Australian Journal of Basic and Applied Sciences, 5(1): 38-47, 2011
ISSN 1991-8178

Application of Analytical and Numerical Methods to the Sequent Depths Problem in Civil Engineering

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Abstract: A classical problem on hydraulic research is to provide explicit equations for evaluating the sequent depths of open channels whose geometry is not rectangular since barring for this section configuration, other solutions are nearly inexistent. Exponential section is the category which includes triangular, parabolic and rectangular channel shapes. For rectangular channel section, as pointed before, it is possible to express the sequent depths analytically by radicals. For other sections, the sequent depths are presently obtained by computational methods. In this paper, in order to find the sequent depths of channels whose sections are exponential or trapezoidal, we apply two different methods: The Lagrange's Inversion Theorem, which is analytical and provides an exact solution by means of an infinite series; and the Householder's Methods, which are numerical and provide approximations of the solutions by using an iterative algorithm. In general, the series obtained from Lagrange's theorem have fast convergence. Otherwise, if the convergence rate is low, we use the Householder's methods. Practical examples are also included.

Key words: Exponential section, Trapezoidal section, Lagrange expansion, Householder's methods, sequent depths.

INTRODUCTION

The governing equation of sequent depths is the specific momentum equation given by (Bakhmeteff, 1932; Chow, 1959).

$$M = \frac{Q^2}{gA} + A\bar{z} \tag{1}$$

Where Q = discharge, g = gravitational acceleration; and A = flow area, M = specific momentum; and $\bar{z}$ = depth of center of gravity of flow area below the free surface. For majority of cross sections (1) has two positive roots that are called sequent depths Henderson, (1966).

The present effort is carried out for finding out explicit solution for sequent depths for exponential sections, which include triangular, rectangular and parabolic channel shapes, besides trapezoidal section. The equations for each of those channels are, in general, high degree polynomials, such as cubics, quartics and quintics. It's known that the first two cases are quite simple to manipulate and give analytical solutions. On the other hand, quintics are far harder to work with, what makes necessary the use of some other methods. Mathematicians have known, for a long time, a theorem that fits perfectly to our needs: The Lagrange's inversion theorem. By giving an infinite series of implicitly defined functions, this theorem provides an exact solution to any of our equations. The major concern of such series expansion is the rate of convergence, which is in general high. Notwithstanding, in a few occasions one shall get slow convergence, what makes this method a bad alternative. To solve this issue, we apply a numerical solution, the Householder's methods. By starting from an initial guess, an iterative formula provides approximations of the solution. One shall choose the order of convergence of such method. By the way, the quadratic convergence expression is the well-known Newton-Rhapson algorithm. Both methods will be stated clearly below:

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Lagrange's Inversion Theorem:

The theorem is stated as: Let y be defined as the following function of constant a, function φ, and a parameter θ

$$y = a + \theta\varphi(y) \tag{2}$$

Then any function $f(y)$ is expressed as the following power series in θ Whittaker and Watson, (1990):

$$f(y) = f(a) + \sum_{n=1}^{\infty} \frac{\theta^n}{n!} \left\{ \frac{d^{n-1}}{dx^{n-1}} \left[f'(x)\varphi^n(x) \right] \right\}_{x=a} \tag{3}$$

It can be noticed that the right hand side of (3) contains y through θ defined in (2).

Householder's Methods:

Consider a function *f(x)* whose roots need to be estimated. By choosing an initial point, x_0, close enough to the value of the root, Householder, (1970) provided the general iteration root-finding recurrence formula:

$$x_{n+1} = x_n + (a+1)\left(\frac{(1/f)^{(a)}}{(1/f)^{(a+1)}} \right)_{x_n} \tag{4}$$

Where a+2 is the desired order of convergence of the iteration and $(1/f)^{(a)}$ is the a-th derivative of the inverse of the function f(x).

Channel Sections:

In this section, exponential and trapezoidal sections are discussed.

Exponential Section:

An exponential section is described by

$$Y = |kX|^p \tag{5}$$

where X and Y are horizontal and vertical coordinate axes respectively; k = coefficient; and p = exponent. For p = 1, the exponential section is a triangle of side slope k vertical to 1 horizontal, for p = 2, it is a parabola of latus rectum k^{-2}. For $p = \infty$, $Y = 0$ when kX is numerically less than unity, and $Y = \infty$ when kX numerically just exceeds unity. Thus a rectangular section of bed width $2/k$ is obtained. The area of the section for depth y is given by

$$A = \frac{2py^{(p+1)/p}}{k(p+1)} \tag{6}$$

The first moment of area about the free surface, $A\bar{z}$ is given by

$$A\bar{z} = \frac{2p^2 y^{(2p+1)/p}}{k(p+1)(2p+1)} \tag{7a}$$

where M = specific momentum; and $\bar{z}$ = depth of center of gravity below free surface.

Swamee, (1993) gave the following equation for critical depth in an exponential channel:

$$y_c = \left[\left(\frac{p+1}{p} \right)^3 \frac{Q^2 k^2}{4g} \right]^{\frac{p}{3p+2}} \tag{7b}$$

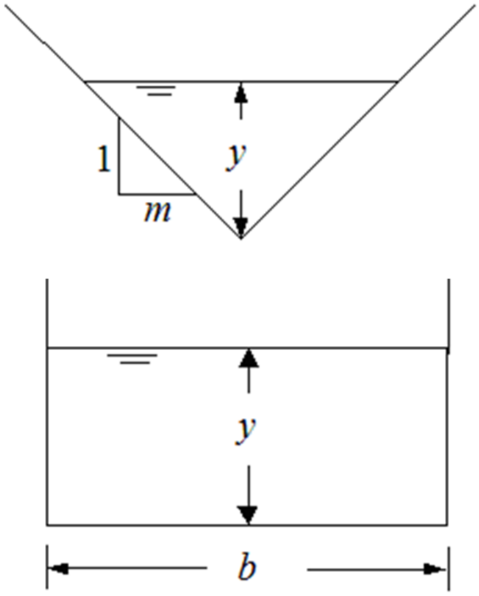

Fig. 1: Canal Sections (a) Triangular (b) Rectangular

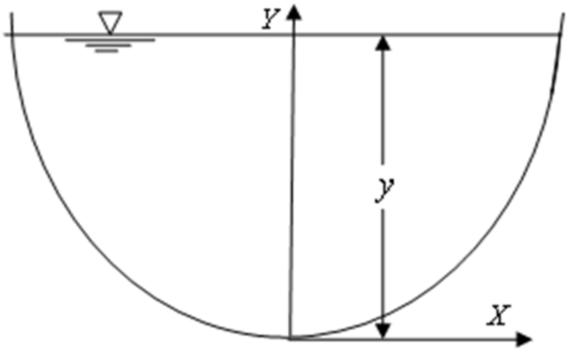

Fig. 2: Parabolic Sections:

Trapezoidal Section:

Due to stability questions, trapezoidal channels are preferred for carrying large discharges. The main issue is the side slope thickness, since vertical side walls require large thickness to resist the earth pressure. On the other hand, sloping side walls require less thickness. For a trapezoidal section of bed width *b* and side slope *m* horizontal to 1 vertical (See Fig. 3), the flow area is

$$A = y\left(b + my\right) \tag{8a}$$

where *m* = side slope *m* horizontal to 1 vertical. See Fig. 3. The moment of the flow area about the free surface is

$$A\overline{z} = \frac{1}{2}by^2 + \frac{1}{3}my^3 \tag{8b}$$

Analytical Considerations- Sequent Depths:

Combining (1), (6) and (7a), one gets

$$M = \frac{Q^2}{g}\frac{k\left(p+1\right)}{2py^{(p+1)/p}} + \frac{2p^2y^{(2p+1)/p}}{k\left(p+1\right)\left(2p+1\right)} \tag{9}$$

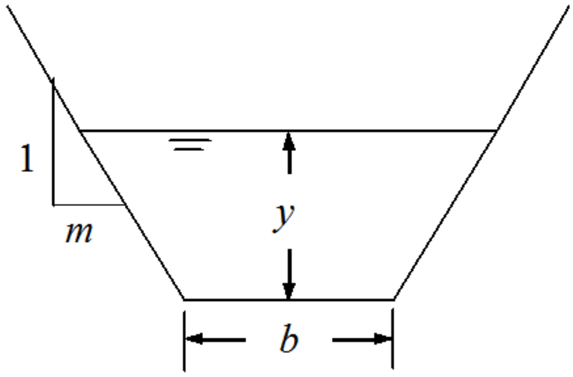

Fig. 3: Trapezoidal Section

Dividing through by y_c^3 and using (7b), following equation is obtained:

$$m_c = \frac{2}{z^{(p+1)/p}}\left(\frac{p}{p+1}\right)^2 + \frac{2p^2 z^{(2p+1)/p}}{(p+1)(2p+1)} \tag{10}$$

where

$$m_c = \frac{Mk}{y_c^{(2p+1)/p}} \tag{11}$$

Triangular Section:

Putting $p = 1$ and $k = 1/m$ for a triangular channel, (10) is converted to the following quintic equation:

$$m_c = \frac{1}{2z^2} + \frac{z^3}{3} \tag{12}$$

Where $z = y/y_c$ and

$$m_c = \frac{M}{my_c^3} \tag{13}$$

Rewriting (11) for obtaining z_1, one gets

$$z_1^2 = \frac{1}{2m_c} + \frac{1}{3m_c} z^5 \tag{14}$$

Taking $z_1^2 = w$ one finds

$$w = \frac{1}{2m_c} + \frac{1}{3m_c} w^{5/2} \tag{15}$$

Using Lagrange series solution for $f(w) = w^{1/2}$, (15) gives

$$z_1 = (2m_c)^{-1/2} + \sum_{n=1}^{\infty} \frac{(3m_c)^{-n}}{n!}\left(\frac{d^{n-1}}{dx^{n-1}} \frac{x^{(5n-1)/2}}{2}\right)_{x=(2m_c)^{-1}} \tag{16}$$

Equation (16) is evaluated as

$$z_1 = \left(2m_c\right)^{-1/2} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{2^{-(3n+1)/2} 3^{-n} m_c^{-(5n+1)/2} \Gamma\left[\left(5n+1\right)/2\right]}{n! \Gamma\left[\left(3n+3\right)/2\right]} \tag{17}$$

Equation (17) is simplified as

$$z_1 = \left(2m_c\right)^{-1/2} + \frac{m_c^{-3}}{24}\left[1 + \left(\frac{0.5880}{m_c}\right)^{5/2} + \left(\frac{0.6274}{m_c}\right)^{5} +. \left(\frac{0.6538}{m_c}\right)^{15/2} + \left(\frac{0.6731}{m_c}\right)^{10} + ...\right] \tag{18}$$

In order to get z_2, (12) is rewritten as

$$z_2^3 = 3m_c + \left(\frac{-3}{2}\right) z_2^{-2} \tag{19}$$

Taking $r = z_2^3$ one get

$$r = 3m_c + \frac{-3}{2} r^{-2/3} \tag{20}$$

Using Lagrange series solution for $f(r) = r^{1/3}$, (20) gives

$$z_2 = \left(3m_c\right)^{1/3} + \sum_{n=1}^{\infty} \frac{\left(-3/2\right)^n}{n!} \left(\frac{d^{n-1}}{dx^{n-1}} \frac{x^{-2(n+1)/3}}{3}\right)_{x=3m_c} \tag{21}$$

Equation (21) is simplified to

$$z_2 = \left(3m_c\right)^{1/3} - \sum_{n=1}^{\infty} \frac{3^{-(2n+2)/3} 2^{-n} m_c^{-(5n-1)/3} \Gamma\left[\left(5n-1\right)/3\right]}{n! \Gamma\left[\left(2n+2\right)/3\right]} \tag{22}$$

Equation (22) is further evaluated as

$$z_2 = \left(3m_c\right)^{1/3} - \frac{m_c^{-4/3}}{2.3^{4/3}}\left[1 + \left(\frac{0.4251}{m_c}\right)^{5/3} + \left(\frac{0.4922}{m_c}\right)^{10/3} +. \left(\frac{0.5368}{m_c}\right)^{5} + \left(\frac{0.5692}{m_c}\right)^{20/3} + ...\right] \tag{23}$$

Parabolic Section:

Putting $p = 2$ for a parabolic channel, (10) is converted to

$$m_c = \frac{8}{9z^{3/2}} + \frac{8z^{5/2}}{15} \tag{24}$$

where

$$m_c = \frac{Mk}{y_c^{5/2}} \tag{25}$$

wherein k^{-2} is latus rectum of the parabola.

For obtaining z_1, rewriting (24) as $z_1^{3/2} = 8/(9m_c) + 8z_1^4/(15m_c)$ and further denoting $q = z_1^{3/2}$, (24) becomes

$$q = \frac{8}{9m_c} + \frac{8}{15m_c} q^{8/3} \tag{26}$$

Using Lagrange's series expansion for $f(q) = q^{2/3}$,(26) provides

$$z_1 = \left(\frac{8}{9m_c}\right)^{2/3} + \frac{2}{3}\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{8}{15m_c}\right)^n \left(\frac{d^{n-1}x^{(8n-1)/3}}{dx^{n-1}}\right)_{x=8/(9m_c)} \tag{27}$$

Equation (27) is simplified to

$$z_1 = \left(\frac{8}{9m_c}\right)^{2/3} + \sum_{n=1}^{\infty}\frac{3^{-(13n+7)/3}2^{8n+3}5^{-n}m_c^{-(8n+2)/3}}{n!}x \quad x\frac{\Gamma\left[(8n+2)/3\right]}{\Gamma\left[(5n+5)/3\right]} \tag{28}$$

By evaluating (28) one gets

$$z_1 = \left(\frac{8}{9m_c}\right)^{2/3} + \left(\frac{0.6753}{m_c}\right)^{10/3}\left[1+\left(\frac{1.0349}{m_c}\right)^{8/3}+\left(\frac{1.0967}{m_c}\right)^{16/3}+.\left(\frac{1.1383}{m_c}\right)^{8}+\left(\frac{1.1687}{m_c}\right)^{32/3}+...\right] \tag{29}$$

To obtain z_2, (24) will be rearranged as

$$s = \frac{15m_c}{8} + \frac{-5}{3}s^{-3/5} \tag{30}$$

where $s = z_2^{5/2}$.

Using Lagrange's theorem for $f(s)=s^{2/5}$, (30) gives:

$$z_2 = \left(\frac{15m_c}{8}\right)^{2/5} + \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{-5}{3}\right)^n\left(\frac{d^{n-1}}{dx^{n-1}}\frac{2x^{-3(n+1)/5}}{5}\right)_{x=15m_c/8} \tag{31}$$

Equation (31) is simplified to

$$z_2 = \left(\frac{15m_c}{8}\right)^{2/5} - \sum_{n=1}^{\infty}\frac{3^{-(13n-2)/5}5^{-3(n+1)/5}2^{(24n-1)/5}}{n!}x \; x\frac{m_c^{-(8n-2)/5}\Gamma\left[(8n-2)/5\right]}{\Gamma\left[(3n+3)/5\right]} \tag{32}$$

Equation (32) is evaluated as

$$z_1 = \left(\frac{15m_c}{8}\right)^{2/5} - \left(\frac{0.0080}{m_c}\right)^{6/5}\left[1+\left(\frac{0.6872}{m_c}\right)^{8/5}+\left(\frac{0.8080}{m_c}\right)^{16/5}+.\left(\frac{0.8883}{m_c}\right)^{24/5}+\left(\frac{0.9467}{m_c}\right)^{32/5}+...\right] \tag{33}$$

Rectangular Section:

Putting p = ¥ for a rectangular, (10) is converted to the following cubic equation:

$$m_c = \frac{2}{z} + z^2 \tag{34}$$

where

$$m_c = \frac{2M}{by_c^2} \tag{35}$$

where $k = 2/b$ has been substituted. Equation (34) is rewritten as

$$z^3 - m_c z + 2 = 0 \tag{36}$$

Solving (36) as cubic Tignol (2007), the sequent depths are obtained as

$$z_1 = 2\left(\frac{m_c}{3}\right)^{1/2} \cos\left(\frac{\delta - 2\pi}{3}\right) \tag{37}$$

$$z_2 = 2\left(\frac{m_c}{3}\right)^{1/2} \cos\left(\frac{\delta}{3}\right) \tag{38}$$

Where

$$\delta = \arccos\left[-\left(\frac{3}{m_c}\right)^{3/2}\right] m_c \geq 3 \tag{39}$$

As an alternate solution, using Lagrange's theorem for z_1, (36) is rewritten as

$$z_1^{-1} = \frac{m_c}{2} - \frac{1}{2} z_1^2 \tag{40}$$

Taking $z_1^{-1} = l$ one gets

$$l = \frac{m_c}{2} - \frac{1}{2} l^{-2} \tag{41}$$

Using Lagrange series solution for $f(l)=l^{-1}$,(41) gives

$$z_1 = \frac{2}{m_c} + \sum_{n=1}^{\infty} \frac{2^{-n}(-1)^{n+1}}{n!} \left(\frac{d^{n-1}}{dx^{n-1}} x^{-2(n+1)}\right)_{x=m_c/2} \tag{42}$$

Equation (42) is simplified to

$$z_1 = \frac{2}{m_c} + \sum_{n=1}^{\infty} \frac{2^{(2n+1)} m_c^{-(3n+1)} \Gamma\left[(3n+1)\right]}{n!\Gamma\left[(2n+2)\right]} \tag{43}$$

Equation (43) is further evaluated as

$$z_1 = \frac{2}{m_c} + \frac{8}{m_c^4}\left[1 + \left(\frac{2.2894}{m_c}\right)^3 + \left(\frac{2.4019}{m_c}\right)^6 +. \left(\frac{2.4778}{m_c}\right)^9 + \left(\frac{2.6840}{m_c}\right)\right. \tag{44}$$

For obtaining z_2, (36) is written as

$$z_2^2 = m_c - 2z_2^{-1} \tag{45}$$

Taking $z_2^2 = u$ one gets

$$u = m_c - 2u^{-1/2} \tag{46}$$

Using Lagrange series solution for $f(u)=u^{1/2}$, (46) gives

$$z_2 = \left(m_c\right)^{1/2} + \sum_{n=1}^{\infty} \frac{(-2)^n}{n!} \left(\frac{d^{n-1}}{dx^{n-1}} \frac{x^{-(n+1)/2}}{2}\right)_{x=m_c} \tag{47}$$

Equation (47) is simplified to

$$z_2 = \left(m_c\right)^{1/2} - \sum_{n=1}^{\infty} \frac{2^{n-1} m_c^{-(3n-1)/2} \Gamma\left[\left(3n-1\right)/2\right]}{n!\Gamma\left[\left(n+1\right)/2\right]} \tag{48}$$

Equation (48) is further evaluated as

$$z_2 = m_c^{1/2} - \frac{1}{m_c}\left[1 + \left(\frac{1.3104}{m_c}\right)^{3/2} + \left(\frac{1.5874}{m_c}\right)^{3} + .\left(\frac{1.7720}{m_c}\right)^{9/2} + \left(\frac{1.9064}{m_c}\right)^{6} + \ldots\right] \tag{49}$$

Trapezoidal channel:

Using (8a) and (8b), (1) reduces to

$$m_b = \frac{G_b^2}{\alpha\left(1+\alpha\right)} + \frac{1}{2}\alpha^2 + \frac{1}{3}\alpha^3 \tag{50}$$

where $m_b = m^2M/b^3$. In order to get α_1, (50) is rewritten as

$$\alpha_1 = \frac{G_b^2}{m_b} + \frac{1}{m_b}\left(\frac{\alpha^5}{3} + \frac{5\alpha^4}{6} + \frac{\alpha^3}{2} - m_b\alpha^2\right) \tag{51}$$

Applying Lagrange's inverse expansion given by (3) to (51), one gets

$$\alpha_1 = \frac{G_b^2}{m_b} + \sum_{n=1}^{\infty} \frac{m_b^{-n}}{n!} \frac{d^{n-1}}{dx^{n-1}}\left[\left(\frac{x^5}{3} + \frac{5x^4}{6} + \frac{x^3}{2} - m_b x^2\right)^n\right]_{x=\frac{G_b^2}{m_b}} \tag{52}$$

Equation (52) does not provide a good value of α_1 due to the slow rate of convergence of the series. In this case, one shall use Householder's methods of iteration by taking $\alpha_* = G_b^2/m_b$ as the first estimative, due to its closeness to the exact value of α_1. Thus, by rearranging (51) and denoting:

$$l(\alpha) = \alpha^5 + \frac{5\alpha^4}{2} + \frac{3\alpha^3}{2} - 3m_b\alpha^2 - 3m_b\alpha + 3G_b^2 \tag{53}$$

The derivatives are:

$$l'(\alpha) = \frac{\partial l(\alpha)}{\partial\alpha} = 5\alpha^4 + 10\alpha^3 + \frac{9\alpha^2}{2} - 6m_b\alpha - 3m_b \tag{54}$$

$$l''(\alpha) = \frac{\partial^2 l(\alpha)}{\partial\alpha^2} = 20\alpha^3 + 30\alpha^2 + 9\alpha - 6m_b \tag{55}$$

$$l'''(\alpha) = \frac{\partial^3 l(\alpha)}{\partial\alpha^3} = 60\alpha^2 + 60\alpha + 9 \tag{56}$$

Finally, considering

$$l_* = -(l/l')(\alpha_*) \tag{57}$$

The value of α_1 with the quadratic convergence method is

$$\alpha_1 = \alpha_* + l_* \tag{58}$$

For cubic convergence one gets

$$\alpha_1 = \alpha_* + \frac{l_*}{1 + \frac{\left[(l''/l')(\alpha_*)\right]}{2} l_*} \tag{59}$$

And, for quadric convergence

$$\alpha_1 = \alpha_* + \frac{l_* \left\{1 + \frac{\left[(l''/l')(\alpha_*)\right]}{2} l_*\right\}}{1 + \left[(l''/l')(\alpha_*)\right] l_* + \frac{\left[(l'''/l')(\alpha_*)\right]}{6} l_*^2} \tag{60}$$

To get α_2, (50) is written as

$$\alpha_2^2 = 2m_b - 2\left(\frac{G_b^2}{\alpha_2\left(1+\alpha_2\right)} + \frac{\alpha_2^3}{3}\right) \tag{61}$$

Taking $u = \alpha_2^2$, one gets

$$u = 2m_b - 2\left(\frac{G_b^2}{u^{1/2}\left(1+u^{1/2}\right)} + \frac{u^{3/2}}{3}\right) \tag{62}$$

Applying Lagrange's inverse expansion given by (8) for $f(u) = u^{1/2}$ to (62), one gets

$$\alpha_2 = (2m_b)^{1/2} + \sum_{n=1}^{\infty} \frac{(-2)^n}{n!} \frac{d^{n-1}}{dx^{n-1}} \left[\frac{1}{2} x^{-1/2} \left(\frac{G_b^2}{x^{1/2}\left(1+x^{1/2}\right)} + \frac{x^{3/2}}{3}\right)^n\right]_{x=2m_b} \tag{63}$$

Again, due to slow convergence, one shall use Householder's methods. Now, following the result of (63) the first estimative shall be taken as $\alpha_* = \left(2m_b\right)^{1/2}$. Since the sequent depths are roots of the same polynomial, equations from (53) to (60) apply for both α_1 and α_2 , thus the results are already given above.

Practical Examples:

For illustrating the use of equations obtained in the foregoing sections, the following examples are presented.

Example 1:

A 4 m wide rectangular channel carries a discharge of 10 m^3/s. Find the sequent depths corresponding to a specific momentum of 10 m^3.

Solution:

In this case g = 9.79 m/s^2 is adopted.

The specific momentum parameter $m_c = 2M/(by_c^2) = 20/(4\text{x}0.861^2)$ = 6.745. Using Lagrange Series, the solution is described as:

Taking 3 terms:

y_1 = 0.259 m; and y_2 = 2.100 m.

Taking 4 terms:

y_1 = 0.259 m; and y_2 = 2.096 m.

Taking 5 terms:

y_1 = 0.259 m; and y_2 = 2.096 m.

On the other hand, the explicit solution provides:

y_1 = 0.259 m; and y_2 = 2.096 m.

Example 2:

A 2 m wide trapezoidal channel having side slope m = 1 carries a discharge of 10 m^3/s. Find the sequent depths corresponding to a specific momentum of 10 m^3.

Solution: In this case g = 9.79 m/s^2 is adopted. For the parameter G_b:

$$G_b = m^{1.5} Q / \left(b^2 \sqrt{gb} \right) = 10/(2^2 \sqrt{9.79x2}) = 0.565$$

The specific momentum parameter m_b = $m^2 M/b^3$ = 10/8 = 1.25. Using Lagrange Series combined with Householder's methods, the solution is described as:

For quadratic convergence:

y_1 = 0.431 m; and y_2 = 2.645 m.

For cubic convergence:

y_1 = 0.430 m; and y_2 = 2.415 m.

For quadric convergence:

y_1 = 0.430 m; and y_2 = 2.317 m.

By trial and error procedures, the following was obtained:

y_1 = 0.429 m; and y_2 = 2.258 m

Conclusion:

Analytical and numerical methods have been applied to solve the sequent depth problems. The results given are useful in situations where no computational resource is available and to give accurate and fast converging algorithms for future civil engineering software. It's known that computational methods are far easier to use than the ones provided in the present paper, however, the methods and ideas presented provide unexplored ways of solving civil engineering problems. The results obtained are very accurate, proving the effectiveness of the formulas given.

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