsto v(\tau, \xi)$ given by

$$u(t,x) = \frac{1}{R^\alpha(t)}e^{i\omega(t)\frac{|x|^2}{2}}v(\tau(t), \xi(t))\ , \quad \xi(t) = \frac{x}{R(t)}$$

is a solution of (NLS) for some $\alpha \in \mathbb{R}$ only in the case $p - 1 = \frac{4}{d}$ (critical case), and $t \mapsto (R(t), \tau(t), \omega(t))$ then solves the system

$$\frac{d\tau}{dt} = \frac{1}{R^2}\ , \quad \frac{dR}{dt} = 2\omega R\ , \quad \frac{d\omega}{dt} = -2\omega^2\ .$$

The solution is given by $\omega(t) = \frac{\omega_0}{1+2\omega_0 t}$, $R(t) = R_0(1+2\omega_0 t)$, $\tau(t) = \frac{t}{R_0^2(1+2\omega_0 t)} + \tau_0$. This transformation can be found in [24] (see also [18] for instance). The conservation of the energy after rescaling (conservation of the energy for v) gives the following conservation law for u:

$$\frac{d}{dt}\left(R^2(t)\int_{\mathbb{R}^d}|\nabla u(t,x) - i\omega(t)xu(t,x)|^2\,dx - \frac{d\varepsilon}{d+2}\int_{\mathbb{R}^d}|u(t,x)|^{\frac{2}{d}(d+2)}\,dx\right) = 0\ . \quad (8.3)$$

This expression clearly corresponds to the case $q = 2 = (p-1)d/2$, and the pseudo-conformal law is nothing else than the expression of dL/dt where L is given by

Eq. (8.2). As we already noticed already several times, one may replace $\omega(t)$ and $R(t)$ by their equivalents as $t \to +\infty$, which is the same as considering the singular solution corresponding to the limit $\omega_0 \to +\infty$ and $R_0\omega_0 \to 1$, and recover instead of Eq. (8.3) the more classical form for the conformal invariance law:

$$\frac{d}{dt}\left(t^2 \int_{\mathbb{R}^d}\left|\nabla u(t,x) - i\frac{x}{2t}u(t,x)\right|^2 dx - \frac{d\varepsilon}{d+2}\int_{\mathbb{R}^d} u(t,x)|^{\frac{2}{d}(d+2)}\, dx\right) = 0\ .$$

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