RECEIVED: *May 03, 2019*
ACCEPTED: *May 20, 2019*
PUBLISHED: *July 01, 2019*

Online Mathematics
OMJ, 01 (03): 15–36, ISSN: 2672-7501
http://www.onlinemathematics.net

RESEARCH ARTICLE

A Logical Formalization of the Notion of Interval Dependency

Towards Reliable Intervalizations of Quantifiable Uncertainties

Hend Dawood[1] **and Yasser Dawood**[2]

[1]*Department of Mathematics, Faculty of Science, Cairo University, Giza 12613, Egypt*
[2]*Department of Astronomy, Faculty of Science, Cairo University, Giza 12613, Egypt*

CORRESPONDING AUTHOR: Hend Dawood (*hend.dawood@sci.cu.edu.eg*)

Abstract

Progress in scientific knowledge discloses an increasingly paramount use of quantifiable properties in the description of states and processes of the real-world physical systems. Through our encounters with the physical world, it reveals itself to us as systems of uncertain quantifiable properties. One approach proved to be most fundamental and reliable in coping with quantifiable uncertainties is interval mathematics. A main drawback of interval mathematics, though, is the persisting problem known as the "interval dependency problem". This, naturally, confronts us with the question: *Formally, what is interval dependency*? Is it a 'meta-concept' or an 'object-ingredient' of interval and fuzzy computations? In other words, what is the fundamental defining properties that characterize the notion of interval dependency as a formal mathematical object? Since the early works on interval mathematics by John Charles Burkill and Rosalind Cecily Young in the dawning of the twentieth century, this question has never been touched upon and remained a question still today unanswered. Although the notion of interval dependency is widely used in the interval and fuzzy literature, it is only illustrated by example, without explicit formalization, and no attempt has been made to put on a systematic basis its meaning, that is, to indicate formally the criteria by which it is to be characterized. Here, we attempt to answer this long-standing question. This article, therefore, is devoted to presenting a complete systematic formalization of the notion of interval dependency, by means of the notions of Skolemization and quantification dependence. A novelty of this formalization is the expression of interval dependency as a logical predicate (or relation) and thereby gaining the advantage of deducing its fundamental properties in a merely logical manner. Moreover, on the strength of the generality of the logical apparatus we adopt, the results of this article are not only about classical intervals, but they are meant to apply also to any possible theory of interval arithmetic. That being so, our concern is to shed new light on some fundamental problems of interval mathematics and to take one small step towards paving the way for developing alternate dependency-aware interval theories and computational methods.

Keywords: Interval mathematics; Interval dependency; Functional dependence; Skolemization; Guaranteed bounds; Interval enclosures; Interval functions; Quantifiable uncertainty; Scientific knowledge; Reliability; Fuzzy mathematics; InCLosure

Mathematics Subject Classification: 65Gxx, 03Exx, 03B80, 08Axx, 68T37, 94D05

> In logic there is no such thing as a hidden connection. You can't get behind the rules because there isn't any behind.
>
> –Ludwig Wittgenstein (1889–1951)

1 Introduction

Acquiring scientific knowledge discloses an ever-increasing use of *quantifiable features* of real-world physical systems. Many features of the physical world represent themselves to us as numerical values. *Three farads capacitance*, *six becquerels radioactivity*, and *nine metres depth* are common examples of quantifiable properties. The numerical values of quantifiable properties are established by means of measurement. Through our encounters with the physical world, measurements cannot, nevertheless, provide perfect exactitude and measurable magnitudes are usually uncertain. In

EMAIL ADDRESS, Hend Dawood: hend.dawood@sci.cu.edu.eg
EMAIL ADDRESS, Yasser Dawood: ydawod@gmail.com

the effort to deal with quantifiable uncertainties, various theoretical approaches have been developed. Among these, one can mention: probabilitization, fuzzification, and intervalization. One approach that proved to be subtle, reliable, and most fundamental in all of mathematics of uncertainty is interval mathematics.

The term "*interval arithmetic*" is reasonably recent: it dates from the 1950s, when the works of Paul S. Dwyer, Ramon Edgar Moore, Raymond E. Boche, Sidney Shayer, and others made the term popular (see, [30], [55], [7], and [64]). But the idea of calculating with intervals is not completely new in mathematics: in the course of history, it has been invented and re-invented several times, under different names, and never been abandoned or forgotten. The concept has been known since the third century BC, when Archimedes used guaranteed lower and upper bounds to compute his constant, π (see [38]).

Early in the twentieth century, the idea seemed to be rediscovered. A form of interval arithmetic perhaps first appeared in 1924 by John Charles Burkill in his paper "*Functions of Intervals*" ([8]), and in 1931 by Rosalind Cecily Young in her paper "*The Algebra of Many-Valued Quantities*" ([73]) that gives rules for calculating with intervals and other sets of real numbers; then later in 1951 by Paul S. Dwyer in his book "*Linear computations*" ([30]) that discusses, in a heuristic manner, certain methods for performing basic arithmetic operations on real intervals, and in 1958 by Teruo Sunaga in his book "*Theory of an Interval Algebra and its Application to Numerical Analysis*" ([66]).

However, it was not until 1959 that new formulations of interval arithmetic were presented. Modern developments of the interval theory began in 1959 with Moore's technical report "*Automatic Error Analysis in Digital Computation*" ([55]) in which he developed a number system and an arithmetic dealing with closed real intervals. He called the numbers "*range numbers*" and the arithmetic "*range arithmetic*" to be the first synonyms of "*interval numbers*" and "*interval arithmetic*". Then later in 1962, Moore developed a theory for exact or infinite precision interval arithmetic in his very influential dissertation entitled "*Interval Arithmetic and Automatic Error Analysis in Digital Computing*" ([56]) in which he used a modified digital (*rounded*) interval arithmetic as the basis for automatic analysis of total error in a digital computation. In his comprehensive book "*Interval Analysis*" ([57]), Moore was the first to define interval analysis in its modern sense and recognize its practical power as a viable computational tool for bounding errors and intervalizing uncertainty. Since then, thousands of research papers and numerous books have appeared on the subject.

By integrating the complementary powers of rigorous mathematics and scientific computing, interval arithmetic is able to offer highly reliable accounts of uncertainty. Not surprisingly, therefore, that the interval theory has been fruitfully applied in diverse areas that deal intensely with uncertain quantitative data (see, e.g., [17], [19], [29], [28], [34], [45], [47], and [59]). In view of its computational power against error, machine realizations of interval arithmetic are of great importance. As a matter of course, there are various software implementations of interval arithmetic. As instances, we may mention INTLAB, Sollya, InCLosure and others (see, e.g., [63], [11], [18], and [53]). Fortunately, computers are getting faster and most existing parallel processors provide a tremendous computing power. So, with little extra hardware, it is very possible to make interval computations as fast as floating point computations (For further reading about machine arithmetizations and hardware circuitries for interval arithmetic, see, e.g., [15], [19], [44], [43], [48], [60], [40], and [41]).

Despite all of the above mentioned advantages of interval mathematics, it has its disadvantages as well. A main drawback of interval mathematics is the persisting problem known as the "*interval dependency problem*". This, naturally, confronts us with the crucial question: *Formally, what is interval dependency*? Is it a meta-concept or an object-ingredient of interval and fuzzy mathematics? In other words, what is the fundamental defining properties that characterize the notion of interval dependency as a formal mathematical object? Since the early works on interval mathematics by Burkill [8] and Young [73] in the dawning of the twentieth century, this question has never been touched upon and remained a question still today unanswered. Although the notion of interval dependency is widely used in the interval and fuzzy literature, it is only illustrated by example, without explicit formalization, and no attempt has been made to put on a systematic basis its meaning, that is, to indicate formally the criteria by which it is to be characterized. Here, we attempt to answer this long-standing question.

To reiterate, what exactly is the sense of saying that two intervals are *dependent* (*independent*)? and how does the dependency of two intervals $2X$ and X differ from that of X^2 and X? Our aim here is to propose a precise logical characterization of the notion of interval dependency and to begin the development of a rigorous mathematical theory which formally characterizes and explains the differences between all cases of interval dependency. That being so, the problem to be dealt with in this text is that of the possibility and the scope of a symbolic formalization of interval dependency. The indispensable role of symbolic formalizations can neither be ignored nor denigrated. Kleene, in [42], best described this by saying:

> "Anyone who doubts the advantages of symbols (in their proper place) is invited to solve the equation $x^2+3x-2=0$ by completing the square (as taught in high school), but doing all the work in words. We start him off by stating the equation in words: The square of the unknown, increased by three times the unknown, and diminished by two, is equal to zero."

With an eye toward making the adopted formal approach clearly comprehensible, before moving on to the main business of the text, we set the stage by fixing our *formalized apparatus* early in section 2. In section 3, we give a characterization of the classical interval theory as a many-sorted algebra over the real field and deduce some of the fundamental properties of interval numbers. Section 4 is devoted to providing a bit of perspective on the need for interval mathematics to deal with quantifiable uncertainties, along with giving a description of the interval dependency problem and the challenges thereof. The objective of section 5 is to justify a symbolic notation suitable for our purpose and to delve into some important fundamentals concerning the logical formulation of the notion of functional dependence and some related notions. In section 6, we attempt to answer the long-standing question concerning the defining properties that characterize the notion of interval dependency as a formal mathematical object. We present a complete *systematization* of the notion of interval dependency, by means of the logical notions of *Skolemization* and *quantification dependence*. Finally, in section 7, we discuss briefly how to compute useful guaranteed enclosures of real-valued functions under functional dependence. The numerical examples of section 7 are computed using version 2.0 of InCLosure. The InCLosure commands [1] to compute the results of the examples are described and an InCLosure input file and its corresponding output containing, respectively, the code and results of the examples are also available as a supplementary material to this text (see Supplementary Materials).

What is then the fundamental importance of a logical formalization of interval dependency? The authors believe that such a formalization, hopefully, might have a worthwhile impact on both fundamental research and real world applications. Effort in pursuit of this aim can have many fruitful consequences. A novelty of this formalization is the expression of interval dependency as a *logical predicate* (or *relation*) and thereby gaining the advantage of deducing its fundamental properties in a merely logical manner. The mathematical theory developed in the present article formally characterizes and explains the differences between all cases of interval dependency, and thus sheds new light on many fundamental problems of interval mathematics. Moreover, taking the passage from the informal treatments to the formal technicalities of mathematical logic, a breakthrough behind our systematization of interval dependency is that it paves the way and provides the systematic apparatus for developing alternate *dependency-aware* interval theories and computational methods with mathematical constructions that better account for dependencies between the quantifiable uncertainties of the real world. Noteworthy also is that on the strength of the generality of the logical apparatus we adopt, the results of this article are not only about Moore's classical intervals, but they are meant to apply also to any *possible* theory of interval arithmetic.

2 A Bit of Formalism: Setting the Stage

In order to be able to give a complete systematic formalization of the notion of interval dependency and related notions, we do need readily available a formal apparatus. So, the adaptation of a particular formal approach, other than that of natural language, is of the utmost importance and has been forced on us by the pursuit of formulating the underlying ideas in a strictly accurate manner that all the results of this work can be generated from clear and distinct elementary concepts. Therefore, before moving on to the main business of this article, we begin in this section by specifying some notational conventions and formalizing some purely logical and algebraic ingredients we shall need throughout this text (For further details about the notions prescribed here, the reader may consult, e.g., [5], [22], [23], [26], [19], [28], and [54]).

Most of our notions are characterized in terms of *ordinals* and *ordinal tuples*. So, we first define what an *ordinal* is. An ordinal is the well-ordered set of all ordinals preceding it. That is, for each ordinal n, there exists an ordinal $S(n)$ called the successor of n such that

$$(\forall n)(\forall k)(k = S(n) \Leftrightarrow (\forall m)(m \in k \Leftrightarrow m \in n \vee m = n)).$$

In other words, we have $S(n) = n \cup \{n\}$. Accordingly, the first *infinite* (*transfinite*) *ordinal* is the set $\omega = \{0, 1, 2, ...\}$. All ordinals preceding ω (all elements of ω) are *finite ordinals*. The idea of *transfinite counting* (*counting beyond the finite*) is due to Cantor (See [9]).

With the aid of ordinals, the notions of *countably finite*, *countably infinite* and *uncountably infinite* sets can be characterized as follows. A set $\mathcal{S}$ is countably finite if there is a bijective mapping from $\mathcal{S}$ onto some finite ordinal $n \in \omega$. A set $\mathcal{S}$ is countably infinite (or denumerable) if there is a bijective mapping from $\mathcal{S}$ onto the infinite ordinal ω. For example the set $\{a_0, a_1, a_2\}$ is countably finite because it can be bijectively mapped onto the finite ordinal $3 = \{0, 1, 2\}$, while the set $\{a_0, a_1, a_2, ...\}$ is denumerable because it can be bijectively mapped onto the infinite ordinal $\omega = \{0, 1, 2, ...\}$. An uncountably infinite set is an infinite set which is not countably infinite. For example the set $\mathbb{R}$ of real numbers is uncountably infinite.

The notion of an *n-tuple* is characterized in the following definition.

[1] InCLosure (Interval enCLosure) is a language and environment for reliable scientific computing, which is coded entirely in Lisp. Latest version of InCLosure is available for free download via *https://doi.org/10.5281/zenodo.2702404*.

Definition 2.1 (Ordinal Tuple). *For an ordinal $n = S(k)$, an n-tuple (ordinal tuple) is any mapping τ whose domain is n. A finite n-tuple is an n-tuple for some finite ordinal n. That is*

$$\begin{aligned}\tau_{S(k)} &= \langle \tau(0), \tau(1), ..., \tau(k) \rangle \\ &= \langle (0, \tau(0)), (1, \tau(1)), ..., (k, \tau(k)) \rangle.\end{aligned}$$

If $n = 0 = \varnothing$, then, for any set $\mathcal{S}$, there is exactly one mapping (*the empty mapping*) $\tau_{\varnothing} = \varnothing$ from $\varnothing$ into $\mathcal{S}$.

An important definition we shall need is that of the Cartesian power of a set.

Definition 2.2 (Cartesian Power). *Let $\varnothing$ denote the empty set. For a set $\mathcal{S}$ and an ordinal n, the n-th Cartesian power of $\mathcal{S}$ is the set $\mathcal{S}^n$ of all mappings from n into $\mathcal{S}$, that is*

$$\mathcal{S}^n = \begin{cases} \{\varnothing\} & n = 0, \\ \text{the set of all n-tuples of elements of } \mathcal{S} & n = 1 \vee 1 \in n. \end{cases}$$

If $\mathcal{S}$ is the empty set $\varnothing$, then [2]

$$\varnothing^n = \begin{cases} \{\varnothing\} & n = 0, \\ \varnothing & n = 1 \vee 1 \in n; \end{cases} \quad \text{and} \quad \varnothing^{\varnothing^n} = \begin{cases} \varnothing & n = 0, \\ \{\varnothing\} & n = 1 \vee 1 \in n. \end{cases}$$

In accordance with the preceding definitions, a set-theoretical relation is a particular type of sets. Let $\mathcal{S}^2$ be the binary Cartesian power of a set $\mathcal{S}$. A binary relation on $\mathcal{S}$ is a subset of $\mathcal{S}^2$. That is, a set $\Re$ is a binary relation on a set $\mathcal{S}$ iff $(\forall \mathbf{r} \in \Re)((\exists x, y \in \mathcal{S})(\mathbf{r} = (x, y))$.

We will continue to follow the formal tradition of Suppes [67] and Tarski [69] in defining, within a set-theoretical framework, the notion of a *finitary relation* and some related concepts. Let $\mathcal{U}^n$ be the n-th Cartesian power of a set $\mathcal{U}$. A set $\Re \subseteq \mathcal{U}^n$ is an n-ary relation on $\mathcal{U}$ iff $\Re$ is a binary relation from $\mathcal{U}^{n-1}$ to $\mathcal{U}$. That is, for $\mathbf{v} = (x_1, ..., x_{n-1}) \in \mathcal{U}^{n-1}$ and $y \in \mathcal{U}$, an n-ary relation $\Re$ is defined to be $\Re \subseteq \mathcal{U}^n = \{(\mathbf{v}, y) \,|\, \mathbf{v} \in \mathcal{U}^{n-1} \wedge y \in \mathcal{U}\}$. In this sense, an n-ary relation is a binary relation (or simply a relation); then its *domain*, *range*, *field*, and *converse* are defined to be, respectively $\mathrm{dom}(\Re) = \{\mathbf{v} \in \mathcal{U}^{n-1} \,|\, (\exists y \in \mathcal{U})(\mathbf{v} \Re y)\}$, $\mathrm{ran}(\Re) = \{y \in \mathcal{U} \,|\, (\exists \mathbf{v} \in \mathcal{U}^{n-1})(\mathbf{v} \Re y)\}$, $\mathrm{fld}(\Re) = \mathrm{dom}(\Re) \cup \mathrm{ran}(\Re)$, and $\widehat{\Re} = \{(y, \mathbf{v}) \in \mathcal{U}^n \,|\, \mathbf{v} \Re y\}$. It is thus obvious that $y \widehat{\Re} \mathbf{v} \Leftrightarrow \mathbf{v} \Re y$ and $\widehat{\widehat{\Re}} = \Re$.

Two important notions, for the purpose at hand, are the *image* and *preimage* of a set, with respect to an n-ary relation. These are defined as follows ([16] and [28]).

Definition 2.3 (Image and Preimage of a Relation). *Let $\Re$ be an n-ary relation on a set $\mathcal{U}$, and for $(\mathbf{v}, y) \in \Re$, let $\mathbf{v} = (x_1, ..., x_{n-1})$, with each x_k is restricted to vary on a set $X_k \subset \mathcal{U}$, that is, $\mathbf{v}$ is restricted to vary on a set $\mathbf{V} \subset \mathcal{U}^{n-1}$. Then, the image of $\mathbf{V}$ (or the image of the sets X_k) with respect to $\Re$, denoted $\mathrm{I}_{\Re}$, is defined to be*

$$\begin{aligned} Y = \mathrm{I}_{\Re}(\mathbf{V}) &= \mathrm{I}_{\Re}(X_1, ..., X_{n-1}) \\ &= \{y \in \mathcal{U} \,|\, (\exists \mathbf{v} \in \mathbf{V})(\mathbf{v} \Re y)\} \\ &= \{y \in \mathcal{U} \,|\, (\exists_{k=1}^{n-1} x_k \in X_k)((x_1, ..., x_{n-1}) \Re y)\}, \end{aligned}$$

where the set $\mathbf{V}$, called the preimage of Y, is defined to be the image of Y with respect to the converse relation $\widehat{\Re}$, that is

$$\mathbf{V} = \mathrm{I}_{\widehat{\Re}}(Y) = \{\mathbf{v} \in \mathcal{U}^{n-1} \,|\, (\exists y \in Y)\left(y \widehat{\Re} \mathbf{v}\right)\}.$$

In accordance with this definition and the fact that $y \widehat{\Re} \mathbf{v} \Leftrightarrow \mathbf{v} \Re y$, we obviously have $Y = \mathrm{I}_{\Re}(\mathbf{V}) \Leftrightarrow \mathbf{V} = \mathrm{I}_{\widehat{\Re}}(Y)$.

Now for a mathematically satisfactory characterization of a finitary function. Within this set-theoretical framework, a completely general definition of the notion of an *n-ary function* can be formulated. A set f is an n-ary function on a set $\mathcal{U}$ iff f is an $(n+1)$-ary relation on $\mathcal{U}$, and $(\forall \mathbf{v} \in \mathcal{U}^n)(\forall y, z \in \mathcal{U})(\mathbf{v} f y \wedge \mathbf{v} f z \Rightarrow y = z)$. Thus, an n-ary function is a *many-one* $(n+1)$-ary relation; that is, a relation, with respect to which, any element in its domain is related exactly to one element in its range. Getting down from relations to the particular case of functions, we have at hand the standard notation: $y = f(\mathbf{v})$ in place of $\mathbf{v} f y$. From the fact that an n-ary function is a special kind of relation, then all the preceding definitions and results, concerning the *domain*, *range*, *field*, and *converse* of a relation, apply to functions as well.

With some criteria satisfied, a function is called *invertible*. A function f has an inverse, denoted f^{-1}, iff its converse relation $\widehat{f}$ is a function, in which case $f^{-1} = \widehat{f}$. In other words, f is invertible if, and only if, it is an *injection* from

[2] Amer in [3] used the n-th Cartesian power of $\varnothing$ to define empty structures, and axiomatized their first-order theory.

its domain to its range, and obviously the inverse f^{-1} is *unique*, from the fact that the converse relation is always *definable* and *unique*.

A formalized theory is characterized by two things; an object language in which the theory is formalized (the symbolism of the theory), and a set of axioms. Let $\mathfrak{L}$ be an object formal language. A formalized theory in $\mathfrak{L}$ (or an $\mathfrak{L}$-theory) is a set of $\mathfrak{L}$-sentences which is closed under its associated deductive apparatus. Let $\Lambda_{\mathfrak{T}}$ denote a finite set of $\mathfrak{L}$-sentences, and let φ denote an $\mathfrak{L}$-sentence. The formalized $\mathfrak{L}$-theory $\mathfrak{T}$ of the set $\Lambda_{\mathfrak{T}}$ is the deductive closure of $\Lambda_{\mathfrak{T}}$ under logical consequence, that is

$$\mathfrak{T} = \{\varphi \in \mathfrak{L} | \varphi \text{ is a consequence of } \Lambda_{\mathfrak{T}}\}.$$

The set $\Lambda_{\mathfrak{T}}$ is called the set of axioms (or postulates) of $\mathfrak{T}$.

A model (or an interpretation) of a theory $\mathfrak{T}$ is some particular (algebraic or relational) structure that satisfies every formula of $\mathfrak{T}$. Finally, we close this section by characterizing some algebraic structures of particular importance to our purpose (see [19] and [28]).

Definition 2.4 (Ringoid). *A ringoid (or a ring-like structure) is a structure $\mathfrak{R} = \langle \mathcal{R}; +_{\mathcal{R}}, \times_{\mathcal{R}} \rangle$ with $+_{\mathcal{R}}$ and $\times_{\mathcal{R}}$ are total binary operations on the universe set $\mathcal{R}$. The operations $+_{\mathcal{R}}$ and $\times_{\mathcal{R}}$ are called respectively the addition and multiplication operations of the ringoid $\mathfrak{R}$.*

Definition 2.5 (S-Ringoid). *An S-ringoid (or a subdistributive ringoid) is a ringoid that satisfies at least one of the following subdistributive criteria.*

(i) $(\forall x, y, z \in \mathcal{R})\,(x \times_{\mathcal{R}} (y +_{\mathcal{R}} z) \subseteq x \times_{\mathcal{R}} y +_{\mathcal{R}} x \times_{\mathcal{R}} z)$,

(ii) $(\forall x, y, z \in \mathcal{R})\,((y +_{\mathcal{R}} z) \times_{\mathcal{R}} x \subseteq y \times_{\mathcal{R}} x +_{\mathcal{R}} z \times_{\mathcal{R}} x)$.

Criteria (i) and (ii) in the preceding definition are called respectively left and right subdistributivity (or S-distributivity).

Definition 2.6 (Semiring). *Let $\mathfrak{R} = \langle \mathcal{R}; +_{\mathcal{R}}, \times_{\mathcal{R}} \rangle$ be a ringoid. $\mathfrak{R}$ is said to be a semiring iff*

(i) *$\langle \mathcal{R}; +_{\mathcal{R}} \rangle$ is a commutative monoid with identity element $0_{\mathcal{R}}$,*

(ii) *$\langle \mathcal{R}; \times_{\mathcal{R}} \rangle$ is a monoid with identity element $1_{\mathcal{R}}$,*

(iii) *Multiplication, $\times_{\mathcal{R}}$, left and right distributes over addition, $+_{\mathcal{R}}$,*

(iv) *$0_{\mathcal{R}}$ is an absorbing element for $\times_{\mathcal{R}}$.*

A commutative semiring is one whose multiplication is commutative.

Definition 2.7 (S-Semiring). *An S-semiring (or a subdistributive semiring) is an S-ringoid that satisfies criteria (i), (ii), and (iv) in definition 2.6. A commutative S-semiring is one whose multiplication is commutative.*

At this point, let us note that the notion of S-semiring is a generalization of the notion of a *near*-semiring; a near-semiring is a ringoid that satisfies the criteria of a semiring except that it is *either* left or right distributive (For detailed discussions of near-semirings and related concepts, the interested reader may consult, e.g., [71], [61], and [12]).

3 The Theory of Interval Algebra over the Real Field

With the formalized apparatus of section 2 at our disposal, the main business of this section is to give a formalized characterization of the theory $\mathsf{Th}_{\mathcal{I}}$ of classical interval arithmetic over the real field. There are many theories of interval arithmetic (see, e.g., [37], [48], [36], [51], [50], [40], [41], [15], [25], [21], and [28]). We are here interested in characterizing classical interval arithmetic as introduced in, e.g., [55], [64], [59], [17], [19], and [27]. Notwithstanding, on the strength of the generality of the logical apparatus we adopt in this work, the formalization of interval dependency presented in the succeeding sections and the results thereof are not only about *Moore's classical intervals*, but they are meant to apply also to *any possible* theory of interval arithmetic.

Our adopted strategy to obtain a concrete system of classical interval numbers is to start with the field of real numbers and to "*intervalize*" it, by defining new interval relations and operations. In other words, the theory $\mathsf{Th}_{\mathcal{I}}$ of real intervals will be constructed as a *definitional extension* of the theory of real numbers. Hereafter and throughout this work, the machinery used, and assumed priori, is the *standard* (*classical*) predicate calculus and axiomatic set theory. Moreover, in all the proofs, elementary facts about operations and relations on the real numbers are usually used without explicit reference.

A theory $\mathsf{Th}_{\mathcal{I}}$ of a *real interval algebra* (a *classical interval algebra* or an *interval algebra over the real field*) is characterized in the following definition (see [17] and [19]).

Definition 3.1 (Theory of Real Interval Algebra). *Take* $\sigma = \{+,\times;-,^{-1};0,1\}$ *as a set of non-logical constants and let* $\mathsf{R} = \langle \mathbb{R};\sigma^{\mathsf{R}}\rangle$ *be the totally* $\leq$*-ordered field of real numbers. The theory* $\mathsf{Th}_{\mathcal{I}}$ *of an interval algebra over the field* R *is the theory of a two-sorted structure* $\mathfrak{I}_{\mathbb{R}} = \langle \mathcal{I}_{\mathbb{R}};\mathbb{R};\sigma^{\mathfrak{I}_{\mathbb{R}}}\rangle$ *prescribed by the following set of axioms.*

(I1) $(\forall X \in \mathcal{I}_{\mathbb{R}})\,(X = \{x \in \mathbb{R} |\, (\exists \underline{x} \in \mathbb{R})\,(\exists \overline{x} \in \mathbb{R})\,(\underline{x} \leq_{\mathbb{R}} x \leq_{\mathbb{R}} \overline{x})\}),$

(I2) $(\forall X,Y \in \mathcal{I}_{\mathbb{R}})\,\big(\circ \in \{+,\times\} \Rightarrow X \circ_{\mathcal{I}_{\mathbb{R}}} Y = \{z \in \mathbb{R} |\, (\exists x \in X)\,(\exists y \in Y)\,(z = x \circ_{\mathbb{R}} y)\}\big),$

(I3) $(\forall X \in \mathcal{I}_{\mathbb{R}})\,\big(\diamond \in \{-\} \vee \big(\diamond \in \{^{-1}\} \wedge 0_{\mathcal{I}_{\mathbb{R}}} \not\subseteq X\big) \Rightarrow \diamond_{\mathcal{I}_{\mathbb{R}}} X = \{z \in \mathbb{R} |\, (\exists x \in X)\,(z = \diamond_{\mathbb{R}} x)\}\big).$

The sentence (I1) of definition 3.1 characterizes what an interval number (or a closed $\mathbb{R}$-interval) is. The sentences (I2) and (I3) prescribe, respectively, the binary and unary operations for $\mathbb{R}$-intervals. Hereafter, the upper-case Roman letters X, Y, and Z (with or without subscripts), or equivalently $[\underline{x},\overline{x}]$, $[\underline{y},\overline{y}]$, and $[\underline{z},\overline{z}]$, shall be employed as variable symbols to denote real interval numbers. A *point* (*singleton*) interval number $\{x\}$ shall be denoted by $[x]$. The letters A, B, and C, or equivalently $[\underline{a},\overline{a}]$, $[\underline{b},\overline{b}]$, and $[\underline{c},\overline{c}]$, shall be used to denote constants of $\mathcal{I}_{\mathbb{R}}$. Also, we shall single out the symbols $1_{\mathcal{I}}$ and $0_{\mathcal{I}}$ to denote, respectively, the singleton $\mathbb{R}$-intervals $\{1_{\mathbb{R}}\}$ and $\{0_{\mathbb{R}}\}$. For the purpose at hand, it is convenient to define two proper subsets of $\mathcal{I}_{\mathbb{R}}$: the sets of *symmetric interval numbers* and *point interval numbers*. Respectively, these are defined and denoted by

$$\mathcal{I}_{\mathrm{S}} = \{X \in \mathcal{I}_{\mathbb{R}} | (\exists x \in \mathbb{R})(0 \leq x \wedge X = [-x,x])\},$$
$$\mathcal{I}_{[x]} = \{X \in \mathcal{I}_{\mathbb{R}} | (\exists x \in \mathbb{R})(X = [x,x])\}.$$

From the fact that real intervals are totally $\leq_{\mathbb{R}}$-ordered subsets of $\mathbb{R}$, equality of $\mathbb{R}$-intervals follows immediately from the axiom of extensionality [3] of set theory. That is,

$$[\underline{x},\overline{x}] =_{\mathcal{I}} [\underline{y},\overline{y}] \Leftrightarrow \underline{x} =_{\mathbb{R}} \underline{y} \wedge \overline{x} =_{\mathbb{R}} \overline{y}.$$

Since $\mathbb{R}$-intervals are *ordered sets* of real numbers, it follows that the next theorem is derivable from definition 3.1 (see [17] and [20]).

Theorem 3.1 (Interval Operations). *For any two interval numbers* $[\underline{x},\overline{x}]$ *and* $[\underline{y},\overline{y}]$*, the binary and unary interval operations are formulated in terms of the intervals' endpoints as follows.*

(i) $[\underline{x},\overline{x}] +_{\mathcal{I}} [\underline{y},\overline{y}] = [\underline{x} +_{\mathbb{R}} \underline{y}, \overline{x} +_{\mathbb{R}} \overline{y}],$

(ii) $[\underline{x},\overline{x}] \times_{\mathcal{I}} [\underline{y},\overline{y}] = [\min\{\underline{x} \times_{\mathbb{R}} \underline{y}, \underline{x} \times_{\mathbb{R}} \overline{y}, \overline{x} \times_{\mathbb{R}} \underline{y}, \overline{x} \times_{\mathbb{R}} \overline{y}\}, \max\{\underline{x} \times_{\mathbb{R}} \underline{y}, \underline{x} \times_{\mathbb{R}} \overline{y}, \overline{x} \times_{\mathbb{R}} \underline{y}, \overline{x} \times_{\mathbb{R}} \overline{y}\}],$

(iii) $-_{\mathcal{I}} [\underline{x},\overline{x}] = [-_{\mathbb{R}}\overline{x}, -_{\mathbb{R}}\underline{x}],$

(iv) $0_{\mathcal{I}} \not\subseteq [\underline{x},\overline{x}] \Rightarrow [\underline{x},\overline{x}]^{-1_{\mathcal{I}}} = \left[\overline{x}^{-1_{\mathbb{R}}}, \underline{x}^{-1_{\mathbb{R}}}\right],$

where min *and* max *are respectively the* $\leq_{\mathbb{R}}$*-minimal and* $\leq_{\mathbb{R}}$*-maximal.*

Wherever there is no confusion, we shall drop the subscripts $\mathcal{I}$ and $\mathbb{R}$. It is obvious that all the interval operations, except interval reciprocal, are *total* operations. The additional operations of interval subtraction and division can be defined respectively as $X - Y = X + (-Y)$ and $X \div Y = X \times \left(Y^{-1}\right)$.

Classical interval arithmetic has a number of peculiar algebraic properties: The point intervals $0_{\mathcal{I}}$ and $1_{\mathcal{I}}$ are identity elements for addition and multiplication, respectively; interval addition and multiplication are both commutative and associative; interval addition is cancellative; interval multiplication is cancellative only for zeroless intervals; an interval number is invertible for addition (respectively, multiplication) if and only if it is a point interval (respectively, a nonzero point interval); and interval multiplication left and right *subdistributes* over interval addition (see definition 2.5 of section 2). To sum up, according to definition 2.7, the structure $\langle \mathcal{I}_{\mathbb{R}};+_{\mathcal{I}},\times_{\mathcal{I}};0_{\mathcal{I}},1_{\mathcal{I}}\rangle$ of classical interval numbers is a commutative S-*semiring* ([17] and [19]).

Throughout this text, we shall employ the following theorem and its corollary (see, [16], and [17]).

Theorem 3.2 (Inclusion Monotonicity in Classical Intervals). *Let* X_1*,* X_2*,* Y_1*, and* Y_2 *be interval numbers such that* $X_1 \subseteq Y_1$ *and* $X_2 \subseteq Y_2$*. Then for any binary operation* $\circ \in \{+,\times\}$ *and any definable unary operation* $\diamond \in \{-,^{-1}\}$*, we have*

(i) $X_1 \circ_{\mathcal{I}} X_2 \subseteq Y_1 \circ_{\mathcal{I}} Y_2,$

[3] The axiom of extensionality asserts that two sets are equal if, and only if they have precisely the same elements, that is, for any two sets $\mathcal{S}$ and $\mathcal{T}$, $\mathcal{S} = \mathcal{T} \Leftrightarrow (\forall z)\,(z \in \mathcal{S} \Leftrightarrow z \in \mathcal{T})$.

(ii) $\diamond_{\mathcal{I}} X_1 \subseteq \diamond_{\mathcal{I}} Y_1$.

In consequence of this theorem, from the fact that $[x,x] \subseteq X \Leftrightarrow x \in X$, we have the following important special case.

Corollary 3.1 (Membership Monotonicity for Classical Intervals). *Let X and Y be real interval numbers with $x \in X$ and $y \in Y$. Then for any binary operation $\circ \in \{+,\times\}$ and any definable unary operation $\diamond \in \{-,^{-1}\}$, we have*

(i) $x \circ_{\mathbb{R}} y \in X \circ_{\mathrm{c}} Y$,

(ii) $\diamond_{\mathbb{R}} x \in \diamond_{\mathrm{c}} X$.

If we endow the classical interval algebra $\langle \mathcal{I}_{\mathbb{R}}; +_{\mathcal{I}}, \times_{\mathcal{I}}; 0_{\mathcal{I}}, 1_{\mathcal{I}} \rangle$ with the *compatible* partial ordering $\subseteq$, then we have a partially-ordered commutative S-semiring. In addition to ordering intervals by the set inclusion relation $\subseteq$, there are many orders presented in the interval literature. Among these is Moore's partial ordering which is defined by $[\underline{x},\overline{x}] <_{\mathrm{M}} [\underline{y},\overline{y}] \Leftrightarrow \overline{x} <_{\mathbb{R}} \underline{y}$. In contrast to the case for $\subseteq$, Moore's partial ordering $<_{\mathrm{M}}$ is not compatible with the algebraic operations on $\mathcal{I}_{\mathbb{R}}$ (see [17] and [24]).

Some numerical examples are shown below.

Example 3.1 (Classical Interval Operations). *For three given interval numbers $[1,2]$, $[3,4]$, and $[-2,2]$, we have*

(i) $[1,2]+[3,4] = [4,6]$,

(ii) $[1,2] \times [3,4] = [3,8]$,

(iii) $[1,2]^{-1} = [1/2,1]$,

(iv) $[-2,2]^2 = [-2,2] \times [-2,2] = [-4,4]$.

Example 3.2 (Moore's and Inclusion Orders). *For four given interval numbers $A = [1,2]$, $B = [1,2]$, $C = [1,3]$, and $D = [4,7]$, we have $A = B <_{\mathrm{M}} D$ and $A \subseteq B \subseteq C$.*

4 Intervalization of Physical Uncertainties and the Dependency Problem

When modelling and predicting a real-world or physical phenomena, we often face the problem that, through our encounters with the real-world phenomena, it reveals its present and future states to us as systems of *uncertain* quantifiable properties. Tackling such uncertainties introduces two different problems: *getting guaranteed bounds* of the exact value of a quantifiable property and *computing an approximation* of the value. The two problems are very different. In many practical applications, the numerical approximations provided by machine real arithmetic are not beneficial. In robotics and control applications, for example, it is important to have guaranteed inclusions of the exact values in order to guarantee stability under uncertainty (see [17] and [28]). The two problems are not equivalent. The implication

"getting guaranteed bounds" $\Rightarrow$ "computing approximations",

is true, but the converse implication is not. Interval arithmetic provides us a reliable way to get guaranteed bounds when modelling physical systems under uncertainty. An interval number (a real closed interval) is then a *reliable enclosure* of an uncertain real-valued quantity, and an interval function is accordingly a reliable enclosure of an uncertain real-valued function.

Error is a *definiens* of uncertainty in measurable magnitudes. *To measure is to err* and to acquire quantitative knowledge about the world involves measurements. Quantitative uncertainty can then be defined to be the quantitative estimation of errors present in measured data; all measurements contain some uncertainty generated through many types of error. Error ("mistaken result", or "mistaken outcome") is common in all scientific practice, and is always a serious threat to the search for a trustworthy scientific knowledge and to reliable epistemic foundations of science. In 1597, Sir Francis Bacon defined science as "*the process used everyday to logically complete thoughts through inference of facts determined by calculated experiments*". The epistemology of scientific knowledge, that is the study of how scientific knowledge is acquired, tells us that in all scientific disciplines, our knowledge of real-world systems (which can be physical, chemical, biological, economical, social, and so forth) is acquired through *observing*, *experimenting*, *measuring identifiable features*, and *formulating hypotheses* with the aid of formal reasoning ([17] and [24]). In their way to acquire this knowledge, scientists and engineers always perform one or both of the following two inquiries:

- **Present-State Inquiry.** To quantify an identifiable property ρ that provides some information about the present state of the real-world system.

- **Future-State Inquiry.** To quantify an identifiable property τ that provides some information about a future state of the real-world system.

In most practical problems, the properties ρ and τ are not directly identifiable, that is, they are not quantifiable by direct measurements or expert estimations. However, we usually know a *present-state function* (or *algorithm*) p that relates the present property ρ to some quantifiable auxiliary properties $x_1,...,x_m$, and a *future-state function* (or *algorithm*) f that relates the future property τ to some quantifiable auxiliary properties $y_1,...,y_n$. So, to quantify the properties ρ and τ, we *measure* or *estimate* the related quantifiable auxiliary properties, and then apply the known present-state and future-state functions

$$\rho = p(x_1,...,x_m),$$
$$\tau = f(y_1,...,y_n).$$

At this point the crucial question is: *Do measurements or expert estimations provide reliable information about the quantifiable properties?* The answer is: "*Unfortunately, no*". In practical situations, uncertainty naturally arises when processing values which come from *measurements* or from *expert estimations*. From both the *epistemological* and *physical* viewpoints, neither measurements nor expert estimations can be exact for the following reasons ([17] and [46]):

- The actual value of the measured quantity is a real number; so, in general, we need *infinitely* many bits to describe the exact value, while after every measurement, we gain only a *finite* number of bits of information (e.g., a finite number of binary digits in the binary expansion of the number).
- There is always some difficult-to-delete noise which is mixed with the measurement results.
- Expert estimates cannot be absolutely exact, because an expert generates only a finite amount of information.
- Experts are usually even less accurate than are measuring instruments.

In addition to measurement and expert estimation errors, there are types of error that arise when doing computations with the measured or estimated values. There are usually three sources of error while performing numerical computations with real numbers:

- Input Errors. Input errors usually arise from human mistakes. An input error is committed when a human inputs to a machine a wrong value for the measured or estimated quantity.
- Truncation Errors. Truncation errors arise when replacing a continuous or infinite operation by a computable discrete operation.
- Rounding Errors. Rounding errors arise when doing arithmetic on a machine. This error is the difference between the result obtained using exact arithmetic and the result computed using finite precision arithmetic.

All of these uncertainties, generated through many types of error, threaten our inquiries about the world, and a mistaken outcome is always a concern in scientific research. So, the question is: *Is there a way out?* Fortunately, interval arithmetic recommends itself to us by its ability to keep track of all error types simultaneously, because an interval arithmetic operation produces an *interval of certainty* within which the true *real-valued* result is guaranteed to lie [17]. As an example, suppose two independent scientific measurements give different uncertain results for the same quantity q. One measurement gives $q = 1.4 \pm 0.2$. The other gives $q = 1.5 \pm 0.2$. These uncertain values of q can be represented as the interval numbers $X = [1.2, 1.6]$ and $Y = [1.3, 1.7]$, respectively. Since q lies in both, it certainly lies in their intersection $X \cap Y = [1.3, 1.6]$. So, if $X \cap Y \neq \varnothing$, we can get a better (tighter) "interval of certainty". If not, we can be certain that at least one of the two measurements is wrong.

In order to further illustrate the power of the guaranteed enclosures of interval arithmetic, it is appropriate to provide here a brief discussion of the limitations and loss of precision of machine real arithmetic. Machine real numbers have finite decimal places of precision. The *finite* precision provided by modern computers is enough in many real life applications, and there is scarcely a physical quantity which can be measured beyond the maximum representable value of this precision. So when is the finite precision not enough? The problem arises when doing arithmetic. The operation of "*subtraction*", for example, results, in many situations, in an inevitable loss of precision. Consider, for instance, the expression $x - y$, with $x = 0.963$ and $y = 0.962$. The exact result of $x - y$ is 0.001, but when evaluating this expression on a machine with 2 significant digits, the values of x and y are rounded *downward* to have the same machine value 0.96, and the machine result of $x - y$ becomes 0, which is a complete loss of precision. Instead, by enclosing the exact values of x and y in interval numbers, with outward rounding, a *guaranteed enclosure* of the exact result of $x - y$ can be obtained easily by computing with machine interval numbers ([19] and [28]).

This problem of machine subtraction has dangerous consequences in numerical computations. To illustrate, consider the problem of calculating the derivative of a real-valued function f at a given point. The method of *finite differences* is a numerical method which can be performed by a computer to approximate the derivative. For a differentiable function f, the first derivative $f^{(1)}$ can be approximated by

$$f^{(1)} \cong \frac{f(x+dx)-f(x)}{dx},$$

for a *small* nonzero value of dx. As dx approaches zero, the derivative is better approximated. But as dx gets smaller, the rounding error increases because of the finite precision of machine real arithmetic, and we accordingly get the problematic situation of $f(x+dx)-f(x)=0$. That is, for small enough values of dx, the derivative will be always computed as *zero*, regardless of the rule of the function f. Using interval enclosures of the function f instead, we can find a way out of this problem, by virtue the *infinite precision* of machine interval arithmetic. For further details on interval enclosures of derivatives, see, e.g., [17] and [29].

Another problem of finite precision arises when *truncating* an infinite operation by a computable finite operation. For example, The exponential function e^x may be written as a Taylor series

$$e^x = 1+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots=\sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

In order to compute this infinite series on a machine, we have to *truncate* it to the partial sum

$$S_k=\sum_{n=0}^{k}\frac{x^n}{n!},$$

for some finite k, and the truncation error then is $|e^x-S_k|$. Using interval bounds for this error term, machine interval arithmetic can provide a *guaranteed enclosure* of the exact value of the exponential function e^x.

In addition to the problematic situations described above, the loss of precision of machine real arithmetic can even lead to fatal and costly disasters, for example, the *failure of the Patriot anti-missile batteries* during the Gulf War which is due to round-off error in floating-point calculations. Another example is the *explosion of the Ariane 5 rocket* on June 4, 1996 (for further details, see [43]).

The preceding examples shed light on the fact that taking the passage from real arithmetic to interval arithmetic opens the way to the rich technicalities and the infinite precision of interval computations. With the power of intervals at our disposal, let us revisit the uncertainty problem that we discussed in the beginning of this section, that of acquiring knowledge about the real world. When applying traditional numerical methods to estimate the error in the measurable auxiliary properties $x_1,\ldots,x_m$ and $y_1,\ldots,y_n$, we get approximate (*non-guaranteed*) bounds to the measurement errors that are, in many cases, *not sufficient*. Moreover, we sometimes face situations in which the probability distribution for the measurement errors cannot be determined, and consequently *probabilitization* is not valid. Furthermore, in some practical situations, fuzzifying the problem suffers from many limitations, and therefore the efficacy of *fuzzification*, in such situations, is questionable.

Now, let us *intervalize* the problems of inquiring the present and future states of a real-world system. We can measure or estimate the *quantifiable* auxiliary properties $x_1,\ldots,x_m$ and $y_1,\ldots,y_n$. With intervals at hand, we have *intervals of certainty* X_i and Y_k for the auxiliary properties x_i and y_k respectively. Knowing the present-state function p and the future-state function f that relate the *directly-unquantifiable* present property ρ and future property τ to their auxiliary properties x_i and y_k respectively, we need to compute the images of X_i and Y_k with respect to the functions p and f. These images are defined and denoted by

$$\mathrm{I}_p(X_1,\ldots,X_m)=\{\rho\in\mathbb{R}|\,(\exists_{i=1}^{m}x_i\in X_i)\,(\rho=p(x_1,\ldots,x_m))\},$$
$$\mathrm{I}_f(Y_1,\ldots,Y_n)=\{\tau\in\mathbb{R}|\,(\exists_{k=1}^{n}y_k\in Y_k)\,(\tau=f(y_1,\ldots,y_n))\}.$$

The functions p and f are usually continuous, and therefore the images are in turn real closed intervals. That is, there are intervals of certainty $\left[\underline{\rho},\overline{\rho}\right]$ and $[\underline{\tau},\overline{\tau}]$ within which the desirable values of the properties ρ and τ are guaranteed to lie respectively.

It is thus natural to think of extending the ordinary arithmetic on *real-valued* quantities to *interval-valued* quantities in such a way that we can do arithmetic on the intervals X_i and Y_k to get, respectively, the intervals $\left[\underline{\rho},\overline{\rho}\right]$ and $[\underline{\tau},\overline{\tau}]$ as results. So, the uncertainty problem of inquiring about real-world systems is now the problem of *interval enclosures of images of real-valued functions*. Now the question is that to what extent interval enclosures can be useful. Toward answering this, we conclude the present section with a peculiarity of interval arithmetic that seems quite strange at first.

Note that the set-theoretic characterization of the interval operations (definition 3.1) implies that interval arithmetic considers *all instances* of variables as *independent*. Accordingly, for two interval variables X and Y assigned the same interval constant A, both the interval operations $X \circ_{\mathcal{I}} X$ and $X \circ_{\mathcal{I}} Y$ are equal and they are the same as the image of the *multivariate* real function $f_{\text{ind}}(x,y) = x \circ_{\mathbb{R}} y$, with $x \in A$ and $y \in A$. In fact, this is one of the strengths of interval mathematics: since images of real functions are inclusion monotonic (see, e.g., [16], [28], and [62]), it follows that the image of the function f_{ind} is an enclosure of the image of a unary real function $f_{\text{dep}}(x) = x \circ_{\mathbb{R}} x$, with $x \in A$, and therefore $X \circ_{\mathcal{I}} X = X \circ_{\mathcal{I}} Y$ is a *guaranteed enclosure* of the image of f_{dep}. However, in many situations, this enclosure might be too wide to be useful. This phenomenon is known as the *interval dependency problem*. The notion of interval dependency and the problems thereof will be logically characterized in the succeeding sections.

5 From Quantifiers and Skolemizations to Functional Dependence

In order to be able to formalize the notion of interval dependency using elementary logical concepts, it is imperative to justify a symbolic notation suitable for the precise expression of ideas and for carrying out proofs in a merely logical manner. An early locus where the indispensability of a symbolic apparatus is emphasized is Frege's "*Begriffsschrift*" [33]. In the spirit of the quote from Wittgenstein given in the beginning of this article, Frege opened his "*Begriffsschrift*" by saying:

> "The most reliable way of carrying out a proof, obviously, is to follow pure logic, a way that, disregarding the particular characteristics of objects, depends solely on those laws upon which all knowledge rests."

Our goal here is thus to fix a symbolic apparatus suitable and adequate for the purpose at hand. Plainly, we mean to reduce the notion of *functional dependence* to the pure logical concepts of *Skolemization* and *quantification dependence*.

Before getting down to particulars, it is apt to commence this section with historic and epistemic generalities at which level the concept of dependence is framed. The notion of *dependency* is one of the most fundamental ingredients that underlie mathematical reasoning and scientific reasoning in general. The notion of *dependency* comes from the notion of a *function*. Not surprisingly, therefore, there is scarcely a mathematical theory which does not involve the notion of a function (See, e.g., [16] and [28]). In ancient mathematics the idea of *functional dependence* was not expressed explicitly and was not an independent object of research, although a wide range of specific functional relations were known and were studied systematically. The concept of a function appears in a rudimentary form in the works of scholars in the Middle Ages, but only in the work of mathematicians in the 17th century, and primarily in those of Pierre de Fermat, Rene Descartes, Isaac Newton, and Gottfried Leibniz, did it begin to take shape as an independent concept. Later, in the 18th century, Euler had a more general approach to the concept of a function as "*dependence of one variable quantity on another*" [31]. By the year 1834, Lobachevskii was writing: "The general concept of a function requires that a function of x is a number which is given for each x and gradually changes with x. The value of a function can be given either by an analytic expression or by a condition which gives a means of testing all numbers and choosing one of them; or finally a *dependence* can exist and *remain unknown*" [49].

On the epistemic side, when scientists observe the world to formulate the defining properties of some physical phenomenon, these defining properties figure as *attributes* (*variables*) depending on some other attributes. Translating this dependence into a formal mathematical language, gives rise to the notion of functional dependence: "*a variable y is absolutely determined by some given variables $x_1,...,x_n$*", or "*a variable y is a function of some given variables $x_1,...,x_n$*", symbolically $y = f(x_1,...,x_n)$. In some cases, such a translation can deterministically result in a certain rule for the function f, for instance $y = x_1 + ... + x_n$. In other cases, we have an approximate rule for f, or we know that a dependence exist but the rule cannot be determined, in which case we write the general usual notation $y = f(x_1,...,x_n)$, without specifying explicitly a rule for the function f. So, in mathematics, *a dependence is formally a function* (For further exhaustive details about the notion of dependence, from the logical and epistemological viewpoints, see, e.g., [4], [39], [70], [15], [22], and [28]).

In the theory of real closed intervals, the notion of interval dependency naturally comes from the idea of functional dependence of real variables. Despite the fact that dependency is an essential and useful notion of real variables, interval dependency is the *main unsolved problem* of the classical theory of interval arithmetic and its modern generalizations (see [17], and [28]). So, a first step in our way to solve (or to cope with) the dependency problem is to answer the question: *Formally, what is interval dependency*? or, in other words, what is the fundamental defining properties that characterize the notion of interval dependency as a formal mathematical object? Providing an answer for this question is the main business of the succeeding section. For now, we will delve deep into some semantical and syntactical fundamentals concerning the logical formulation of the notion of functional dependence and some related notions.

Two pure logical notions we shall need are those of a quantification matrix and a prenex sentence. A *quantification matrix* $\mathcal{Q}$ is a sequence $(Q_1x_1)\dots(Q_nx_n)$, where $x_1,\dots,x_n$ are variable symbols and each Q_i is $\forall$ or $\exists$. A *prenex sentence* is a sentence of the form $\mathcal{Q}\varphi$, where $\mathcal{Q}$ is a quantification matrix and φ is a quantifier-free formula.

As is well known, the most fundamental part of all mathematical sciences is formal logic. So, getting down to the most elementary fundamentals, it can be clarified that in all mathematical theories, any type of dependence can be reduced to the following simple logical definition (see, e.g., [16] and [28]).

Definition 5.1 (Quantification Dependence). *Let $\mathcal{Q}$ be a quantification matrix and let $\varphi(x_1,\dots,x_m;y_1,\dots,y_n)$ be a quantifier-free formula. For any universal quantification $(\forall x_i)$ and any existential quantification $(\exists y_j)$ in $\mathcal{Q}$, the variable y_j is dependent on the variable x_i in the prenex sentence $\mathcal{Q}\varphi$ iff $(\exists y_j)$ is in the scope of $(\forall x_i)$ in $\mathcal{Q}$. Otherwise x_i and y_j are independent.*

That is, the order of quantifiers in a quantification matrix determines the mutual dependence between the variables in a sentence.

Let us illustrate this by the following two examples.

Example 5.1. *Consider the prenex sentence*

$$(\exists x)(\forall y)(\exists z)(y = x\circ y \wedge x = z\circ y),$$

which asserts that there exists an identity element x, for the operation $\circ$, with respect to which every element possesses an inverse z.

According to the order in which quantifiers are written, the variable z depends only on y, while there is no dependency between x and y or between x and z.

Example 5.2. *In the prenex sentence*

$$(\forall x)(\exists y)(\forall z)(\exists u)\,\varphi(x,y,z,u),$$

the variable y depends on x, and the variable u depends on both x and z.

By means of a *Skolem equivalent form* or a *Skolemization*[4], a quantification dependence is translated into a functional dependence. The notion of a Skolem equivalent form is characterized in the following definition (see, e.g., [28], [32], and [70]).

Definition 5.2 (Skolem Equivalent Form). *Let σ be a sentence that takes the prenex form*

$$\left(\forall_{i=1}^{m}x_i\right)\left(\exists_{j=1}^{n}y_j\right)\varphi(x_1,\dots,x_m;y_1,\dots,y_n).$$

where φ is a quantifier-free formula.

The Skolem equivalent form of σ is defined to be

$$\left(\exists_{j=1}^{n}f_j\right)\left(\forall_{i=1}^{m}x_i\right)\varphi(x_1,\dots,x_m;f_1,\dots,f_n),$$

where $f_j(x_1,\dots,x_m)=y_j$ are the dependency functions of y_j upon $x_1,\dots,x_m$, for $i\in\{1,\dots,m\}$ and $j\in\{1,\dots,n\}$.

It comes therefore as no surprise that in all mathematics, any instance of a dependence is, in fact, a functional dependence.

In order to clarify the matters, let us consider the following example.

Example 5.3 (Skolemization of a Sentence). *Let a sentence σ take the prenex form*

$$(\forall x)(\exists y)(\forall z)(\exists u)\,\varphi(x,y,z,u).$$

The Skolem equivalent form of σ is

$$(\exists f)(\exists g)(\forall x)(\forall z)\,\varphi(x,f(x),z,g(x,z)).$$

[4] *Skolemization* is named after the Norwegian logician Thoralf Skolem (1887–1963), who first presented the notion in [65].

6 What Interval Dependency Formally is: Putting on a Systematic Basis All Together

Now we are proposing to study interval dependency, and indeed by the formal apparatus we fixed in the preceding sections. With the logical and set-theoretical fundamentals of sections 2 and 5 at our disposal, this section, is devoted to presenting a formalized treatment of the notion of interval dependency, that is, putting on a *systematic basis* its meaning, and thus gaining the advantage of indicating formally the criteria by which it is to be characterized and, accordingly, deducing its fundamental properties in a merely logical manner. On the strength of the generality of the logical apparatus we adopt, the results of this section are not only about Moore's classical intervals, but they are meant to apply also to *any possible* theory of interval arithmetic.

Before we proceed, it is convenient here to introduce some notational conventions. By a *finitary* real-valued function in real arguments (in short, a *real function* or $\mathbb{R}$-function), we understand a function $f_{\mathbb{R}} : \mathcal{D}_{\mathbb{R}} \subseteq \mathbb{R}^n \mapsto \mathbb{R}$, and by an *interval function* (or $\mathcal{I}$-function) we understand a function $f_{\mathcal{I}} : \mathcal{D}_{\mathcal{I}} \subseteq \mathcal{I}^n \mapsto \mathcal{I}$. The $\mathbb{R}$-subscripted letters $f_{\mathbb{R}}$, $g_{\mathbb{R}}$, $h_{\mathbb{R}}$ shall be employed to denote real-valued functions, while the $\mathcal{I}$-subscripted letters $f_{\mathcal{I}}$, $g_{\mathcal{I}}$, $h_{\mathcal{I}}$ shall be employed to denote interval-valued functions. If the type of function is clear from its arguments, and if confusion is not likely to ensue, we shall usually drop the subscripts "$\mathbb{R}$" and "$\mathcal{I}$". Thus, we may, for instance, write $f(x_1,...,x_n)$ and $f(X_1,...,X_n)$ for, respectively, a real-valued function and an interval-valued function, which are both defined by the same rule.

An important notion we shall need is that of the image set of real closed intervals, under an n-ary real-valued function. This notion is a special case of that of the corresponding $(n+1)$-ary relation on $\mathbb{R}$. More precisely, we have the following definition.

Definition 6.1 (Image of Real Closed Intervals). *Let f be an n-ary function on $\mathbb{R}$, and for $(\mathbf{v},y) \in f$, let $\mathbf{v} = (x_1,...,x_n)$, with each x_k is restricted to vary on a real closed interval $X_k \subset \mathbb{R}$, that is, $\mathbf{v}$ is restricted to vary on a set $\mathbf{V} \subset \mathbb{R}^n$. Then, the image of the closed intervals X_k with respect to f, denoted I_f, is defined to be*

$$\begin{aligned} Y = \mathrm{I}_f(\mathbf{V}) &= \mathrm{I}_f(X_1,...,X_n) \\ &= \{y \in \mathbb{R} \mid (\exists \mathbf{v} \in \mathbf{V})(\mathbf{v} f y)\} \\ &= \{y \in \mathbb{R} \mid (\exists_{k=1}^n x_k \in X_k)(y = f(x_1,...,x_n))\} \subseteq \mathbb{R}, \end{aligned}$$

where the set $\mathbf{V}$, called the preimage[5] of Y, is defined to be the image of Y with respect to the converse relation $\widehat{f}$, that is

$$\mathbf{V} = \mathrm{I}_{\widehat{f}}(Y) = \{\mathbf{v} \in \mathbb{R}^n \mid (\exists y \in Y)\left(y \widehat{f} \mathbf{v}\right)\}.$$

The fact formulated in the following theorem is well-known (see, e.g., [16] and [28]).

Theorem 6.1 (Extreme Value Theorem). *Let X_k be real closed intervals and let $f(x_1,...,x_n)$ be an n-ary real-valued function with $x_k \in X_k$. If f is continuous in X_k, in symbols $\mathrm{Cont}(f,X_k)$, then f must attain its minimum and maximum value, that is*

$$\begin{aligned} (\forall f)(\mathrm{Cont}(f,X_k) \Rightarrow (\exists_{k=1}^n a_k \in X_k)(\exists_{k=1}^n b_k \in X_k)(\forall_{k=1}^n x_k \in X_k) \\ (f(a_1,...,a_n) \leq f(x_1,...,x_n) \leq f(b_1,...,b_n))), \end{aligned}$$

where $\min\limits_{x_k \in X_k} f = f(a_1,...,a_n)$ and $\max\limits_{x_k \in X_k} f = f(b_1,...,b_n)$ are respectively the minimum and maximum of f.

An immediate consequence of definition 6.1 and theorem 6.1, is the following important property [28].

Theorem 6.2 (Main Theorem of Image Evaluation). *Let an n-ary real-valued function f be continuous in the real closed intervals X_k. The (accurate) image $\mathrm{I}_f(X_1,...,X_n)$, of X_k, is in turn a real closed interval such that*

$$\mathrm{I}_f(X_1,...,X_n) = \left[\min_{x_k \in X_k} f(x_1,...,x_n), \max_{x_k \in X_k} f(x_1,...,x_n)\right].$$

A cornerstone result from the above theorem, that should be stressed at once, is that the *best* way to evaluate the *accurate* image of a continuous real-valued function is to apply minimization and maximization directly to determine the exact lower and upper endpoints of the image. For rational[6] real-valued functions, this optimization problem is,

[5] From the fact that the converse relation $\widehat{f}$ is always definable, the preimage of a function f is always definable, regardless of the definability of the inverse function f^{-1}.

[6] A *rational* real-valued function is a function obtained by means of a finite number of the basic real algebraic operations $\circ_{\mathbb{R}} \in \{+,\times\}$ and $\diamond_{\mathbb{R}} \in \{-,^{-1}\}$.

in general, computationally solvable, by applying *Tarski's algorithm*, which is also known as *Tarski's real quantifier elimination* (see, e.g., [10] and [68]). For algebraic [7] real-valued functions, the problem is computable, by applying the *cylindrical algebraic decomposition algorithm* (*CAD algorithm*, or *Collins' algorithm*) [8], which is a more effective version of Tarski's algorithm (see, e.g., [13] and [52]).

Before turning to the notion of interval dependency, we first prove the following indispensable result.

Theorem 6.3 (Image Inclusions in Prenex Sentences). *Let σ_1 and σ_2 be the two prenex sentences such that*

$$\sigma_1 \Leftrightarrow (\forall_{i=1}^{m} x_i \in X_i)\left(\exists_{j=1}^{n} y_j \in Y_j\right)(\exists z)\left(z = f(x_1,...,x_m;y_1,...,y_n)\right),$$
$$\sigma_2 \Leftrightarrow (\forall_{i=1}^{m} x_i \in X_i)\left(\forall_{j=1}^{n} y_j \in Y_j\right)(\exists z)\left(z = f(x_1,...,x_m;y_1,...,y_n)\right),$$

where X_i and Y_j are real closed intervals, and f is a continuous real-valued function with $x_i \in X_i$ and $y_j \in Y_j$.

If $\mathrm{I}_f^{\sigma_1}$ and $\mathrm{I}_f^{\sigma_2}$ are the images of f, respectively, in σ_1 and σ_2, then $\mathrm{I}_f^{\sigma_1} \subseteq \mathrm{I}_f^{\sigma_2}$.

Proof. According to definition 5.1, in the sentence σ_1, all y_j are dependent upon all x_i, and in the sentence σ_2, all x_i and y_j are pairwise independent.

By definition 5.2, there are some functions $g_j(x_1,...,x_m)$ such that σ_1 has the Skolem equivalent form

$$\left(\exists_{j=1}^{n} g_j\right)(\forall_{i=1}^{m} x_i \in X_i)(\exists z)\left(z = f(x_1,...,x_m;g_1,...,g_n)\right).$$

Finally, employing theorem 6.2, we therefore have $\mathrm{I}_f^{\sigma_1} \subseteq \mathrm{I}_f^{\sigma_2}$. □

From the fact that existential quantification over a nonempty set $\mathcal{S}$ defines a set $\mathcal{T}$ such that $\mathcal{T} \subseteq \mathcal{S}$, the previous theorem entails, as a special case, the following important result of "real analysis".

Corollary 6.1 (Inclusion Monotonicity of Real Images). *Let $\mathbf{V}_X = (X_1,...,X_i,...,X_n)$ and $\mathbf{V}_Y = (Y_1,...,Y_i,...,Y_n)$ be two preimages of a continuous real-valued function f. Then, the image I_f is inclusion monotonic. That is*

$$(\forall_{i=1}^{n} X_i, Y_i)\left(X_i \subseteq Y_i \Rightarrow \mathrm{I}_f(\mathbf{V}_X) \subseteq \mathrm{I}_f(\mathbf{V}_Y)\right).$$

The following example makes the statement of theorem 6.3 clear.

Example 6.1 (Image Inclusion in Two Prenex Sentences). *Let σ_1 and σ_2 be the two prenex sentences such that*

$$\sigma_1 \Leftrightarrow (\forall x \in [1,2])(\exists y \in [1,2])(\exists z \in \mathbb{R})(z = f(x,y) = y - x),$$
$$\sigma_2 \Leftrightarrow (\forall x \in [1,2])(\forall y \in [1,2])(\exists z \in \mathbb{R})(z = f(x,y) = y - x).$$

In the sentence σ_1, the variable y depends on x, and therefore there is some function $g(x)$ such that σ_1 has the Skolem equivalent form

$$(\exists g)(\forall x \in [1,2])(\exists z \in \mathbb{R})(z = f(x,g(x)) = g(x) - x).$$

Let g be the identity function. Consequently, the image of f in σ_1 is $\mathrm{I}_f^{\sigma_1} = \{0\}$.

Obviously, the image of f in σ_2 is $\mathrm{I}_f^{\sigma_2} = [-1,1]$, and therefore $\mathrm{I}_f^{\sigma_1} \subseteq \mathrm{I}_f^{\sigma_2}$.

Next we define the notion of an *exact* (or *generalized*) interval operation.

Definition 6.2 (Exact Interval Operation). *Let $\circ_{\mathbb{R}} \in \{+,\times\}$ be a binary real operation, and let $\mathrm{I}_f = \mathrm{I}_f^{\sigma_{\mathrm{Dep}}} \vee \mathrm{I}_f = \mathrm{I}_f^{\sigma_{\mathrm{Ind}}}$, where $\mathrm{I}_f^{\sigma_{\mathrm{Dep}}}$ and $\mathrm{I}_f^{\sigma_{\mathrm{Ind}}}$ are the images of a function f for two real closed intervals X and Y in, respectively, two prenex sentences σ_{Dep} and σ_{Ind} such that*

$$\sigma_{\mathrm{Dep}} \Leftrightarrow (\forall x \in X)(\exists y \in Y)(\exists z \in \mathbb{R})(z = f(x,y) = x \circ_{\mathbb{R}} y),$$
$$\sigma_{\mathrm{Ind}} \Leftrightarrow (\forall x \in X)(\forall y \in Y)(\exists z \in \mathbb{R})(z = f(x,y) = x \circ_{\mathbb{R}} y).$$

Then, an exact interval operation $\circ_{\mathcal{J}} \in \{+,\times\}$ is defined by

$$X \circ_{\mathcal{J}} Y = \mathrm{I}_f(X,Y).$$

We have then the following obvious result for the classical interval operations.

[7] An *algebraic* function is a function that satisfies a polynomial equation whose coefficients are polynomials with rational coefficients.

[8] The *CAD algorithm* is efficient enough for being one of the most important optimization algorithms of computational real algebraic geometry (see, e.g., [6]).

Theorem 6.4 (Inexactness of Classical Interval Operations). *The value of a classical interval operation $X \circ_{\mathcal{I}} Y$ is exact only when the real variables $x \in X$ and $y \in Y$ are independent, that is*

$$X \circ_{\mathcal{I}} Y = \mathrm{I}_f^{\sigma_{\mathrm{Ind}}}(X,Y).$$

Proof. The theorem is immediate from definition 3.1 of the classical interval algebra. □

With the help of the preceding notions and the deductions from them, we are now ready to pass to our formal characterization of the notion of interval dependency.

Definition 6.3 (Interval Dependency Relation). *Let $S_1,...,S_m$ be some arbitrary real closed intervals. For two interval variables X and Y, we say that Y is dependent on X, in symbols $Y\mathfrak{D}X$, iff there is some given real-valued function f such that Y is the image of $(X;S_1,...,S_m)$ with respect to f. That is*

$$Y\mathfrak{D}X \Leftrightarrow Y = \mathrm{I}_f(X;S_1,...,S_m),$$

where f is called the dependency function of Y on X. Otherwise Y is not dependent on X, in symbols $Y\mathfrak{I}X$, that is

$$Y\mathfrak{I}X \Leftrightarrow \neg Y\mathfrak{D}X \Leftrightarrow \neg Y = \mathrm{I}_f(X;S_1,...,S_m).$$

From now on, and throughout the text, the following notational convention shall be adopted. We write $Y\mathfrak{D}_f X$ (with the subscript f) to mean that Y is dependent on X by some given dependency function f, and we write $\mathfrak{I}(X,Y)$ to mean that X and Y are mutually independent. In general, the notation $\mathfrak{I}(X_1,...,X_n)$ shall be employed to mean "all $X_1,...,X_n$ are pairwise mutually independent". Hereafter, for simplicity of the language, we shall always make use of the following abbreviation.

$$\mathfrak{I}_{k=1}^{n}(X_k) \Leftrightarrow \mathfrak{I}(X_1,...,X_n).$$

So, to say that an interval variable Y is dependent on an interval variable X, we must be *given* some real-valued function f such that Y is the image of X under f. This characterization of interval dependency is completely compatible with the concept of functional dependence of real variables: for two real variables x and y, the variable y is *functionally* dependent on x if there is some *given* function f such that $y = f(x)$, and to keep the dependency information, between x and y, in an algebraic expression $x \circ_{\mathbb{R}} y$, it suffices to write $x \circ_{\mathbb{R}} f(x)$. If x and y are mutually dependent by an *idempotence* $y = f(x)$ and $x = g(y)$, then, to keep the dependency information, it suffices to write either $x \circ_{\mathbb{R}} f(x)$ or $g(y) \circ_{\mathbb{R}} y$. In case there is neither such a given function f nor such a given function g, then it is obvious that the real variables x and y are *not* functionally dependent. Definition 6.3 extends this concept to the set of real closed intervals.

The preceding definition, along with two deductions that we shall presently make (theorem 6.5 and corollary 6.2), touches the notion of interval dependency in a way which copes with all possible cases. Noteworthy also is that the dependency relation characterized in definition 6.3 is a *meta-concept*, not an *object-ingredient* of the numerous interval (and fuzzy) theories heretofore presented in the literature. In other words, on the strength of the generality of the logical apparatus we adopt, the results of our formalization of interval dependency are not only about Moore's classical intervals, but they are meant to apply also to any possible theory of interval arithmetic.

To illustrate, let us give the following example.

Example 6.2 (Dependency Relation for Two Variables). *Let X and Y be two interval variables that both are assigned the same individual constant $[0,1]$. Then, we may have one of the following cases.*

(i) *Y is not dependent on X (there is no given dependency function).*

(ii) *Y is dependent on X, by the identity function $y = f(x) = x$.*

(iii) *Y is dependent on X, by the square function $y = f(x) = x^2$.*

This example shows that if two interval *variables* X and Y both are assigned the same *individual constant* (both have the same *value*), it does not necessarily follow that X and Y are *identical*, unless they are dependent by the identity function[9]. Thereupon, for an interval theory to be *dependency-aware*, it must incorporate in its symbolism the dependency relation as an *object-ingredient* in such a way that two intervals are equal iff they are "*one and the same*".

[9]The notions of *identity* and *equality* are commonly confused and treated as synonyms. However, they are two *distinct* logical concepts. Despite the fact that equality implies identity in the theory of real numbers, this is not always the case. Two line halves are *equal* but not *identical* (one and the same). Every line *equals* infinitely many other lines, but no line *is* (identical to) any other line (see [14] and [69]). Identity, which is the most fundamental ingredient of any mathematical theory, is characterized by *Leibniz's principle of the identity of indiscernibles* which states that two entities x and y are identical iff any property of x is also a property of y and vice versa.

This fact itself calls for a new characterization of the *equality relation for interval numbers*: Two interval variables X and Y are equal (identical) iff they are *dependent by identity*, that is

$$X = Y \Leftrightarrow X\mathfrak{D}_{\mathrm{Id}}Y.$$

In consequence of this definition, in a possible *dependency-aware* interval theory, for any two interval variables $[\underline{x},\overline{x}]$ and $[\underline{y},\overline{y}]$, it should be the case that

$$[\underline{x},\overline{x}] = [\underline{y},\overline{y}] \Leftrightarrow \underline{x} = \underline{y} \wedge \overline{x} = \overline{y} \wedge (\forall x \in [\underline{x},\overline{x}])\left(\exists y \in [\underline{y},\overline{y}]\right)(y = \mathrm{Id}(x)).$$

In any case, the formalization proposed here provides a tool for investigating the dependency problem rigorously. As a consequence of our characterization of interval dependency, we have the next immediate theorem that establishes that the interval dependency relation is a *quasi-ordering* relation.

Theorem 6.5 (Quasi-Orderness of the Dependency Relation). *The interval dependency relation is a quasi-ordering relation on the set of real closed intervals. That is, for any three interval variables X, Y, and Z, the following statements are true:*

(i) *$\mathfrak{D}$ is reflexive, in symbols $(X\mathfrak{D}X)$,*

(ii) *$\mathfrak{D}$ is transitive, in symbols $(X\mathfrak{D}Y \wedge Y\mathfrak{D}Z \Rightarrow X\mathfrak{D}Z)$.*

In accordance with this theorem and definition 6.2, we also have the following corollary.

Corollary 6.2 (Dependency Relation Properties). *For any interval operation $\circ_{\mathcal{J}}$, and for any three interval variables X, Y, and Z, the following two assertions are true:*

(i) $(X \circ_{\mathcal{J}} Y)\mathfrak{D}X$,

(ii) $(X \circ_{\mathcal{J}} Y)\mathfrak{D}Y$.

The interval dependency problem can now be formulated in the following theorem.

Theorem 6.6 (Dependency Problem). *Let X_k be real closed intervals and let $f(x_1,\ldots,x_n)$ be a continuous real-valued function with $x_k \in X_k$. Evaluating the accurate image of f for the interval numbers X_k, using classical interval arithmetic, is not always possible if there exist X_i and X_j such that $X_j\mathfrak{D}X_i$ for $i \neq j$. That is,*

(i) $(\exists f)\left(\mathrm{I}_f(X_1,\ldots,X_n) \neq f(X_1,\ldots,X_n)\right)$.

In general,

(ii) $(\forall f)\left(\mathrm{I}_f(X_1,\ldots,X_n) \subseteq f(X_1,\ldots,X_n)\right)$.

Proof. For (i), it suffices to give a counterexample.

For two interval variables X_1 and X_2 that both are assigned the same individual constant $[-a,a]$, let f be a function defined by the rule $f(x_1,x_2) = x_1x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$. If $X_2\mathfrak{D}_gX_1$, with g is the identity function $x_2 = g(x_1) = x_1$, then f has the equivalent rule $f(x) = x^2$, with $x \in [-a,a]$.

According to theorem 6.2, the (accurate) image of $[-a,a]$ under the real-valued function f is

$$\mathrm{I}_f([-a,a]) = \left[\min_{x\in[-a,a]} x^2, \max_{x\in[-a,a]} x^2\right] = [0,a^2].$$

If we evaluate the image of $[-a,a]$ using classical interval arithmetic, by theorem 3.1, we obtain the interval-valued function,

$$\begin{aligned} f([-a,a]) &= [-a,a] \times [-a,a] \\ &= [-a^2,a^2], \end{aligned}$$

which is not the actual image of $[-a,a]$ under f, that is, there is some function f, for which

$$\mathrm{I}_f(X_1,\ldots,X_n) \neq f(X_1,\ldots,X_n),$$

and therefore evaluating the accurate image of real-valued functions is not always possible, using classical interval arithmetic.

Toward proving (ii), let

$$\mathrm{I}_f(X_1,...,X_n) = \mathrm{I}_f^{\sigma_1}(X_1,...,X_n) \vee \mathrm{I}_f(X_1,...,X_n) = \mathrm{I}_f^{\sigma_2}(X_1,...,X_n),$$

where $\mathrm{I}_f^{\sigma_1}$ and $\mathrm{I}_f^{\sigma_2}$ are the images of f, respectively, in two prenex sentences σ_1 and σ_2 such that in σ_1, there exist X_i and X_j such that $X_j \mathfrak{D} X_i$ for $i \neq j$, and in σ_2, all X_k are pairwise independent, that is $\Im_{k=1}^{n}(X_k)$. Employing theorem 6.3, We accordingly have

$$\mathrm{I}_f^{\sigma_1}(X_1,...,X_n) \subseteq \mathrm{I}_f^{\sigma_2}(X_1,...,X_n).$$

According to definition 3.1, of the classical interval algebra, all interval variables are assumed to be independent. We consequently have

$$f(X_1,...,X_n) = \mathrm{I}_f^{\sigma_2}(X_1,...,X_n).$$

Thus

$$(\forall f)\left(\mathrm{I}_f(X_1,...,X_n) \subseteq f(X_1,...,X_n)\right),$$

and therefore (ii) is verified. □

Obviously, the result $\left[-a^2,a^2\right]$, obtained using classical interval arithmetic, has an overestimation of

$$\left|w\left(\left[-a^2,a^2\right]\right) - w\left(\left[0,a^2\right]\right)\right| = a^2,$$

where w is the width of the interval. This overestimated result is due to the fact that the classical interval theory assumes independence of all interval variables, even when dependencies exist.

A numerical example is shown below.

Example 6.3 (Overestimation due to Dependency). *Consider the real-valued function*

$$f(x) = x(x-1),$$

with $x \in [0,1]$.

The actual image of $[0,1]$ *under* f *is* $[-1/4,0]$. *Evaluating the image using classical interval arithmetic, we get*

$$f([0,1]) = [0,1] \times ([0,1]-1) = [-1,0],$$

which has an overestimation of

$$|w([-1,0]) - w([-1/4,0])| = 3/4.$$

Finally, in consequence of theorem 6.6, we are led to the following immediate result.

Theorem 6.7 (Occurrences of Variables and Dependency). *Let* $f(x)$ *and* $g(x)$ *be two real functions of different rules such that* $f(x) = g(x)$, *and let* $f(X)$ *and* $g(X)$ *be the interval extensions of* $f(x)$ *and* $g(x)$ *respectively. Define* $\mathsf{O}_f(X)$ *and* $\mathsf{O}_g(X)$ *to be the numbers of occurrences of the variable* X *in* f *and* g *respectively. Then*

$$\mathsf{O}_f(X) \leq \mathsf{O}_g(X) \Rightarrow f(X) \subseteq g(X).$$

Due to interval dependency, classical interval arithmetic has a number of peculiar properties. For example, we have *subdistributive inverses* (or S-*inverses*) with respect to interval addition and multiplication. Precisely

$$(\forall X)(0_{\mathcal{I}} \subseteq X - X),$$
$$(\forall X)(0_{\mathcal{I}} \not\subseteq X \Rightarrow 1_{\mathcal{I}} \subseteq X/X).$$

In general, for two interval numbers X and Y, if $X \cap Y \neq \varnothing$, then

$$(0_{\mathcal{I}} \subseteq X - Y) \wedge (0_{\mathcal{I}} \subseteq Y - X),$$
$$(0_{\mathcal{I}} \not\subseteq Y) \Rightarrow (1_{\mathcal{I}} \subseteq X/Y).$$
$$(0_{\mathcal{I}} \not\subseteq X) \Rightarrow (1_{\mathcal{I}} \subseteq Y/X).$$

The problem of computing the image $\mathrm{I}_f(X_1,...,X_n)$, using interval arithmetic, is the main problem of interval computations. This problem is, in general, NP-hard[10] (see, e.g., [35], [46], and [62]). That is, for the classical interval theory, there is no efficient algorithm to make the identity

$$(\forall f)\left(\mathrm{I}_f(X_1,...,X_n) = f(X_1,...,X_n)\right),$$

[10] In principle, this result is not necessarily applicable to other theories of interval arithmetic (present or future) because each theory has its peculiar set of algorithms, where each algorithm is a sequence of elementary relations and functions of the foundational level of the theory.

always hold unless NP = P, which is widely believed to be false. However, a considerable scientific effort is put into finding a way out from the interval dependency problem. There are many special methods and algorithms, based on the classical interval theory, that successfully compute useful narrow bounds to the desirable accurate image. In the succeeding section, we shall give a bit of perspective on how to compute useful guaranteed interval enclosures under functional dependence.

Beyond the techniques based on the classical interval theory, various proposals for possible alternate theories of interval arithmetic were introduced to reduce the dependency effect or to enrich the algebraic structure of interval numbers. Among these alternate theories of intervals, we can mention as examples: *Hansen's generalized intervals*, *Kulisch's complete intervals*, *directed intervals*, *modal intervals*, *parametric intervals*, *universal intervals*, and others (see, e.g., [37], [48], [51], [36], [50], [16], [27], and [28]).

7 Guaranteed Enclosures Under Interval Dependency

In this section, we shall discuss briefly how to compute useful guaranteed enclosures of real-valued functions under functional dependence. The numerical examples of this section are computed using version 2.0 of InCLosure. The InCLosure commands to compute the results of the examples are described and an InCLosure input file and its corresponding output containing, respectively, the code and results of the examples are also available as a supplementary material to this text (see Supplementary Materials).

As mentioned in the preceding section, there are many special methods and algorithms, based on the classical interval theory, that provide a way out of the dependency problem to get narrower enclosures. With a knowledge of regions of monotonicity, most elementary interval functions can be defined to be the exact images of the corresponding real functions. As instances, for an interval number $X=[\underline{x},\overline{x}]$ and a nonnegative integer n, we can define

$$e^X=\left[e^{\underline{x}},e^{\overline{x}}\right],\quad \ln(X)=\left[\ln(\underline{x}),\ln(\overline{x})\right] \text{ if } \underline{x}>0;$$

$$\sqrt{X}=\left[\sqrt{\underline{x}},\sqrt{\overline{x}}\right] \text{ if } \underline{x}\geq 0,\quad \sin(X)=\left[\min_{x\in X}(\sin(x)),\max_{x\in X}(\sin(x))\right];$$

$$X^n=\begin{cases}[\underline{x}^n,\overline{x}^n] & \text{iff } \underline{x}>0 \text{ or } n \text{ is odd},\\ [\overline{x}^n,\underline{x}^n] & \text{iff } \overline{x}<0 \text{ and } n \text{ is even},\\ [0,|X|^n] & \text{iff } 0\in X \text{ and } n \text{ is even};\end{cases}$$

where $|X|=\max\{|\underline{x}|,|\overline{x}|\}$ is the absolute value of the interval number X.

Applying the *naive* (*pure*) interval operations on these exact evaluations, we can get better (narrower) enclosures of the images of the algebraic combinations of the corresponding real functions. Moreover, many techniques are used to improve the results obtained from the naive method by reducing the width of the resulting interval. Among these techniques *centered forms*, *generalized centered forms*, *circular complex centered forms*, *Hansen's method*, *remainder forms*, the *subdivision method*, and many others (see, e.g., [17], [28], [57], [58], [62], [48], [1], and [2]). For instance, we can get *arbitrarily* narrower enclosures of images of real functions using the subdivision method, which is due to Moore (see, e.g., [57] and [58]). This method can be described as follows. Let $X=[\underline{x},\overline{x}]$ be an interval number and let $w(X)=\overline{x}-\underline{x}$ be the width of X. First, we subdivide the interval X into n subintervals X_i such that

$$X_i=[\underline{x}+(i-1)w(X)/n,\underline{x}+(i)w(X)/n],$$

where $w(X_i)=w(X)/n$. Hence $X=\cup_{i=1}^n X_i$. Then, we evaluate the interval-valued function f for each subinterval X_i, $f(X_i)$. Accordingly, we have [28]

$$\mathrm{I}_f(X)\subseteq\cup_{i=1}^n f(X_i)\subseteq f(X).$$

That is, the subdivision method produces better enclosures than the naive method. Moreover, the larger the number of subintervals n, the narrower the enclosure of the image $\mathrm{I}_f(X)$. Next, we shall describe how to make use of the interval capability of InCLosure to get *arbitrarily* better enclosures of the images of real functions.

Example 7.1 (Interval Subdivision in InCLosure). *Consider the interval function*

$$f(X)=X^9-X^6-X^3-369.$$

InCLosure provides arbitrarily narrower interval results which are limited only by the computational power of the host machine. We can compute the value of the above interval function at the interval $[-3,6]$, $f([-3,6])$, *using the following InCLosure command.*

```
EvalInt "X^9-X^6-X^3-369" "X=[-3,6]" 1 30
```

This will result in

$$[-66924.0, 1.0077354E7].$$

*The last parameter, "*30*", in the previous command, is the precision of the result (default is* 20*). The parameter "*1*" indicates the number of subintervals if the subdivision method is applied (the number "*1*" means no subdivision of the interval). To get narrower intervals, the number of subintervals can be increased arbitrarily. Table 1 shows* InCLosure *results obtained for no subdivision, subdivision with* 10*,* 50*,* 100*,* 500*, and* 1000 *subintervals of* $[-3, 6]$.

Number of subintervals	`InCLosure` result $f([-3,6])$
1 (no subdivision)	$[-66924.0, 1.0077354E7]$
10	$[-20771.739, 1.0059598061199E7]$
50	$[-20758.574232, 1.0038266720770032576E7]$
100	$[-20756.357829, 1.0034509264991644959E7]$
500	$[-20754.483089832, 1.0031290472751452950435776E7]$
1000	$[-20754.242271729, 1.0030874303048468185232559E7]$

Table 1. InCLosure Result for Different Numbers of Subdivisions

The exact image of the corresponding real function $f(x) = x^9 - x^6 - x^3 - 369$ on $[-3, 6]$ is 'approximately' $[-20754, 10030455]$ according to Wolfram Mathematica [72]. All the results of table 1 are *guaranteed enclosures* of the exact image, and example 7.1 clearly shows that classical interval arithmetic with the *arbitrarily* narrower interval results of InCLosure so markedly surpasses the ordinary numerical methods.

8 Conclusion

One approach that proved to be subtle, reliable, and most fundamental in all of mathematics of uncertainty is interval mathematics. By integrating the complementary powers of rigorous mathematics and scientific computing, interval arithmetic is able to offer highly reliable accounts of uncertainty. Despite all of advantages of interval mathematics detailed in the introduction and elsewhere, it has its disadvantages as well. A main drawback of interval mathematics is the persisting problem known as the "*interval dependency problem*". This, naturally, confronts us with the crucial question: *Formally, what is interval dependency*? Is it a meta-concept or an object-ingredient of interval and fuzzy mathematics? In other words, what is the fundamental defining properties that characterize the notion of interval dependency as a formal mathematical object? what exactly is the sense of saying that two intervals are *dependent*? and how does the dependency of two intervals $2X$ and X differ from that of X^2 and X? Since the early works on interval mathematics by Burkill and Young in the dawning of the twentieth century, this question has never heretofore been touched upon and remained a question still today unanswered. Although the notion of interval dependency is widely used in the interval and fuzzy literature, it is only illustrated by example, without explicit formalization, and no attempt has been made to put on a systematic basis its meaning, that is, to indicate formally the criteria by which it is to be characterized. This article has been devoted to answering this long-standing question, and, that being so, the problem dealt with in this text is that of the possibility and the scope of a symbolic formalization of interval dependency. We proposed a precise metatheoretic characterization of the notion of interval dependency, deduced its fundamental properties in a merely logical manner, and thereupon we developed a rigorous mathematical theory thereof which formally characterizes and explains the differences between all cases of interval dependency.

We would also remark that, by virtue of our formalization, many nice consequences of real and interval analysis come for free, for example: it has been shown that for an interval theory to be *dependency-aware*, it must incorporate in its symbolism the dependency relation as an *object-ingredient* in such a way that two intervals are equal iff they are *one and the same* and a new definition of the *equality relation for interval numbers* is proposed accordingly, the main theorem of image evaluation and the inclusion monotonicity of real images follow immediately from a generalized theorem about image inclusions in prenex sentences, and a generalized theorem concerning the dependency problem is easily established.

To sum up, what is so important about a logical formalization of interval dependency? In fact, effort in pursuit of this aim has many fruitful consequences. A novelty of this formalization is the expression of interval dependency as

a *logical predicate* (or *relation*) and thus gaining the advantage of deducing its fundamental properties in a merely logical manner. Taking the passage from the informal treatments to the formal technicalities of mathematical logic, this result sheds new light on many fundamental problems of interval mathematics. Moreover, a breakthrough behind our formalization of interval dependency is that it paves the way and provides the systematic apparatus for developing alternate *dependency-aware* interval theories and computational methods with mathematical constructions that better account for dependencies between the quantifiable uncertainties of the real world. Noteworthy also is that on the strength of the generality of the logical apparatus we adopt, the results of this article are not only about Moore's classical intervals, but they are meant to apply also to any *possible* theory of interval arithmetic.

Supplementary Materials

To reproduce the results of the calculations in this article, latest version of InCLosure is available for free download via *https://doi.org/10.5281/zenodo.2702404*. An InCLosure input file and its corresponding output containing, respectively, the code and results of the examples are also available as a supplementary material to this article, via *http://doi.org/10.5281/zenodo.3466032*.

Acknowledgments

The authors would like to thank Prof. Nefertiti Megahed for all the algebraic discussions that greatly contributed to polishing this article.

Competing Interests

The authors have no competing interests.

Funding

The research work has no funding support.

References

[1] G. Alefeld and J. Herzberger. *Introduction to Interval Computation*. Academic Press, New York, first edition, 1983.

[2] G. Alefeld and G. Mayer. *Interval Analysis: Theory and Applications*. Journal of Computational and Applied Mathematics, 121(1):421–464, 2000.

[3] M. A. Amer. *First Order Logic with Empty Structures*. Studia Logica, 48:169–177, 1989. 2.

[4] W. W. Armstrong. *Dependency Structures of Data Base Relationships*. In *Proceedings of IFIP Congress*, pp. 580–583, 1974.

[5] D. W. Barnes and J. M. Mack. *An Algebraic Introduction to Mathematical Logic*. Springer Verlag, first edition, 1975.

[6] S. Basu, R. Pollack, and M.-F. Roy. *Algorithms in Real Algebraic Geometry*. Springer Verlag, first edition, 2003.

[7] R. E. Boche. *An Operational Interval Arithmetic*. In *IEEE National Electronics Conference, University of Illinois*, volume 111, Chicago, August 1963.

[8] J. C. Burkill. *Functions of Intervals*. Proceedings of the London Mathematical Society, 2(1):275–310, 1924.

[9] G. Cantor. *Beitrage zur Begrundung der transfiniten Mengenlehre II*. Mathematische Annalen, 49:207–246, 1897. Translated with introduction and commentary by Philip E. B. Jourdain as "*Contributions to the Founding of the Theory of Transfinite Numbers*", Dover Publications, New York, 1955.

[10] G. Chen and T. T. Pham. *Introduction to Fuzzy Sets, Fuzzy Logic, and Fuzzy Control Systems*. CRC Press, first edition, 2000.

[11] S. Chevillard, M. Joldes, and C. Lauter. *Sollya: An Environment for the Development of Numerical Codes*. In K. Fukuda, J. van der Hoeven, M. Joswig, and N. Takayama, editors, *Mathematical Software - ICMS 2010*, volume 6327 of *Lecture Notes in Computer Science*, pp. 28–31, Springer, Heidelberg, Germany, September 2010.

[12] J. R. Clay. *Nearrings: Geneses and Applications*. Oxford University Press, 1992.

[13] G. E. Collins. *Quantifier Elimination for the Elementary Theory of Real Closed Fields by Cylindrical Algebraic Decomposition*. In *Automata Theory and Formal Languages 2nd GI Conference Kaiserslautern*, volume 33 of *Lecture Notes in Computer Science*, pp. 134–183, Springer Verlag, 1975.

[14] J. Corcoran and A. Ramnauth. *Equality and Identity*. Bulletin of Symbolic Logic, 19(3):255–256, 2013.

[15] H. Dawood. *Theories of Interval Arithmetic: Mathematical Foundations and Applications*. LAP Lambert Academic Publishing, Saarbrücken, 2011, ISBN 978-3-8465-0154-2.

[16] H. Dawood. *Interval Mathematics: Foundations, Algebraic Structures, and Applications*. Master's thesis, Department of Mathematics, Faculty of Science, Cairo University, Giza, 2012, doi: 10.13140/RG.2.2.24252.13449. Doi:10.13140/RG.2.2.24252.13449, URL *http://dx.doi.org/10.13140/RG.2.2.24252.13449*.

[17] H. Dawood. *Interval Mathematics as a Potential Weapon against Uncertainty*. In S. Chakraverty, editor, *Mathematics of Uncertainty Modeling in the Analysis of Engineering and Science Problems*, chapter 1, pp. 1–38, IGI Global, Hershey, PA, January 2014, ISBN 978-1-4666-4991-0, doi: 10.4018/978-1-4666-4991-0.ch001. URL *http://dx.doi.org/10.4018/978-1-4666-4991-0.ch001*.

[18] H. Dawood. *InCLosure (Interval enCLosure)–A Language and Environment for Reliable Scientific Computing*. Computer Software, Version 2.0, Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt, December 2018, doi: 10.5281/zenodo.2757278. InCLosure Support: *http://scholar.cu.edu.eg/henddawood/software/InCLosure*, URL *https://doi.org/10.5281/zenodo.2757278*.

[19] H. Dawood. *On Some Algebraic and Order-Theoretic Aspects of Machine Interval Arithmetic*. Online Mathematics Journal, 1(2):1–13, April 2019, doi: 10.5281/zenodo.2656089. URL *http://doi.org/10.5281/zenodo.2656089*.

[20] H. Dawood and Y. Dawood. *On the Metamathematics of the Theory of Interval Numbers*. Technical Report CU-Math-2010-06-MTIN, Department of Mathematics, Faculty of Science, Cairo University, Giza, June 2010.

[21] H. Dawood and Y. Dawood. *A Dependency-Aware Interval Algebra*. Technical Report CU-Math-2013-09-DAIA, Department of Mathematics, Faculty of Science, Cairo University, September 2013.

[22] H. Dawood and Y. Dawood. *Logical Aspects of Interval Dependency*. Technical Report CU-Math-2013-03-LAID, Department of Mathematics, Faculty of Science, Cairo University, March 2013.

[23] H. Dawood and Y. Dawood. *On Some Order-theoretic Aspects of Interval Algebras*. Technical Report CU-Math-2014-06-OTAIA, Department of Mathematics, Faculty of Science, Cairo University, June 2014.

[24] H. Dawood and Y. Dawood. *The Form of the Uncertain: On the Mathematical Structures of Uncertainty*. Technical Report CU-Math-2016-09-FUMSU, Department of Mathematics, Faculty of Science, Cairo University, September 2016.

[25] H. Dawood and Y. Dawood. *Interval Algerbras: A Formalized Treatment*. Technical Report CU-Math-2016-06-IAFT, Department of Mathematics, Faculty of Science, Cairo University, June 2016.

[26] H. Dawood and Y. Dawood. *Investigations into a Formalized Theory of Interval Differentiation*. Technical Report CU-Math-2017-03-IFTID, Department of Mathematics, Faculty of Science, Cairo University, March 2017.

[27] H. Dawood and Y. Dawood. *Parametric Intervals: More Reliable or Foundationally Problematic?* Online Mathematics Journal, 1(3), July 2019, doi: 10.5281/zenodo.3234186. URL *http://doi.org/10.5281/zenodo.3234186*.

[28] H. Dawood and Y. Dawood. *Universal Intervals: Towards a Dependency-Aware Interval Algebra*. In S. Chakraverty, editor, *Mathematical Methods in Interdisciplinary Sciences*, John Wiley & Sons, Hoboken, New Jersey, March 2020, ISBN 978-1-119-58550-3.

[29] H. Dawood and N. Megahed. *A Consistent and Categorical Axiomatization of Differentiation Arithmetic Applicable to First and Higher Order Derivatives*. Punjab University Journal of Mathematics, 51, 2019.

[30] P. S. Dwyer. *Linear computations*. Chapman & Hall, New York, 1951.

[31] L. Euler. *Foundations of Differential Calculus*. Springer Verlag, 2000.

[32] S. Feferman. *What Kind of Logic is Independence-Friendly Logic?* In R. E. Auxier and L. E. Hahn, editors, *The Philosophy of Jaakko Hintikka*, volume 30 of *Library of Living Philosophers*, pp. 453–469, Open Court Publishing Company, 2006.

[33] G. Frege. *Begriffsschrift: Eine der Arithmetischen Nachgebildete Formelsprache des Reinen Denkens*. Verlag von Louis Nebert, Halle, 1879. Translated by Stefan Bauer-Mengelberg as '*Begriffsschrift, a Formula Language, Modeled upon that of Arithmetic, for Pure Thought*'. In Jean van Heijenoort, editor, "*From Frege to Gödel: A Source Book in Mathematical Logic 1879-1931*", pages 1-82. Harvard University Press, 1967.

[34] C. Fu, Y. Liu, and Z. Xiao. *Interval Differential Evolution with Dimension-Reduction Interval Analysis Method for Uncertain Optimization Problems*. Applied Mathematical Modelling, 69:441–452, 2019.

[35] A. A. Gaganov. *Computational Complexity of the Range of the Polynomial in Several Variables*. Cybernetics, 21:418–421, 1985.

[36] E. Gardenyes, H. Mielgo, and A. Trepat. *Modal Intervals: Reason and Ground Semantics*. In *Interval Mathematics*, volume 212 of *Lecture Notes in Computer Science*, pp. 27–35, Springer Verlag, 1985.

[37] E. R. Hansen. *A Generalized Interval Arithmetic*. In *Interval Mathematics*, volume 29 of *Lecture Notes in Computer Science*, pp. 7–18, Springer Verlag, 1975.

[38] T. L. Heath, editor. *The Works of Archimedes: Edited in Modern Notation with Introductory Chapters*. Cambridge University Press, Cambridge, 2009, doi: 10.1017/CBO9780511695124. URL *https://doi.org/10.1017/CBO9780511695124*.

[39] J. Hintikka. *The Principles of Mathematics Revisited*. Cambridge University Press, 1996.

[40] IEEE 1788 Committee. *IEEE Standard for Interval Arithmetic*. IEEE Std 1788-2015, pp. 1–97, June 2015, doi: 10.1109/IEEESTD.2015.7140721. URL *https://ieeexplore.ieee.org/document/7140721*.

[41] IEEE 1788 Committee. *IEEE Standard for Interval Arithmetic (Simplified)*. IEEE Std 1788.1-2017, pp. 1–38, January 2018, doi: 10.1109/IEEESTD.2018.8277144. URL *https://ieeexplore.ieee.org/document/8277144*.

[42] S. C. Kleene. *Mathematical Logic*. Dover Publications, New York, 2002.

[43] R. T. Kneusel. *Numbers and Computers*. Springer International Publishing, Switzerland, 2015, doi: 10.1007/978-3-319-17260-6.

[44] L. V. Kolev. *Interval Methods for Circuit Analysis*. World Scientific Publishing Company, 1993.

[45] O. Kosheleva and V. Kreinovich. *Physics Need for Interval Uncertainty and How It Explains Why Physical Space Is (at Least) 3-Dimensional*. Technical Report UTEP-CS-19-05, University of Texas at El Paso, January 2019.

[46] V. Kreinovich. *Interval Computations as an Important Part of Granular Computing: An Introduction*. In W. Pedrycz, A. Skowron, and V. Kreinovich, editors, *Handbook of Granular Computing*, chapter 1, Wiley-Interscience, first edition, 2008.

[47] V. Kreinovich and S. P. Shary. *Interval Methods for Data Fitting Under Uncertainty: A Probabilistic Treatment*. Reliable Computing, 23:105–141, 2016.

[48] U. W. Kulisch. *Computer Arithmetic and Validity: Theory, Implementation, and Applications*. Walter de Gruyter, 2008.

[49] N. I. Lobachevskii. *Complete works 5*. Moscow-Leningrad, 1951.

[50] W. A. Lodwick. *Constrained Interval Arithmetic*. Technical Report 138, University of Colorado at Denver, Center for Computational Mathematics, Denver, USA, February 1999.

[51] S. M. Markov. *On Directed Interval Arithmetic and its Applications*. Journal of Universal Computer Science, 1(7):514–526, 1995.

[52] S. McCallum. *On Propagation of Equational Constraints in CAD-Based Quantifier Elimination*. In *Proceedings of the International Symposium on Symbolic and Algebraic Computation*, pp. 223–231, 2001.

[53] G. Melquiond. *Proving Bounds on Real-Valued Functions with Computations*. In A. Armando, P. Baumgartner, and G. Dowek, editors, *International Joint Conference on Automated Reasoning IJCAR*, volume 5195 of *Lecture Notes in Artificial Intelligence*, pp. 2–17, Springer-Verlag, August 2008, doi: 10.1007/978-3-540-71070-7_2.

[54] C. Menini and F. V. Oystaeyen. *Abstract Algebra: A Comprehensive Treatment*. CRC Press, first edition, 2004.

[55] R. E. Moore. *Automatic Error Analysis in Digital Computation*. Technical Report LMSD-48421, Lockheed Missiles and Space Company, Lockheed Corporation, Palo Alto, CA, 1959.

[56] R. E. Moore. *Interval Arithmetic and Automatic Error Analysis in Digital Computing*. Ph.D. thesis, Stanford University, Stanford, 1962.

[57] R. E. Moore. *Interval Analysis*. Prentice Hall, 1966.

[58] R. E. Moore. *Methods and Applications of Interval Analysis*. Number 2 in SIAM studies in Applied Mathematics, SIAM, Philadelphia, 1979.

[59] R. E. Moore, R. B. Kearfott, and M. J. Cloud. *Introduction to Interval Analysis*. SIAM, 2009.

[60] J.-M. Muller, N. Brisebarre, F. D. Dinechin, C.-P. Jeannerod, V. Lefevre, G. Melquiond, and N. Revol. *Handbook of Floating-Point Arithmetic*. Birkhäuser Boston, first edition, 2009.

[61] G. Pilz. *Near-Rings: The Theory and its Applications*, volume 23 of *North-Holland Mathematics Studies*. North-Holland Publishing Company, 1983.

[62] J. Rokne and H. Ratschek. *Computer Methods for the Range of Functions*. Ellis Horwood Publications, 1984.

[63] S. M. Rump. *INTLAB–INTerval LABoratory*. In T. Csendes, editor, *Developments in Reliable Computing*, pp. 77–104, Kluwer Academic Publishers, Dordrecht, 1999.

[64] S. Shayer. *Interval Arithmetic with Some Applications for Digital Computers*. Technical Report LMSD-5136512, Lockheed Missiles and Space Company, Lockheed Corporation, Palo Alto, CA, 1965.

[65] T. A. Skolem. *Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Satze nebst einem Theoreme über dichte Mengen*. Skrifter utgit av Videnskabsselskapet i Kristiania, I. Matematisk-Naturvidenskabelig Klasse No. 4, pp. 1–36, 1920. Translated from Norwegian as "*Logico-combinatorial investigations on the satisfiability or provability of mathematical propositions: A simplified proof of a theorem by Loewenheim*" by Stefan Bauer-Mengelberg.

[66] T. Sunaga. *Theory of an Interval Algebra and its Application to Numerical Analysis*. In *RAAG Memoirs*, volume 2, pp. 29–46, 1958.

[67] P. Suppes. *Axiomatic Set Theory*. Dover Publications, New York, 1972.

[68] A. Tarski. *A Decision Method for Elementary Algebra and Geometry*. University of California Press, second edition, 1951.

[69] A. Tarski. *Introduction to Logic and to the Methodology of the Deductive Sciences*. Oxford University Press, New York, fourth edition, 1994. Translated from Polish by Olaf Helmer.

[70] J. Vaananen. *Dependence Logic: A New Approach to Independence Friendly Logic*. Cambridge University Press, 2007.

[71] W. G. van Hoorn and B. van Rootselaar. *Fundamental Notions in the Theory of Seminearrings*. Compositio Mathematica, 18(1-2):65–78, 1967.

[72] Wolfram Research, Inc.. *Mathematica, Version 11.3, 'Computer Software'*. Wolfram Research, Inc., Champaign, Illinois, 2018. URL *https://www.wolfram.com*.

[73] R. C. Young. *The Algebra of Many-Valued Quantities*. Mathematische Annalen, 104:260–290, 1931.