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A Combinatorial Approach For the Spanning Tree Entropy in Complex Network

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Abstract: The goal of this paper is to propose the combinatorial method to facilitate the calculation of the number of spanning trees for complex networks. In particular, we derive the explicit formulas for the triangular snake, double triangular snake, four triangular snake, the total graph of path, the generalized friendship graphs and the subdivision of double triangular snake. Finally, we calculate their spanning trees entropy and we compare it between them.

Key Words: Entropy, cyclic snakes, total graph, number of spanning trees.

AMS(2010): 05C05, 05C30.

§1. Introduction

In real life, most of the systems are represented by graphs, such that the nodes denote the basic constituents of the system and edges describe their interaction. The Internet, electric, bioinformatics, telephone calls, social networks and many other systems are now represented by complex graphs [1].

There are many different types of networks and their classification depends on the properties such as nodes degrees, clustering coefficients, shortest paths. Another concern in studying complex network is how to evaluate the robustness of a network and its ability to adapt to changes [21]. The robustness of a network is correlated to its ability to deal with internal feedbacks within the network and to avoid malfunctioning when a fraction of its constituents is damaged. We use the entropy of spanning trees or what is called the asymptotic complexity [4] in order to quantify the robustness and to characterize the structure. The number of spanning trees in G, also called, the complexity of the graph is a well-studied quantity (for long time) and appear in a number of applications. Most notable application fields are network reliability [15, 16, 17], enumerating certain chemical isomers [18] and counting the number of Eulerian circuits in a graph [19].

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A graph G has different subgraphs. In fact a graph having $|V(G)|$ nodes has

$$2^{(\frac{|V(G)|(|V(G)-1|)}{2})}$$

possible distinct subgraphs. Some of these subgraphs are trees and the others are not trees. We are focused certain kinds of trees called spanning trees. The history of determining the number of spanning trees $\tau(G)$ of a graph G, dates back to the year 1842 in which the German Mathematician Gustav Kirchhoff [2] introduced a relation between the number of spanning trees of a graph G, and the determinant of a specific submatrix associated with G. This method is infeasible for large graphs. For this reason scientists have developed techniques to get around the difficulties and have paid more attention to deriving explicit and simple formulas for special classes, see [3 - 13].

The basic combinatorial idea, Feussners recursive formula [20], for counting $\tau(G)$ in a graph G is quite intuitive. For an undirected simple graph G, let e be any edge of G. All spanning trees in G can be separated into two parts: one part contains all spanning trees without e as a tree edge; the other part contains all spanning trees with e as a tree edge. The first part has the same number of spanning trees as graph $G - e$, but leaving all other edges and vertices as they are. The second part has the same number of spanning trees as graph $G \odot e$, where $G \odot e$is the graph (not a subgraph) obtained from G by contracting the edge $e = \{u, v\}$ until the two vertices u and v coincide. Call this new vertex uv. Both $G - e$ and $G \odot e$ have fewer edges, than G. So the number of spanning trees in G can be counted recursively in this way.
In this paper, we propose the combinatorial method to facilitate the calculation of the number of spanning trees for complex networks. In particular, we derive the explicit formulas for the triangular snake ($\Delta_k - snake$), double triangular snake ($2\Delta_k - snake$), four triangular snake ($4\Delta_k - snake$), the total graph of path $P_n(T(P_n))$, the graph $nC_4 \odot 2P_n$, the generalized friendship graphs ${}^kF_n$ and the subdivision of double triangular snake ($S(2\Delta_n - snake)$). Finally, we calculate their spanning trees entropy and we compare it between them.

§2. Preliminary Notes

The combinatorial method involves the operation of contraction of an edge. An edge e of a graph G is said to be contracted if it is deleted and its ends are identified. The resulting graph is denoted by $G \bullet e$. Also we denote by $G - e$ the graph obtained from G by deleting the edge e.

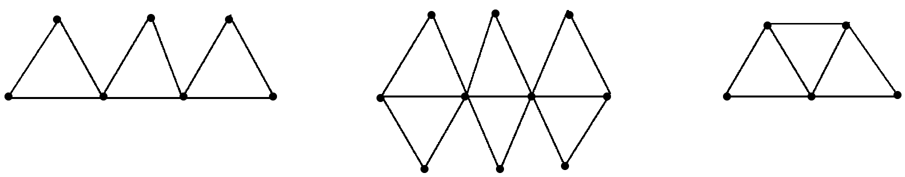
Theorem 2.1([13-20]) *Let G be a planar graph (multiple edges are allowed in here). Then for any edge $\tau(G) = \tau(G - e) + \tau(G \bullet e)$.*

Definition 2.2([22]) *A triangular snake($\Delta - snake$) is a connected graph in which all blocks are triangles and the block-cut-point graph is a path, as shown in Figure 1.*

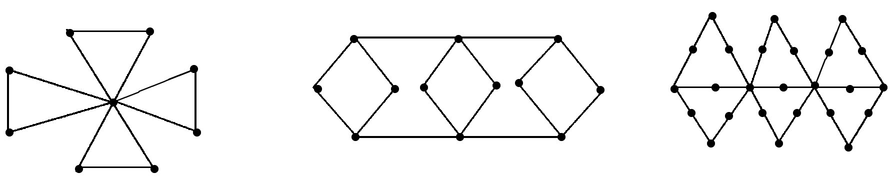
Definition 2.3 *For an integer number m, an m-triangular snake is a graph formed by m triangular snakes having a common path. If $m = 2$ that graph is called the double triangular* *snake is denoted by* $2\Delta - snake$*, as shown in Figure 1.*

Definition 2.4 *The friendship graph* $F_{n,k}$ *is a collection of* k*-cycles (all of order* n*), meeting at a common vertex, as shown in Figure 1.*

Definition 2.5 *The graph* $nC_m \odot 2P_n$ *is a connected graph obtained from* n *copies of* C_m *(* nC_m *is a disconnected graph) and two paths where each path connects with one vertex* u_i $(i = 1, 2, \cdots, 2n)$ *of each copy of* C_m *. All the vertices* u_i $(i = 1, 2, \cdots, 2n)$ *are distinct as shown in Figure 1.*



(a) Δ_k-snake (b) $2\Delta_3$-sanke (c) $T(P_3)$



(d) $F_{3,4}$ (e) $3C_4 \odot 2P_3$ (f) $S(2\Delta_3 - snake)$

Figure 1 Triangular snake, double triangular snake, four triangular snake, total graph of path, generalized friendship and subdivision of double triangular snake

Definition 2.6 *The total graph of a graph* G *is the graph whose vertex set is* $V(G) \cup E(G)$ *and two vertices are adjacent whenever they are either adjacent or incident in* G*. The total graph of* G *denoted by* $T(G)$*.*

§3. Main Results

Theorem 3.1 *The number of spanning trees of triangular snake graph is*

$$\tau(\Delta_n) = 3^n.$$

Proof Consider a triangular snake graph $\Delta_n^{'}$ constructed from Δ_n by deleting one edge. See Figure 2.

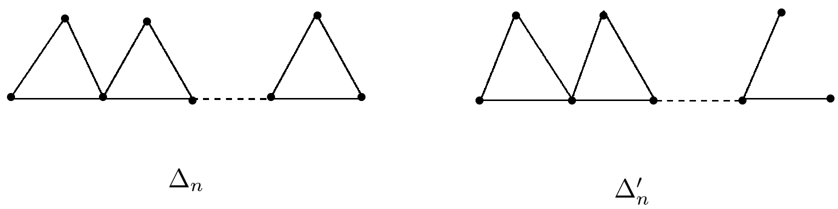


Figure 2 Triangular snake graph (Δ_n)

We put

$$\Delta_n = \tau(\Delta_n) \text{ and } \Delta_n^{'} = \tau(\Delta_n^{'}).$$

It is clear that

$$\Delta_n = 2(\Delta_{n-1}) + 3(\Delta_{n-1}^{'}) \text{ and } \Delta_n^{'} = 2(\Delta_{n-1}) - 3(\Delta_{n-1}^{'})$$

with initial conditions $\Delta_1 = 3, \Delta_n^{'} = 1$ thus we have

$$\begin{pmatrix} \Delta_n \\ \Delta_n^{'} \end{pmatrix} = A \begin{pmatrix} \Delta_n \\ 4\Delta_n^{'} \end{pmatrix},$$

where,

$$A = \begin{pmatrix} 2 & 3 \\ 2 & -3 \end{pmatrix}; \quad \begin{pmatrix} \Delta_n \\ \Delta_n^{'} \end{pmatrix} = A \begin{pmatrix} \Delta_{n-1} \\ \Delta_{n-1}^{'} \end{pmatrix} = \cdots = A^{n-1} \begin{pmatrix} \Delta_1 \\ \Delta_1^{'} \end{pmatrix},$$

we compute A^{n-1}as follows:

$$\det(A - \lambda I_2) = \lambda^2 - \lambda - 12 = 0, \ \lambda_1 = -4 \text{ and } \lambda_2 = 3, \lambda_1 \neq \lambda_2.$$

Then there is a matrix M is invertible such that $A = MBM^{-1}$, where

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{3} \end{pmatrix}; \quad M^{-1} \begin{pmatrix} \frac{1}{7} & \frac{-3}{7} \\ \frac{6}{7} & \frac{3}{7} \end{pmatrix}; \quad A^{n-1} = MB^{n-1}M^{-1},$$

where

$$B^{n-1} = \begin{pmatrix} (-4)^{n-1} & 0 \\ 0 & (3)^{n-1} \end{pmatrix}$$

From which, we obtain

$$A^{n-1} = \begin{pmatrix} \frac{(-4)^{n-1}}{7} + \frac{2*3^n}{7} & \frac{-3*(-4)^{n-1}}{7} + \frac{3^n}{7} \\ \frac{-2*(-4)^{n-1}}{7} + \frac{2*(3)^{n-1}}{7} & \frac{6*(-4)^{n-1}}{7} + \frac{3^{n-1}}{7} \end{pmatrix}$$

and hence the result follows. □

Theorem 3.2 *The number of spanning trees of the double triangular snake is*

$$\tau(2\Delta_n - snake) = 8^n.$$

Proof Consider a double triangular snake graph $2\Delta'_n$-snake constructed from $2\Delta_n$-snake by deleting two edges. See Figure 3.

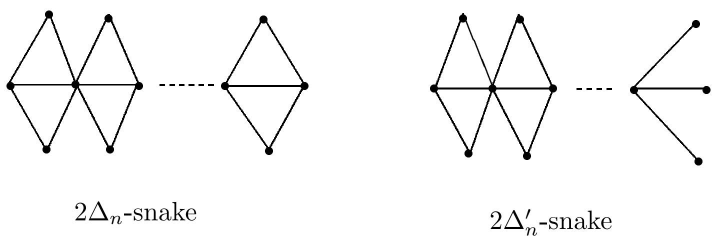


Figure 3 Triangular snake graph (Δ_n)

We put

$$2\Delta_n - snake = \tau(2\Delta_n - snake) \text{ and } 2\Delta_2^{'} - snake = \tau(2\Delta_2^{'} - snake).$$

It is clear that

$$\begin{aligned} 2\Delta_n - snake &= 7(2\Delta_{n-1} - snake) + 8(2\Delta_2^{'} - snake) \\ 2\Delta_2^{'} - snake &= 2(2\Delta_{n-1} - snake) - 8(2\Delta_{n-1}^{'} - snake) \end{aligned}$$

with initial conditions $2\Delta_1 - snake = 8$, $2\Delta_1^{'} - snake = 1$. Thus we have

$$\begin{pmatrix} 2\Delta_n - snake \\ 2\Delta_n^{'} - snake \end{pmatrix} = A \begin{pmatrix} 2\Delta_{n-1} - snake \\ 2\Delta_n^{'} - snake \end{pmatrix}, \text{ where } A = \begin{pmatrix} 7 & 8 \\ 2 & -8 \end{pmatrix},$$

$$\begin{pmatrix} 2\Delta_n - snake \\ 2\Delta_n^{'} - snake \end{pmatrix} = A \begin{pmatrix} 2\Delta_{n-1} - snake \\ 2\Delta_n^{'} - snake \end{pmatrix} = \cdots = A^{n-1} \begin{pmatrix} 2\Delta_1 - snake \\ 2\Delta_1^{'} - snake \end{pmatrix}.$$

We compute A^{n-1} as follows:

$$\det(A - \lambda I_2) = \lambda^2 - \lambda - 72 = 0, \ \lambda_1 = -9 \text{ and } \lambda_2 = 8, \ \lambda_1 \neq \lambda_2.$$

Then there is a matrix M is invertible such that $A = MBM^{-1}$, where

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{8} \end{pmatrix}; \quad M^{-1} = \begin{pmatrix} \frac{-1}{7} & \frac{8}{7} \\ \frac{8}{7} & \frac{-8}{7} \end{pmatrix}; \quad A^{n-1} = MB^{n-1}M^{-1},$$

where

$$B^{n-1} = \begin{pmatrix} (8)^{n-1} & 0 \\ 0 & (-9)^{n-1} \end{pmatrix}.$$

From which, we obtain

$$A^{n-1} = \begin{pmatrix} \frac{(-8)^{n-1}}{7} + \frac{8*(-9)^{n-1}}{7} & \frac{8^n}{7} + \frac{-8*(-9)^{n-1}}{7} \\ \frac{-2*(8)^{n-1}}{7} + \frac{(-9)^{n-1}}{7} & \frac{-2*(8)^n}{7} + \frac{-(-9)^{n-1}}{7} \end{pmatrix}$$

and hence the result follows. □

Theorem 3.3 *The number of spanning trees in* $4\Delta_n - snake$ *is* $\tau(2\Delta_n - snake)$=48^n*, where* n *is the number of blocks.*

Proof Consider a double triangular snake graph $2\Delta_2^{'}-snake$ constructed from $2\Delta_n-snake$ by deleting four edges. See Figure 4.

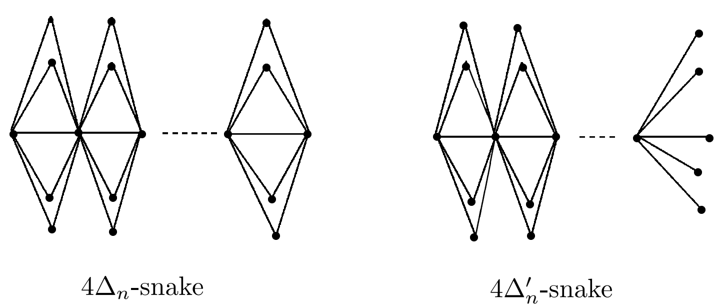


Figure 4 Friendship graph $F_{4,k}$

We put

$$4\Delta_n - snake = \tau(4\Delta_n - snake) \text{ and } 4\Delta_n^{'} - snake = \tau(4\Delta_n^{'} - snake).$$

It is clear that

$$4\Delta_n - snake = 47(4\Delta_{n-1} - snake) + 48(4\Delta_2^{'} - snake)$$

and

$$4\Delta_n^{'} - snake = 2(4\Delta_{n-1} - snake) - 48(4\Delta_{n-1}^{'} - snake)$$

with initial conditions $4\Delta_1 - snake = 48$, $4\Delta_1^{'} - snake = 1$. Thus, we have

$$\begin{pmatrix} 4\Delta_n - snake \\ 4\Delta_n^{'} - snake \end{pmatrix} = A \begin{pmatrix} 4\Delta_{n-1} - snake \\ 4\Delta_{n-1}^{'} - snake \end{pmatrix},$$

where

$$A = \begin{pmatrix} 47 & 48 \\ 2 & -48 \end{pmatrix}, \quad \begin{pmatrix} 4\Delta_n - snake \\ 4\Delta_n^{'} - snake \end{pmatrix} = A \begin{pmatrix} 4\Delta_{n-1} - snake \\ 4\Delta_{n-1}^{'} - snake \end{pmatrix} = \cdots = A^{n-1} \begin{pmatrix} 4\Delta_1 - snake \\ 4\Delta_1^{'} - snake \end{pmatrix}.$$

We compute A^{n-1} as follows:

$$\det(A - \lambda I_2) = \lambda^2 + 4\lambda - 2352 = 0, \ \lambda_1 = 48 \text{ and } \lambda_2 = -49, \ \lambda_1 \neq \lambda_2.$$

Then there is a matrix M is invertible such that A=MBM^{-1}, where

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ \frac{1}{48} & -2 \end{pmatrix}; \ M^{-1} = \begin{pmatrix} \frac{96}{97} & \frac{48}{97} \\ \frac{1}{97} & \frac{-48}{97} \end{pmatrix}; \ A^{n-1} = MB^{n-1}M^{-1},$$

where

$$B^{n-1} = \begin{pmatrix} (48)^{n-1} & 0 \\ 0 & (-49)^{n-1} \end{pmatrix}.$$

From which, we obtain

$$A^{n-1} = \begin{pmatrix} \frac{2*(48)^n}{97} + \frac{(-49)^{n-1}}{97} & \frac{48^n}{97} + \frac{-48}{97} * (-49)^{n-1} \\ \frac{2*(48)^{n-1}}{97} + \frac{-2}{97} * (-49)^{n-1} & \frac{(48)^n}{97} + \frac{96}{97} * (-49)^{n-1} \end{pmatrix}$$

and hence the result follows. □

Theorem 3.4 *The number of spanning trees of the total graph of path P_n is*

$$\tau(T(P_n)) = \frac{1}{\sqrt{5}} \left[(\frac{7+3\sqrt{5}}{2})^n - (\frac{7-3\sqrt{5}}{2})^n \right].$$

Proof Consider a total graph of path $P_n T(P_n^{'})$ constructed from $T(P_n)$ by deleting one edge. See Figure 5.

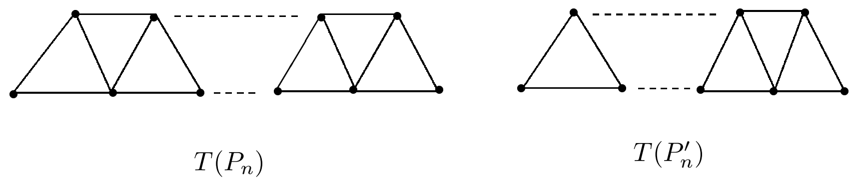


Figure 5 Total graph of path

We put

$$T(P_n) = \tau(T(P_n)) \text{ and } T(P_n^{'}) = \tau(T(P_n^{'}).$$

It is clear that

$$T(P_n) = 7T(P_{n-1}) - T(P_{n-2}^{'}),$$

where $T(P_n)$ is the number of even block and

$$T(P_n^{'}) = 48T(P_{n-2}) - 7T(P_{n-3}^{'}),$$

where $T(P_n^{'})$ is the number of odd block with initial conditions $T(P_2) = 3, T(P_2^{'}) = 1$. Thus, we have

$$\begin{pmatrix} T(P_n) \\ T(P_n^{'}) \end{pmatrix} = A \begin{pmatrix} T(P_{n-1}) \\ T(P_{n-1}^{'}) \end{pmatrix},$$

where

$$A = \begin{pmatrix} 7 & -1 \\ 48 & -7 \end{pmatrix}, \quad \begin{pmatrix} T(P_n) \\ T(P_n^{'}) \end{pmatrix} = A \begin{pmatrix} T(P_{n-1}) \\ T(P_{n-1}^{'}) \end{pmatrix} = \cdots = A^{n-2} \begin{pmatrix} T(P_2) \\ T(P_2^{'}) \end{pmatrix},$$

$\lambda_1 = 1$ and $\lambda_2 = -1$, $\lambda_1 \neq \lambda_2$. Then there is a matrix M is invertible such that A=MDM^{-1}, where

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ 6 & 8 \end{pmatrix}; \quad M^{-1} \begin{pmatrix} 4 & \frac{-1}{2} \\ & \\ -3 & \frac{1}{2} \end{pmatrix}; \quad A^{n-2} = MB^{n-2}M^{-1},$$

where

$$B^{n-2} = \begin{pmatrix} (1)^{n-2} & 0 \\ 0 & (-1)^{n-2} \end{pmatrix}.$$

From which, we obtain

$$A^{n-2} = \begin{pmatrix} 4*(1)^{n-2} - 3*(-1)^{n-2} & (\frac{-1}{2})*(1)^{n-2} + (\frac{1}{2})*(-1)^{n-2} \\ 24*(1)^{n-2} - 24*(-1)^{n-2} & -3*(1)^{n-2} + 4*(-1)^{n-2} \end{pmatrix}$$

and hence the result follows. □

Theorem 3.5 *The number of spanning trees in the graph $nC_4 \circ 2P_n$ is $\tau(nC_4 \circ 2P_n) = 4^n$.*

Proof Consider a graph B_n constructed from $nC_4 \circ 2P_n = A_n$ by deleting two edges. See Figure 6.

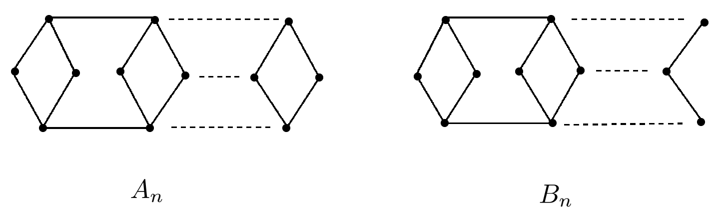


Figure 6 $nC_4 \circ 2P_n$ graph

We put

$$A_n = \tau(A_n) \text{ and } B_n = \tau_(B_n).$$

It is clear that

$$A_n = 3A_{n-1} + 4B_{n-1} \text{ and } B_n = 2A_{n-1} - 4B_{n-1}$$

with initial conditions $A_1 = 4$ and $B_1 = 1$ thus we have

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = A \begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix},$$

where

$$A = \begin{pmatrix} 3 & 4 \\ 2 & -4 \end{pmatrix}, \quad \begin{pmatrix} A_n \\ B_n \end{pmatrix} = A \begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix} = \cdots = A^{n-1} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}.$$

We compute A^{n-1} as follows:

$$\det(A - \lambda I_2) = \lambda^2 + \lambda - 20 = 0, \ \lambda_1 = -5 \text{ and } \lambda_2 = 4, \ \lambda_1 \neq \lambda_2.$$

Then there is a matrix M is invertible such that A=MBM^{-1}, where

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{4} \end{pmatrix};\ M^{-1} = \frac{\frac{1}{9}}{4}\begin{pmatrix} \frac{1}{4} & -1 \\ 2 & 1 \end{pmatrix};\ A^{n-1} = MB^{n-1}M^{-1},$$

where

$$B^{n-1} = \begin{pmatrix} (-5)^{n-1} & 0 \\ 0 & (4)^{n-1} \end{pmatrix}.$$

From which, we obtain

$$A^{n-1} = \begin{pmatrix} \frac{(-5)^{n-1}}{9} + \frac{2*(4)^n}{9} & \frac{-4*(-5)^{n-1}}{9} + \frac{4^n}{9} \\ \frac{-2*(-5)^{n-1}}{9} + \frac{2*4^{n-1}}{9} & \frac{8*(-5)^{n-1}}{9} + \frac{4^{n-1}}{9} \end{pmatrix}$$

and hence the result follows. □

Theorem 3.6 *The number of spanning trees of friendship graph $F_{3,k}$ is $\tau(F_{3,k})$=3^k.*

Proof Consider a friendship graph $F'_{3,k}$ constructed from $F_{3,k}$ by deleting one edge. See Figure 7.

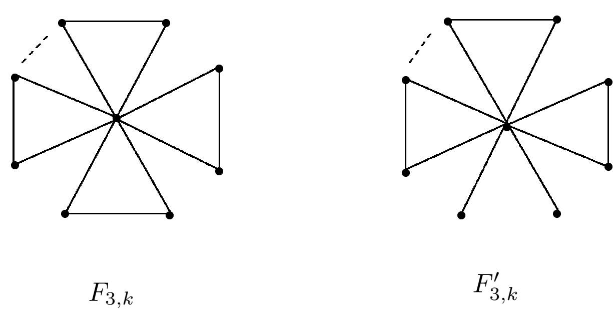


Figure 7 Friendship graph $F_{3,k}$

We put

$$F_{3,k} = \tau(F_{3,k}) \text{ and } F'_{3,k} = \tau(F'_{3,k}).$$

It is clear that

$$\tau(F_{3,k}) = 2\tau(F_{3,k-1}) + 3\tau(F'_{3,k-1}) \text{ and } \tau(F'_{3,k}) = 2\tau(F_{3,k-1}) - 3\tau(F'_{3,k-1})$$

with initial conditions $(F_{3,1}) = 3$, $(F'_{3,1}) = 1$. Thus we have

$$\begin{pmatrix} F_{3,k} \\ F'_{3,k} \end{pmatrix} = A \begin{pmatrix} F_{3,k-1} \\ F'_{3,k-1} \end{pmatrix},$$

where

$$A = \begin{pmatrix} 2 & 3 \\ 2 & -3 \end{pmatrix}, \quad \begin{pmatrix} F_{3,k} \\ F'_{3,k} \end{pmatrix} = A \begin{pmatrix} F_{3,k-1} \\ F'_{3,k-1} \end{pmatrix} = \cdots = A^{k-1} \begin{pmatrix} F_{3,1} \\ F'_{3,1} \end{pmatrix}.$$

We compute A^{k-1} as follows:

$$\det(A - \lambda I_2) = \lambda^2 - \lambda - 12 = 0, \ \lambda_1 = -4 \text{ and } \lambda_2 = 3, \ \lambda_1 \neq \lambda_2.$$

Then there is a matrix M is invertible such that A=MBM^{-1}, where

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{3} \end{pmatrix}; \ M^{-1} = \frac{\frac{1}{9}}{4} \begin{pmatrix} \frac{1}{7} & \frac{-3}{7} \\ \frac{6}{7} & \frac{3}{7} \end{pmatrix}; \ A^{k-1} = MB^{k-1}M^{-1},$$

where

$$B^{k-1} = \begin{pmatrix} (-4)^{k-1} & 0 \\ 0 & (3)^{k-1} \end{pmatrix}.$$

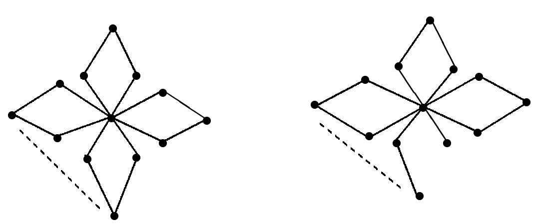
From which, we obtain

$$A^{k-1} = \begin{pmatrix} \frac{(-4)^{k-1}}{7} + \frac{2*(3)^k}{7} & \frac{-3*(-4)^{k-1}}{7} + \frac{3^k}{7} \\ \frac{-2*(-4)^{k-1}}{7} + \frac{2*3^{k-1}}{7} & \frac{6*(-4)^{k-1}}{7} + \frac{3^{k-1}}{7} \end{pmatrix}$$

and hence the result follows. □

Theorem 3.7 *The number of spanning trees of friendship graph $F_{4,k}$ is $\tau(F_{4,k})$=4^k.*

Proof Consider a friendship graph $F'_{4,k}$ constructed from $F_{4,k}$ by deleting one edge. See Figure 8.



$F_{4,k}$ $F'_{4,k}$

Figure 8 Friendship graph $F_{4,k}$

We put

$$\tau(F_{4,k}) = 3\tau(F_{4,k-1}) + 4\tau(F'_{4,k-1}) \text{ and } tau(F'_{4,k}) = 2\tau(F_{4,k-1}) - 4\tau(F'_{4,k-1})$$

with initial conditions $(F_{4,1}) = 4$, $(F'_{4,1}) = 1$. Thus, we have

$$\begin{pmatrix} F_{4,k} \\ F'_{4,k} \end{pmatrix} = A \begin{pmatrix} F_{4,k-1} \\ F'_{4,k-1} \end{pmatrix},$$

where

$$A = \begin{pmatrix} 3 & 4 \\ 2 & -4 \end{pmatrix}, \begin{pmatrix} F_{4,k} \\ F'_{4,k} \end{pmatrix} = A \begin{pmatrix} F_{4,k-1} \\ F'_{4,k-1} \end{pmatrix} = \cdots = A^{k-1} \begin{pmatrix} F_{4,1} \\ F'_{4,1} \end{pmatrix}.$$

We compute A^{k-1} as follows:

$$\det(A - \lambda I_2) = \lambda^2 + \lambda - 20 = 0,\ \lambda_1 = -5 \text{ and } \lambda_2 = 4,\ \lambda_1 \neq \lambda_2.$$

Then there is a matrix M is invertible such that $A = MBM^{-1}$, where

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{4} \end{pmatrix};\ M^{-1} = \frac{4}{9} \begin{pmatrix} \frac{1}{4} & -1 \\ 2 & 1 \end{pmatrix};\ A^{k-1} = MB^{k-1}M^{-1},$$

where

$$B^{k-1} = \begin{pmatrix} (-5)^{k-1} & 0 \\ 0 & (4)^{k-1} \end{pmatrix}.$$

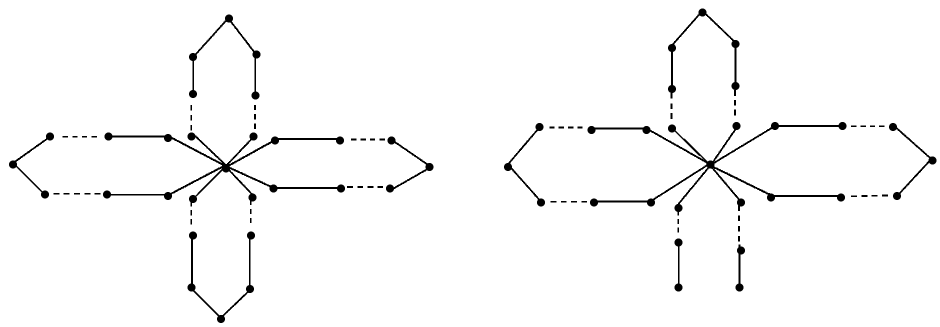
From which, we obtain

$$A^{k-1} = \begin{pmatrix} \frac{(-5)^{k-1}}{9} + \frac{2*(4)^k}{9} & \frac{-4*(-5)^{k-1}}{9} + \frac{4^k}{9} \\ \frac{-2*(-5)^{k-1}}{9} + \frac{2*4^{k-1}}{9} & \frac{8*(-5)^{k-1}}{9} + \frac{4^{k-1}}{9} \end{pmatrix}$$

and hence the result follows. □

Theorem 3.8 *The number of spanning trees of friendship graph $F_{n,k}$ is $\tau(F_{n,k})$=n^k.*

Proof Consider a friendship graph $F'_{n,k}$ constructed from $F_{n,k}$ by deleting one edge. See Figure 9.



$F_{n,k}$ $F'_{n,k}$

Figure 9 Friendship graph $F_{4,k}$

We put

$$F_{n,k} = \tau(F_{n,k}) \ \text{ and } \ F^{'}_{n,k} = \tau(F^{'}_{n,k}).$$

It is clear that

$$\tau(F_{n,k}) = (n-1)\tau(F_{n,k-1}) + n\tau(F^{'}_{n,k-1}) \ \text{ and} \tau(F^{'}_{n,k}) = 2\tau(F_{n,k-1}) - n\tau(F^{'}_{n,k-1})$$

with initial conditions $(F_{n,1}) = n$, $(F^{'}_{n,1}) = 1$. Thus, we have

$$\begin{pmatrix} F_{n,k} \\ F^{'}_{n,k} \end{pmatrix} = A \begin{pmatrix} F_{n,k-1} \\ F^{'}_{n,k-1} \end{pmatrix},$$

where

$$A = \begin{pmatrix} n-1 & n \\ 2 & -n \end{pmatrix}, \ \begin{pmatrix} F_{n,k} \\ F^{'}_{n,k} \end{pmatrix} = A \begin{pmatrix} F_{n,k-1} \\ F^{'}_{n,k-1} \end{pmatrix} = \cdots = A^{k-1} v \begin{pmatrix} n-1 & n \\ 2 & -n \end{pmatrix}.$$

We compute A^{k-1} as follows:

$$\det(A - \lambda I_2) = \lambda^2 + \lambda - n(n-1) = 0, \ \lambda_1 = -(n+1) \text{ and } \lambda_2 = n, \ \lambda_1 \neq \lambda_2.$$

Then there is a matrix M is invertible such that A=MBM^{-1}, where

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{n} \end{pmatrix}; \ M^{-1} = \frac{n}{2n+1} \begin{pmatrix} \frac{1}{n} & -1 \\ 2 & 1 \end{pmatrix}; \ A^{k-1} = MB^{k-1}M^{-1},$$

where

$$B^{k-1} = \begin{pmatrix} -(n+1)^{k-1} & 0 \\ 0 & (n)^{k-1} \end{pmatrix}.$$

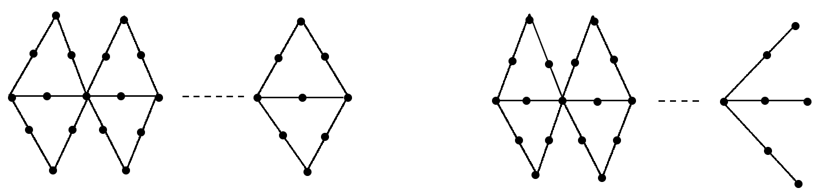
From which, we obtain

$$A^{k-1} = \begin{pmatrix} \frac{(-n-1)^{k-1}}{2n+1} + \frac{2*(n)^k}{2n+1} & \frac{-n*(-n-1)^{k-1}}{2n+1} + \frac{n^k}{2n+1} \\ \frac{-2*(-n-1)^{k-1}}{2n+1} + \frac{2*n^{k-1}}{2n+1} & \frac{2n*(-n-1)^{k-1}}{2n+1} + \frac{n^{k-1}}{2n+1} \end{pmatrix}$$

and hence the result follows. □

Theorem 3.9 *The number of spanning trees of the subdivision of double triangular snake graph is* $\tau(S(2\Delta_n - snake)) = 32^n$.

Proof Consider a double triangular snake graph $S(2\Delta_n^{'} - snake)$ constructed from $S(2\Delta_n - snake)$ by deleting one edges. See Figure 10,



$S(2\Delta_n$-snake) $S(2\Delta_n'$-snake)

Figure 10 Friendship graph $F_{4,k}$

We put

$$S(2\Delta_n - snake) = \tau(S(2\Delta_n - snake)) \text{ and } S(2\Delta_n^{'} - snake) = \tau(S(2\Delta_n^{'} - snake)).$$

It is clear that

$$S(2\Delta_n - snake) = 31(S(2\Delta_{n-1} - snake)) + 32(S(2\Delta_2^{'} - snake))$$

and

$$S(2\Delta_2^{'} - snake) = 2(S(2\Delta_{n-1} - snake)) - 32(S(2\Delta_{n-1}^{'} - snake))$$

with initial conditions $S(2\Delta_1 - snake) = 32$, $S(2\Delta_1^{'} - snake) = 1$. Thus, we have

$$\begin{pmatrix} S(2\Delta_n - snake) \\ S(2\Delta_n^{'} - snake) \end{pmatrix} = A \begin{pmatrix} S(2\Delta_{n-1} - snake) \\ S(2\Delta_n^{'} - snake) \end{pmatrix},$$

where

$$A = \begin{pmatrix} 31 & 32 \\ 2 & -32 \end{pmatrix},$$

$$\begin{pmatrix} S(2\Delta_n - snake) \\ S(2\Delta_n^{'} - snake) \end{pmatrix} = A \begin{pmatrix} S(2\Delta_{n-1} - snake) \\ S(2\Delta_n^{'} - snake) \end{pmatrix} = \cdots = A^{n-1} \begin{pmatrix} S(2\Delta_1 - snake) \\ S(2\Delta_1^{'} - snake) \end{pmatrix}.$$

We compute A^{n-1} as follows:

$$\det(A - \lambda I_2) = \lambda^2 + \lambda - 1056 = 0,\ \lambda_1 = -33 \text{ and } \lambda_2 = 32,\ \lambda_1 \neq \lambda_2.$$

Then there is a matrix M is invertible such that A=MBM^{-1}, where

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{32} \end{pmatrix};\ M^{-1} = \begin{pmatrix} \frac{1}{65} & \frac{-32}{65} \\ \frac{64}{65} & \frac{32}{65} \end{pmatrix};\ A^{n-1} = MB^{n-1}M^{-1},$$

where

$$B^{n-1} = \begin{pmatrix} (32)^{n-1} & 0 \\ 0 & (-33)^{n-1} \end{pmatrix}.$$

From which, we obtain

$$A^{n-1} = \begin{pmatrix} \frac{(32)^{n-1}}{65} + \frac{64*(-33)^{n-1}}{65} & \frac{(-32)^n}{65} + \frac{-32*(-33)^{n-1}}{65} \\ \frac{-2*(32)^{n-1}}{65} + \frac{2*(-33)^{n-1}}{65} & \frac{2*(32)^n}{65} + \frac{(-33)^{n-1}}{65} \end{pmatrix}$$

and hence the result follows. □

§4. **Spanning Tree Entropy**

The entropy of spanning trees of a network or the asymptotic complexity is a quantitative measure of the number of spanning trees and it characterizes the network structure. We use this entropy to quantify the robustness of networks. The most robust network is the network that has the highest entropy. We can calculate its spanning tree entropy which is a finite number and a very interesting quantity characterizing the network structure, defined in [15, 16] as

$$Z(G) = \lim_{V(G)\to\infty} \frac{\ln \tau(G)}{|V(G)|};$$

$$
\begin{aligned}
Z(\Delta_k - snake) &= \lim_{V(G)\to\infty} \frac{\ln \tau(G)}{|V(G)|} = \lim_{n\to\infty} \frac{3^n}{2n+1} = 0.5493;\\
Z(2\Delta_k - snake) &= \lim_{V(G)\to\infty} \frac{\ln \tau(G)}{|V(G)|} = \lim_{n\to\infty} \frac{\ln(8^n)}{3n+1} = 0.6931;\\
Z(4\Delta_k - snake) &= \lim_{V(G)\to\infty} \frac{\ln \tau(G)}{|V(G)|} = \lim_{n\to\infty} \frac{\ln(48^n)}{5n+1} = 0.7742;\\
Z(T(P_n)) &= \lim_{n\to\infty} \frac{\ln \frac{1}{\sqrt{5}}[(\frac{7+3\sqrt{5}}{2})^n - (\frac{7-3\sqrt{5}}{2})]}{2n-1} = \ln(\sqrt{\frac{7+3\sqrt{5}}{2}}) = 0.7650;\\
Z(nC_4 \odot 2P_n) &= \lim_{V(G)\to\infty} \frac{\ln \tau(G)}{|V(G)|} = \lim_{n\to\infty} \frac{\ln(4^n)}{4n} = \frac{\ln 4}{4} = 0.3466;\\
Z(F_3^k) &= \lim_{V(G)\to\infty} \frac{\ln \tau(G)}{|V(G)|} = \lim_{k\to\infty} \frac{\ln(3^k)}{2k+1} = 0.5493;\\
Z(F_4^k) &= \lim_{V(G)\to\infty} \frac{\ln \tau(G)}{|V(G)|} = \lim_{k\to\infty} \frac{\ln(4^k)}{3k+1} = 0.4621;\\
Z(F_n^k) &= \lim_{V(G)\to\infty} \frac{\ln \tau(G)}{|V(G)|} = \lim_{k\to\infty} \frac{\ln(n^k)}{(n-1)k+1} = \ln \frac{(n)}{n-1};\\
Z(S(2\Delta_k - snake)) &= \lim_{V(G)\to\infty} \frac{\ln \tau(G)}{|V(G)|} = \lim_{n\to\infty} \frac{\ln(32^n)}{8n+1} = \ln \frac{(32)}{8} = 0.4332.
\end{aligned}
$$

§5. Conclusion

In this paper, we proposed the combinatorial method to facilitate the calculation of the number of spanning trees for complex networks. In particular, we derive the explicit formulas for the triangular snake ($\Delta_k - snake$), double triangular snake ($2\Delta_k - snake$), four triangular snake ($4\Delta_k - snake$), the total graph of path $P_n(T(P_n))$, the graph $nC_4 \odot 2P_n$, the generalized friendship graphs F_n^k and the subdivision of double triangular snake ($S(2\Delta_n - snake)$). Finally, we calculate their spanning trees entropy and we compare it between them.

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