A theorem on the outer product of input and output Stokes vectors for deterministic optical systems

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Abstract

The Jones matrix transforms two dimensional complex Jones vectors into complex Jones vectors and accounts for the phase introduced by the deterministic optical system. On the other hand, the Mueller-Jones matrix of the deterministic optical system transforms four dimensional real Stokes vectors into real Stokes vectors which contains no information about the phase. Previously, a 4×4 complex matrix ($\mathbf{Z}$ matrix), akin to the Mueller-Jones matrix ($\mathbf{M} = \mathbf{Z}\mathbf{Z}^*$) was introduced and it was shown that $\mathbf{Z}$ matrix transforms four dimensional real Stokes vectors into four dimensional complex vectors which contain the relevant phase besides the other information. In this note it is shown that, for deterministic optical systems, there exists a relation between the outer products of experimentally measured real input-output Stokes vectors and theoretically calculated four dimensional complex vectors that obtained as a result of the transformation of real Stokes vectors by the $\mathbf{Z}$ matrix.

1 Introduction

In previous works [1,2] two different representations of Mueller-Jones states were introduced: the vector state $|h\rangle$ and the matrix state $\mathbf{Z}$. Vector state, $|h\rangle$, can be defined in terms of a Hermitian matrix $\mathfrak{H}$ associated with the Mueller matrix $\mathbf{M}$ according to the following transformation:

$$\mathfrak{H} = \frac{1}{4}\sum_{i,j=0}^{3} M_{ij}\Sigma_{ij}, \tag{1}$$

where $M_{ij}(i,j = 0,1,2,3)$ are the elements of the Mueller matrix $\mathbf{M}$ and $\Sigma_{ij} = \mathbf{U}(\sigma_i \otimes \sigma_j^*)\mathbf{U}^{-1}$ with,

$$\mathbf{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \end{pmatrix}, \quad \mathbf{U}^{-1} = \mathbf{U}^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & 1 & i \\ 1 & -1 & 0 & 0 \end{pmatrix}. \tag{2}$$

The superscript † indicates the complex conjugate and transpose, the superscript * indicates complex conjugate, $\otimes$ is the Kronecker product and σ_i are the Pauli matrices with the 2×2 identity in the following order:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{3}$$

If and only if the Mueller matrix of the system is nondepolarizing, the associated $\mathfrak{H}$ matrix will be of rank 1. In this case it is always possible to define a covariance vector $|h\rangle$ such that [4,5]:

$$\mathfrak{H} = |h\rangle\langle h|, \tag{4}$$

where $|h\rangle$ is the eigenvector of $\mathfrak{H}$ corresponding to the single nonzero eigenvalue.

The dimensionless components of $|h\rangle$ can be parametrized as τ, α, β and γ:

$$|h\rangle = \begin{pmatrix} \tau \\ \alpha \\ \beta \\ \gamma \end{pmatrix}, \tag{5}$$

α, β and γ are generally complex numbers, while τ can always be chosen as real and positive if the overall phase is not taken into account.

The $\mathbf{Z}$ matrix is also defined in terms of the parameters τ, α, β and γ and it is just another form of the Mueller-Jones state in matrix form:

$$\mathbf{Z} = \begin{pmatrix} \tau & \alpha & \beta & \gamma \\ \alpha & \tau & -i\gamma & i\beta \\ \beta & i\gamma & \tau & -i\alpha \\ \gamma & -i\beta & i\alpha & \tau \end{pmatrix}. \tag{6}$$

In terms of the $\mathbf{Z}$ matrices the Mueller matrix of nondepolarizing optical media can be written as:

$$\mathbf{M} = \mathbf{Z}\mathbf{Z}^* = \mathbf{Z}^*\mathbf{Z}. \tag{7}$$

Eq.(7) leads to an expression for the Mueller-Jones matrix in terms of the parameters τ, α, β and γ [1]:

$$\mathbf{M} = \left(\begin{array}{c|c|c|c} \begin{matrix}\tau\tau^* + \alpha\alpha^* \\ \beta\beta^* + \gamma\gamma^*\end{matrix} & \begin{matrix}\tau\alpha^* + \alpha\tau^* \\ +i(\gamma\beta^* - \beta\gamma^*)\end{matrix} & \begin{matrix}\tau\beta^* + \beta\tau^* \\ +i(\alpha\gamma^* - \gamma\alpha^*)\end{matrix} & \begin{matrix}\tau\gamma^* + \gamma\tau^* \\ +i(\beta\alpha^* - \alpha\beta^*)\end{matrix} \\ \hline \begin{matrix}\tau\alpha^* + \alpha\tau^* \\ -i(\gamma\beta^* - \beta\gamma^*)\end{matrix} & \begin{matrix}\tau\tau^* + \alpha\alpha^* \\ -\beta\beta^* - \gamma\gamma^*\end{matrix} & \begin{matrix}\alpha\beta^* + \beta\alpha^* \\ +i(\tau\gamma^* - \gamma\tau^*)\end{matrix} & \begin{matrix}\alpha\gamma^* + \gamma\alpha^* \\ +i(\beta\tau^* - \tau\beta^*)\end{matrix} \\ \hline \begin{matrix}\tau\beta^* + \beta\tau^* \\ -i(\alpha\gamma^* - \gamma\alpha^*)\end{matrix} & \begin{matrix}\alpha\beta^* + \beta\alpha^* \\ -i(\tau\gamma^* - \gamma\tau^*)\end{matrix} & \begin{matrix}\tau\tau^* - \alpha\alpha^* \\ +\beta\beta^* - \gamma\gamma^*\end{matrix} & \begin{matrix}\beta\gamma^* + \gamma\beta^* \\ +i(\tau\alpha^* - \alpha\tau^*)\end{matrix} \\ \hline \begin{matrix}\tau\gamma^* + \gamma\tau^* \\ -i(\beta\alpha^* - \alpha\beta^*)\end{matrix} & \begin{matrix}\alpha\gamma^* + \gamma\alpha^* \\ -i(\beta\tau^* - \tau\beta^*)\end{matrix} & \begin{matrix}\beta\gamma^* + \gamma\beta^* \\ -i(\tau\alpha^* - \alpha\tau^*)\end{matrix} & \begin{matrix}\tau\tau^* - \alpha\alpha^* \\ -\beta\beta^* + \gamma\gamma^*\end{matrix} \end{array}\right). \tag{8}$$

The Mueller matrix $\mathbf{M}$ transforms the Stokes vector $|s\rangle$ into the Stokes vector $|s'\rangle$:

$$|s'\rangle = \mathbf{M}|s\rangle, \tag{9}$$

where $|s\rangle = (s_0, s_1, s_2, s_3)^T$; $|s'\rangle = (s'_0, s'_1, s'_2, s'_3)^T$ (s_i and s'_i are real numbers).

If the Mueller matrix is nondepolarizing and $|s\rangle$ represents totally polarized light $[(s_0)^2 = (s_1)^2 + (s_2)^2 + (s_3)^2]$ then $|s'\rangle$ is also totally polarized, i.e., $(s'_0)^2 = (s'_1)^2 + (s'_2)^2 + (s'_3)^2$.

On the other hand, $\mathbf{Z}$ matrix transforms the real stokes vector $|s\rangle$ into a four dimensional complex vector $|\tilde{s}\rangle$:

$$|\tilde{s}\rangle = \mathbf{Z}|s\rangle, \tag{10}$$

where $|\tilde{s}\rangle = (\tilde{s}_0, \tilde{s}_1, \tilde{s}_2, \tilde{s}_3)^T$ and $\tilde{s}_0, \tilde{s}_1, \tilde{s}_2$ and $\tilde{s}_3$ are, in general, complex numbers.

It can be shown that $|\tilde{s}\rangle$ bears the total phase introduced by the optical element [3]:

$$\langle E|E'\rangle = \langle E|\mathbf{J}|E\rangle = \frac{1}{2}\langle s|\mathbf{Z}|s\rangle = \frac{1}{2}\langle s|\tilde{s}\rangle, \tag{11}$$

where $\mathbf{J}$ is the Jones matrix associated with the optical system and $|E\rangle$ is the Jones vector representing the polarization state of the incoming light and $|E'\rangle$ is the transformed Jones vector.

2 Theorem for outer product of input and output Stokes vectors

In this note it is shown that there exists a relation between the vectors $|s\rangle, |s'\rangle$ and $|\tilde{s}\rangle$. The relation can be formulated between real $\mathbf{K}$ and complex $\mathfrak{S}$ matrices defined by the following outer products:

$$\mathbf{K} = |s'\rangle\langle s|, \tag{12}$$

$$\mathfrak{S} = |\tilde{s}\rangle\langle\tilde{s}|. \tag{13}$$

It is worth noting that $\mathfrak{S}$ is a Hermitian matrix and it should not be confused with the $\tilde{\mathbf{S}}$ matrix that was previously introduced [3] (see Appendix). Also note that $|s'\rangle$ and $|s\rangle$ vectors are real and directly measurable quantities and $\mathbf{K}$ has the following explicit form:

$$\mathbf{K} = \begin{pmatrix} s'_0 s_0 & s'_0 s_1 & s'_0 s_2 & s'_0 s_3 \\ s'_1 s_0 & s'_1 s_1 & s'_1 s_2 & s'_1 s_3 \\ s'_2 s_0 & s'_2 s_1 & s'_2 s_2 & s'_2 s_3 \\ s'_3 s_0 & s'_3 s_1 & s'_3 s_2 & s'_3 s_3 \end{pmatrix}. \tag{14}$$

On the other hand, $\mathfrak{S}$ matrix is a complex-Hermitian matrix that contains information about the phase introduced by the optical system which cannot be measured by simple polarimetric methods relying on only intensity measurements and it has the following explicit form:

$$\mathfrak{S} = \begin{pmatrix} \tilde{s}_0\tilde{s}_0 & \tilde{s}_0\tilde{s}_1 & \tilde{s}_0\tilde{s}_2 & \tilde{s}_0\tilde{s}_3 \\ \tilde{s}_1\tilde{s}_0 & \tilde{s}_1\tilde{s}_1 & \tilde{s}_1\tilde{s}_2 & \tilde{s}_1\tilde{s}_3 \\ \tilde{s}_2\tilde{s}_0 & \tilde{s}_2\tilde{s}_1 & \tilde{s}_2\tilde{s}_2 & \tilde{s}_2\tilde{s}_3 \\ \tilde{s}_3\tilde{s}_0 & \tilde{s}_3\tilde{s}_1 & \tilde{s}_3\tilde{s}_2 & \tilde{s}_3\tilde{s}_3 \end{pmatrix}. \tag{15}$$

In this note it will be shown that $\mathbf{K}$ and $\mathfrak{S}$ matrices can be bridged by means of a complex-Hermitian $\mathfrak{T}$ matrix. Theorem states that $\mathfrak{S}$ matrix is equal to the $\mathfrak{T}$ matrix:

$$\mathfrak{S} = \mathfrak{T}, \tag{16}$$

where $\mathfrak{T}$ is defined by the following transformation:

$$\mathfrak{T} = \frac{1}{2}\sum_{i,j=0}^{3} K_{ij}\Sigma_{ij}, \tag{17}$$

or, in an explicit form:

$$\mathfrak{T} = \frac{1}{2}\left(\begin{array}{c|c|c|c} \begin{matrix}K_{00}+K_{11}\\ K_{22}+K_{33}\end{matrix} & \begin{matrix}K_{01}+K_{10}\\ -i(K_{23}-K_{32})\end{matrix} & \begin{matrix}K_{02}+K_{20}\\ +i(K_{13}-K_{31})\end{matrix} & \begin{matrix}K_{03}+K_{30}\\ -i(K_{12}-K_{21})\end{matrix} \\ \hline \begin{matrix}K_{01}+K_{10}\\ +i(K_{23}-K_{32})\end{matrix} & \begin{matrix}K_{00}+K_{11}\\ -K_{22}-K_{33}\end{matrix} & \begin{matrix}K_{12}+K_{21}\\ +i(K_{03}-K_{30})\end{matrix} & \begin{matrix}K_{13}+K_{31}\\ -i(K_{02}-K_{20})\end{matrix} \\ \hline \begin{matrix}K_{02}+K_{20}\\ -i(K_{13}-K_{31})\end{matrix} & \begin{matrix}K_{12}+K_{21}\\ -i(K_{03}-K_{30})\end{matrix} & \begin{matrix}K_{00}-K_{11}\\ +K_{22}-K_{33}\end{matrix} & \begin{matrix}K_{23}+K_{32}\\ +i(K_{01}-K_{10})\end{matrix} \\ \hline \begin{matrix}K_{03}+K_{30}\\ +i(K_{12}-K_{21})\end{matrix} & \begin{matrix}K_{13}+K_{31}\\ +i(K_{02}-K_{20})\end{matrix} & \begin{matrix}K_{23}+K_{32}\\ -i(K_{01}-K_{10})\end{matrix} & \begin{matrix}K_{00}-K_{11}\\ -K_{22}+K_{33}\end{matrix} \end{array}\right) \tag{18}$$

The relation $\mathfrak{S} = \mathfrak{T}$ can be proved by using Eq.(8) and calculating each element of the matrices to show that $\mathfrak{S}_{i,j} = \mathfrak{T}_{i,j}$.

As an example, let $|h\rangle$ be the following vector state:

$$|h\rangle = \begin{pmatrix} 1+i \\ 1-2i \\ 2+3i \\ 0 \end{pmatrix} \tag{19}$$

Corresponding $\mathbf{Z}$ matrix is

$$\mathbf{Z} = \begin{pmatrix} 1+i & 1-2i & 2+3i & 0 \\ 1-2i & 1+i & 0 & -3+2i \\ 2+3i & 0 & 1+i & -2-i \\ 0 & 3-2i & 2+i & 1+i \end{pmatrix} \tag{20}$$

The Mueller matrix of the deterministic optical system can be easily obtained from the relation $\mathbf{M} = \mathbf{Z}\mathbf{Z}^*$:

$$\mathbf{M} = \begin{pmatrix} 20 & -2 & 10 & -14 \\ -2 & -6 & -8 & -2 \\ 10 & -8 & 10 & -6 \\ 14 & 2 & 6 & -16 \end{pmatrix} \tag{21}$$

For a given input Stokes vector all relevant vectors and matrices can be calculated. For example let $|s\rangle = (5, 3, 0, 4)^T$:

$$|s'\rangle = \begin{pmatrix} 38 \\ -36 \\ 2 \\ 12 \end{pmatrix}; \quad |\tilde{s}\rangle = \begin{pmatrix} 8-i \\ -4+i \\ 2+11i \\ 13-2i \end{pmatrix}. \tag{22}$$

$$\mathbf{K} = |s'\rangle\langle s| = \begin{pmatrix} 190 & 114 & 0 & 152 \\ -180 & -108 & 0 & -144 \\ 10 & 6 & 0 & 8 \\ 60 & 36 & 0 & 48 \end{pmatrix} \tag{23}$$

Then, according to Eq.(17) or transformation table (18):

$$\mathfrak{T} = \frac{1}{2}\sum_{i,j=0}^{3} K_{ij}\Sigma_{ij} = \begin{pmatrix} 65 & -33-4i & 5-90i & 106+3i \\ -33+4i & 17 & 3+46i & -54+5i \\ 5+90i & 3-46i & 125 & 4+147i \\ 106-3i & -54-5i & 4-147i & 173 \end{pmatrix}. \tag{24}$$

It is now easy to show $\mathfrak{S} = \mathfrak{T}$ by calculating the outer product, $\mathfrak{S} = |\tilde{s}\rangle\langle\tilde{s}|$:

$$\mathfrak{S} = \begin{pmatrix} 8-i \\ -4+i \\ 2+11i \\ 13-2i \end{pmatrix} \begin{pmatrix} 8+i, & -4-i, & 2-11i, & 13+2i \end{pmatrix} \tag{25}$$

It is worth noting that $rank(\mathfrak{S}) = rank(\mathfrak{T}) = 1$, hence, all column vectors of $\mathfrak{S}$ matrix (and $\mathfrak{T}$ matrix) are equivalent to each other, i.e., column vectors differ from each other only by respective phases. These column vectors are also equivalent to the the $|\tilde{s}\rangle$ vector apart from overall phase factors. For example the first column vector of the $\mathfrak{T}$ matrix, $|c_1\rangle$, differs from the $|\tilde{s}\rangle$ vector by a factor $8+4i$:

$$|c_1\rangle = \begin{pmatrix} 65 \\ -33+4i \\ 5+90i \\ 106-3i \end{pmatrix} = (8+i)|\tilde{s}\rangle \tag{26}$$

Now, suppose that $|h\rangle$ vector or $\mathbf{Z}$ matrix or the Mueller matrix $\mathbf{M}$ are not given but $|s\rangle$ and $|s'\rangle$ vectors are known as a result of the measurement, then $|\tilde{s}\rangle$ vector can be calculated from the outer product of $|s\rangle$ and $|s'\rangle$ vectors apart from its original overall phase. Situation is very similar to the overall phase issue encountered while trying to calculate the Jones matrix of the optical system from the associated nondepolarizing Mueller matrix [6].

Complex components of the $|\tilde{s}\rangle$ vector are not directly accessible by measurement, but $|\tilde{s}\rangle$ vector plays a very important role in the mathematical formalism of polarization optics. If $|\tilde{s}\rangle$ is given or calculated it is possible to extract $|s\rangle$ and $|s'\rangle$ vectors from $|\tilde{s}\rangle$ by the following inverse transformation:

$$\mathbf{K} = \frac{1}{2}\sum_{i,j=0}^{3} \mathfrak{S}_{ij}\Sigma_{ij}. \tag{27}$$

Once $\mathbf{K}$ matrix is calculated from $\mathfrak{S}$ matrix, it can be shown that $|s'\rangle$ and $|s\rangle$ vectors (input and output Stokes vectors) can be obtained from $\mathbf{K}$ matrix which is by definition equal to the outer product $|s'\rangle\langle s|$. In the above example $|s\rangle$ vector can be read from the first row of $\mathbf{K}$ matrix and $|s'\rangle$ can be read from the first column of $\mathbf{K}$ matrix:

$$(first - row)^T = \begin{pmatrix} 190 \\ 114 \\ 0 \\ 152 \end{pmatrix} = s_0' |s\rangle = 38 \begin{pmatrix} 5 \\ 3 \\ 0 \\ 4 \end{pmatrix} \tag{28}$$

$$(first - column) = \begin{pmatrix} 190 \\ -180 \\ 10 \\ 60 \end{pmatrix} = s_0 |s\rangle = 5 \begin{pmatrix} 38 \\ -36 \\ 2 \\ 12 \end{pmatrix} \tag{29}$$

In order to recover the original $|s\rangle$ and $|s'\rangle$ vectors actual values of the parameters s_0' and s_0 are needed. But, usually, absolute values of the input and output Stokes vectors are not important, hence, $|s\rangle$ and $|s'\rangle$ vectors can be re-normalized for further calculations.

As a result, it is shown that there exists two Hermitian matrices bridging between the experimentally measured real valued input and output Stokes vectors and theoretically calculated associated complex valued vectors. In the next manuscript an application of this theorem will be introduced and it will be shown that if the Mueller matrix of a nondepolarizing optical system has one of the symmetries described in [1] the Jones matrix of the optical system, apart from an overall phase, can be obtained by two polarimetric measurements.

3 Appendix

$\mathfrak{S}$ and $\tilde{\mathbf{S}}$ matrices are different mathematical objects. $\tilde{\mathbf{S}}$ matrix is defined as follows [3]:

$$\tilde{\mathbf{S}} = \mathbf{ZS} \tag{30}$$

where $\mathbf{S}$ and $\tilde{\mathbf{S}}$ matrices are defined as,

$$\mathbf{S} = \begin{pmatrix} s_0 & s_1 & s_2 & s_3 \\ s_1 & s_0 & -is_3 & is_2 \\ s_2 & is_3 & s_0 & -is_1 \\ s_3 & -is_2 & is_1 & s_0 \end{pmatrix} \tag{31}$$

$$\tilde{\mathbf{S}} = \begin{pmatrix} \tilde{s}_0 & \tilde{s}_1 & \tilde{s}_2 & \tilde{s}_3 \\ \tilde{s}_1 & \tilde{s}_0 & -i\tilde{s}_3 & i\tilde{s}_2 \\ \tilde{s}_2 & i\tilde{s}_3 & \tilde{s}_0 & -i\tilde{s}_1 \\ \tilde{s}_3 & -i\tilde{s}_2 & i\tilde{s}_1 & \tilde{s}_0 \end{pmatrix} \tag{32}$$

Still, there is a connection between $\mathfrak{S}$ and $\tilde{\mathbf{S}}$ matrices: First column of $\tilde{\mathbf{S}}$ matrix is equal to $|\tilde{s}\rangle$ vector, and it is also equal to the first column of the $\mathfrak{S}$ matrix if s_0 is chosen to be equal to 1.

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