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On Certain Coloring Parameters of Graphs

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Abstract: Coloring the vertices of a graph G according to certain conditions can be considered as a random experiment and a discrete random variable X can be defined as the number of vertices having a particular color in the proper coloring of G. In this paper, we extend the concepts of mean and variance, two important statistical measures, to the theory of graph coloring and determine the values of these parameters for a number of standard graphs.

Key Words: Graph coloring, Smarandachely Λ-coloring, coloring sum of graphs, coloring mean, coloring variance, χ-chromatic mean, χ^+-chromatic.

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§1. Introduction

Investigations on graph coloring problems have attracted wide interest among researchers since its introduction in the second half of the nineteenth century. The vertex coloring or simply a coloring of a graph is an assignment of colors or labels to the vertices of a graph subject to certain conditions. For example, Smarandachely Λ-coloring of graph G by colors in $\mathscr{C}$ such that $\varphi(u) \neq \varphi(v)$ if u and v are elements of a subgraph isomorphic to graph Λ in G. In a proper coloring of a graph, its vertices are colored in such a way that no two adjacent vertices in that graph have the same color.

Different types of graph colorings have been introduced during several subsequent studies. Many practical and real life situations paved paths to different graph coloring problems.

Several researchers have also introduced various parameters related to different types of graph coloring and studied their properties extensively. The first and the most important parameter in the theory of graph coloring is the *chromatic number* of graphs which is defined as the minimum number of colors required in a proper coloring of the given graph. The concept of chromatic number has been extended to almost all types of graph colorings.

The notion of chromatic sums of graphs related to various graph colorings have been

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introduced and studied extensively. Some of these studies can be found in [9, 10, 11]. The notion of a general coloring sum of a graph has been explained in [9] as follows:

Let $\mathcal{C} = \{c_1, c_2, c_3, \cdots, c_k\}$ be a particular type of proper k-coloring of a given graph G and $\theta(c_i)$ denotes the number of times a particular color c_i is assigned to vertices of G. Then, the *coloring sum* of a coloring $\mathcal{C}$ of a given graph G, denoted by $\omega_{\mathcal{C}}(G)$, is defined to be $\omega_{\mathcal{C}}(G) = \sum\limits_{i=1}^{k} i\,\theta(c_i)$.

Motivated by the studies on different types of graph coloring problems, corresponding parameters and their applications, we discuss the concepts of mean and variance, two important statistical parameters, to the theory of graph coloring in this paper.

For all terms and definitions, not defined specifically in this paper, we refer to [2, 3, 4, 6, 15, 16] and for the terminology of graph coloring, we refer to [5, 7, 8]. For the concepts in Statistics, please see [12, 13]. Unless mentioned otherwise, all graphs considered in this paper are simple, finite, connected and non-trivial.

§2. Coloring Mean and Variance of Graphs

We can identify the coloring of the vertices of a given graph G with a random experiment. Let $\mathcal{C} = \{c_1, c_2, c_3, \cdots, c_k\}$ be a proper k-coloring of G and let X be the random variable ($r.v$) which denotes the color of an arbitrarily chosen vertex in G. Since the sum of all weights of colors of G is the order of G, the real valued function $f(i)$ defined by

$$f(i) = \begin{cases} \frac{\theta(c_i)}{|V(G)|}; & i = 1, 2, 3, \cdots, k \\ 0; & \text{elsewhere} \end{cases}$$

is the probability mass function ($p.m.f$)of the $r.v$ X. If the context is clear, we can also say that $f(i)$ is the $p.m.f$ of the graph G with respect to the given coloring $\mathcal{C}$.

Hence, analogous to the definitions of the mean and variance of random variables, the mean and variance of a graph G, with respect to a general coloring of G can be defined as follows.

Definition 2.1 *Let $\mathcal{C} = \{c_1, c_2, c_3, \cdots, c_k\}$ be a certain type of proper k-coloring of a given graph G and $\theta(c_i)$ denotes the number of times a particular color c_i is assigned to vertices of G. Then, the coloring mean of a coloring $\mathcal{C}$ of a given graph G, denoted by $\mu_{\mathcal{C}}(G)$, is defined to be*

$$\mu_{\mathcal{C}}(G) = \frac{\sum\limits_{i=1}^{k} i\,\theta(c_i)}{\sum\limits_{i=1}^{k} \theta(c_i)}.$$

Definition 2.2 *For a positive integer r, the r-th moment of the coloring $\mathcal{C}$ is denoted by $\mu_{\mathcal{C}^r}(G)$*

and is defined as

$$\mu_{\mathcal{C}^r}(G) = \frac{\sum\limits_{i=1}^{k} i^r\,\theta(c_i)}{\sum\limits_{i=1}^{k} \theta(c_i)}.$$

Definition 2.3 *The coloring variance of a coloring $\mathcal{C}$ of a given graph G, denoted by $\sigma^2_{\mathcal{C}}(G)$, is defined to be*

$$\sigma^2_{\mathcal{C}}(G = \frac{\sum\limits_{i=1}^{k} i^2\,\theta(c_i)}{\sum\limits_{i=1}^{k} \theta(c_i)} - \left(\frac{\sum\limits_{i=1}^{k} i^2\,\theta(c_i)}{\sum\limits_{i=1}^{k} \theta(c_i)}\right)^2.$$

2.1 χ-Chromatic Mean and Variance of Graphs

Coloring mean and variance corresponding to a particular type of minimal proper coloring of the vertices of G are defined as follows.

Definition 2.4 *A coloring mean of a graph G, with respect to a proper coloring $\mathcal{C}$ is said to be a χ-chromatic mean of G, if $\mathcal{C}$ is the minimum proper coloring of G and the coloring sum ω_G is also minimum. The χ-chromatic mean of a graph G is denoted by μ_χ.*

Definition 2.5 *The χ-chromatic variance of G, denoted by $\sigma^2_\chi(G)$, is a coloring variance of G with respect to a minimal proper coloring $\mathcal{C}$ of G which yields the minimum coloring sum.*

Let us now determine the χ-chromatic mean and variance of certain standard graph classes. The following result discusses the χ-chromatic mean and variance of a complete graph K_n.

Proposition 2.6 *The χ-chromatic mean of a complete graph K_n is $\frac{n+1}{2}$ and its χ-chromatic variance is $\frac{n^2-1}{12}$.*

Proof Note that all vertices of a complete graph K_n must have different colors as they are all adjacent to each other. That is, $\theta(c_i) = 1$ for color c_i, $1 \leqslant i \leqslant n$. Therefore,

$$\mu_\chi(K_n) = \frac{1}{n}\sum_{i=1}^{n} i = \frac{n+1}{2}$$

and

$$\sigma^2_\chi(K_n) = \frac{1}{n}\sum_{i=1}^{n} i^2 - \left(\frac{n+1}{2}\right)^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{2} = \frac{n^2-1}{12}. \qquad \square$$

The following theorem gives the probability distribution of a proper coloring of a complete graph.

Theorem 2.7 *Any proper coloring of a complete graph K_n has the discrete uniform distribution on $\{1, 2, \cdots, k\}(DU(k))$.*

Proof Let X be the *r.v* representing the number of colors in a proper k-coloring of a complete graph K_n. For any proper k-coloring $\mathcal{C}$ of the complete graph K_n, $\theta(c_i) = 1$ and $k = n$. Hence, the corresponding *p.m.f* is

$$f(i) = \begin{cases} \frac{1}{n}; & n = 1, 2, 3, \ldots, n, \\ 0; & \text{elsewhere} \end{cases}$$

which is that of the discrete uniform distribution on $\{1, 2, \cdots, k\}$. Hence, $X \sim DU(k)$. □

The following result determines the χ-chromatic mean and variance for a path P_n.

Proposition 2.8 *The χ-chromatic mean of a path P_n is*

$$\mu_\chi(P_n) = \begin{cases} \frac{3}{2}; & \text{if } n \text{ is even}, \\ \frac{3n-1}{2n}; & \text{if } n \text{ is odd}, \end{cases}$$

and the χ-chromatic variance of P_n is

$$\sigma^2_\chi(P_n) = \begin{cases} \frac{1}{4}; & \text{if } n \text{ is even}, \\ \frac{n^2-1}{4n^2}; & \text{if } n \text{ is odd}. \end{cases}$$

Proof Consider a path P_n on n vertices. Being a bipartite graph, the vertices of P_n can be colored using two colors, say c_1 and c_2. Then, we have the following cases.

(i) If n is even, exactly $\frac{n}{2}$ vertices of P_n have color c_1 and $\frac{n}{2}$ vertices have color c_2. Then, the *p.m.f* of the corresponding *r.v* X is

$$f(i) = \begin{cases} \frac{1}{2}; & i = 1, 2, \\ 0; & \text{elsewhere}. \end{cases}$$

Hence, the χ-chromatic mean is

$$\mu_\chi(P_n) = \sum_{i=1}^{2} i \, \frac{1}{2} = \frac{3}{2}$$

and the χ-chromatic variance is

$$\sigma^2_\chi(P_n) = \sum_{i=1}^{2} i^2 \, \frac{1}{2} - (\mu_\chi)^2 = \frac{5}{2} - \left(\frac{3}{2}\right)^2 = \frac{1}{4}.$$

(ii) If n is odd, then the *p.m.f* of the corresponding *r.v* X is

$$f(i) = \begin{cases} \frac{n+1}{2n}; & i = 1, \\ \frac{n-1}{2n}; & i = 2, \\ 0; & \text{elsewhere}. \end{cases}$$

Then, the χ-chromatic mean of P_n is

$$\mu_\chi(P_n) = 1 \cdot \frac{n+1}{2n} + 2 \cdot \frac{n-1}{2n} = \frac{3n-1}{2n}$$

and its χ-chromatic variance is

$$\sigma^2_\chi(P_n) = 1^2 \cdot \frac{n+1}{2n} + 2^2 \cdot \frac{n-1}{2n} - \left(\frac{3n-1}{2n}\right)^2 = \frac{n^2-1}{4n^2}. \qquad \square$$

The following result determines the values of these parameters for a cycle C_n.

Proposition 2.9 *The χ-chromatic mean of a cycle C_n is*

$$\mu_\chi(C_n) = \begin{cases} \frac{3}{2}; & \textit{if } n \textit{ is even,} \\ \frac{3n+3}{2n}; & \textit{if } n \textit{ is odd,} \end{cases}$$

and the χ-chromatic variance of C_n is

$$\sigma^2_\chi(C_n) = \begin{cases} \frac{1}{4}; & \textit{if } n \textit{ is even,} \\ \frac{n^2-8n+9}{4n^2}; & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof Consider a cycle C_n on n vertices. Then, we have the following cases.

(i) If n is even, then C_n is bipartite and is 2-colorable. Then, exactly $\frac{n}{2}$ vertices of C_n also have color c_1 and c_2 each. Then, as explained in the first part of previous theorem, we have $\mu_\chi(C_n) = \frac{3}{2}$ and $\sigma^2_\chi(C_n) = \frac{1}{4}$.

(ii) If n is odd, then C_n is 3-colorable. Let $\mathcal{C} = \{c_1, c_2, c_3\}$ be the minimal proper coloring of C_n. Then, the *p.m.f* of the *r.v* X is given by

$$f(i) = \begin{cases} \frac{n-1}{2n}; & \text{if } \ i = 1, 2, \\ \frac{1}{n}; & \text{if } \ i = 3, \\ 0; & \text{elsewhere.} \end{cases}$$

Then, the χ-chromatic mean of G is

$$\mu_\chi(C_n) = 1 \cdot \frac{n-1}{2n} + 2 \cdot \frac{n-1}{2n} + 3 \cdot \frac{1}{n} = \frac{3n+3}{2n}$$

and the χ-chromatic variance of C_n is

$$\sigma^2_\chi(C_n) = \left(1^2 \cdot \frac{n-1}{2n} + 2^2 \cdot \frac{n-1}{2n} + 3^2 \cdot \frac{1}{n}\right) - \left(\frac{3n+3}{2n}\right)^2 = \frac{n^2-8n+9}{4n^2}. \qquad \square$$

In the following theorem, we determine the χ-chromatic mean and variance of a wheel graph $W_n = K_1 + C_{n-1}$.

Proposition 2.10 *The χ-chromatic mean of a wheel graph W_n is*

$$\mu_\chi(W_n) = \begin{cases} \frac{3n+3}{2n}; & \text{if } n \text{ is odd,} \\ \frac{3n+1}{2n+2}; & \text{if } n \text{ is even,} \end{cases}$$

and the χ-chromatic variance of W_n is

$$\sigma^2_\chi(W_n) = \begin{cases} \frac{n^2+8n-9}{4n^2}; & \text{if } n \text{ is odd,} \\ \frac{n^2+32n-64}{4n^2}; & \text{if } n \text{ is even.} \end{cases}$$

Proof Note that the wheel graph W_n is 3-colorable, when n is odd and 4-colorable when is even. Then, we have the following cases.

(i) First, assume that n is an odd integer. Then, the outer cycle C_{n-1} of W_n is an even cycle. Hence, $\frac{n-1}{2}$ vertices of C_{n-1} have color c_1, $\frac{n-1}{2}$ vertices of C_{n-1} have color c_2 and the central vertex of W_n has color c_3. Hence the corresponding $p.m.f$ for W_n is given by

$$f(i) = \begin{cases} \frac{n-1}{2n}; & \text{if} \quad i = 1, 2, \\ \frac{1}{n}; & \text{if} \quad i = 3, \\ 0; & \text{elsewhere.} \end{cases}$$

Hence, the corresponding χ-chromatic mean is

$$\mu_\chi(W_n) = 1 \cdot \frac{n-1}{2n} + 2 \cdot \frac{n-1}{2n} + 3 \cdot \frac{1}{n} = \frac{3n+3}{2n}.$$

Now, the χ-chromatic variance is

$$\sigma^2_\chi(W_n) = (1^2+2^2) \cdot \frac{n-1}{2n} + 3^2 \cdot \frac{1}{n} - (\mu_\chi(W_n))^2 = \left(\frac{5(n-1)}{2n} + \frac{9}{n}\right) - \left(\frac{3n+3}{2n}\right)^2 = \frac{n^2+8n-9}{4n^2}.$$

(ii) Next, assume that n is an even integer. Then, the outer cycle C_{n-1} of W_n is an odd cycle. Hence, $\frac{n-2}{2}$ vertices of the outer cycle C_{n-1} have color c_1, $\frac{n-2}{2}$ vertices of C_{n-1} have color c_2 and one vertex of C_{n-1} has color c_3 and the central vertex of W_n has the c_4. Hence, the $p.m.f$ for W_n is given by

$$f(i) = \begin{cases} \frac{n-2}{2n}; & \text{if} \quad i = 1, 2, \\ \frac{1}{n}; & \text{if} \quad i = 3, 4 \\ 0; & \text{elsewhere.} \end{cases}$$

Hence, the corresponding χ-chromatic mean is

$$\mu_\chi(W_n) = 1 \cdot \frac{n-2}{2n} + 2 \cdot \frac{n-2}{2n} + 3 \cdot \frac{1}{n} + 4 \cdot \frac{1}{n} = \frac{3n+8}{2n}$$

and the χ-chromatic variance is

$$\begin{aligned}\sigma^2_\chi(W_n) &= (1^2+2^2)\cdot\frac{n-2}{2n}+(3^2+4^2)\cdot\frac{1}{n}-(\mu_\chi(W_n))^2\\ &= \left(\frac{5(n-2)}{2n}+\frac{3^2+4^2}{n}\right)-\left(\frac{3n+8}{2n}\right)^2=\frac{n^2+32n-64}{4n^2}.\end{aligned}$$ □

Remark 2.1 From the above discussion, we observe that the minimum proper coloring of bipartite graph follows a two-point distribution. In general, for a bipartite graph $G(V_1, V_2, E)$, with $|V_1| = m_1 > |V_2| = m_2, m_1 + m_2 = n$, the *p.m.f* can be defined as

$$f(i)=\begin{cases}\frac{m_1}{n}; & \text{if} \quad i=1,\\ \frac{m_2}{n}; & \text{if} \quad i=2,\\ 0; & \text{elsewhere.}\end{cases}$$

Hence, we have $\mu_\chi(G) = \frac{m_1+2m_2}{n} = 1+\frac{m_2}{n}$ and $\sigma^2_\chi(G) = \frac{m_1+4m_2}{n} - \left(1+\frac{m_2}{n}\right)^2 = \frac{1}{n^2}\left[(n-1)m_1 + 2(2n-1)m_2\right]$.

Remark 2.2 If G is a regular bipartite graph on n vertices, then there will be $\frac{n}{2}$ vertices in each partition and hence with respect to a minimal proper coloring, exactly $\frac{n}{2}$ vertices having the colors c_1 and c_2 each. Hence the *p.m.f* is

$$f(i)=\begin{cases}\frac{1}{2}; & i=1,2,\\ 0; & \text{elsewhere.}\end{cases}$$

Hence, $\mu_\chi(G) = \frac{3}{2}$ and $\sigma^2_\chi(G) = \frac{1}{4}$ as mentioned in Proposition 2.9.

2.2 χ^+-Chromatic Mean and Variance of Graphs

Coloring mean and variance corresponding to another type of a minimal proper coloring of the vertices of G are defined as follows.

Definition 2.11 *A coloring mean of a graph G, with respect to a proper coloring $\mathcal{C}$ is said to be a χ^+-chromatic mean of G, if $\mathcal{C}$ is a minimum proper coloring of G such that the corresponding coloring sum ω_G is maximum. The χ^+-chromatic number of a graph G is denoted by $\mu_{\chi^+}(G)$.*

Definition 2.12 *The χ^+-chromatic variance of G, denoted by $\sigma^2_{\chi^+}(G)$, is a coloring variance of G with respect to a minimal proper coloring $\mathcal{C}$ of G such that the corresponding coloring sum is maximum.*

Invoking the definitions of two types of chromatic means and variances mentioned above, we can infer the following.

Remark 2.3 For any arbitrary minimal proper coloring $\mathcal{C}$ of a given graph G, we have $\mu_\chi(G) \leqslant \mu_{\mathcal{C}}(G) \leqslant \mu_{\chi^+}(G)$ and $\sigma^2_\chi(G) \leqslant \sigma^2_{\mathcal{C}}(G) \leqslant \sigma^2_{\chi^+}(G)$.

Remark 2.4 Since all vertices of a complete graph have different colors, the χ-chromatic mean and the χ^+-chromatic mean are equal and the χ-chromatic variance and the χ^+-chromatic variance are equal.

Let us now discuss the χ^+-chromatic mean and variance of the graph classes mentioned in the previous section.

Proposition 2.13 *The χ^+-chromatic mean of a path P_n is*

$$\mu_{\chi^+}(P_n) = \begin{cases} \frac{3}{2}; & \text{if } n \text{ is even,} \\ \frac{3n-1}{2n}; & \text{if } n \text{ is odd,} \end{cases}$$

and the χ^+-chromatic variance of P_n is

$$\sigma^2_{\chi^+}(P_n) = \begin{cases} \frac{1}{4}; & \text{if } n \text{ is even,} \\ \frac{n^2-1}{4n^2}; & \text{if } n \text{ is odd.} \end{cases}$$

Proof As in Proposition 2.8, we consider the following cases.

(i) If n is even, as mentioned in Proposition 2.8, exactly $\frac{n}{2}$ vertices of P_n have color c_1 and $\frac{n}{2}$ vertices have color c_2. Then, the *p.m.f* of the corresponding *r.v* X is also as defined there. Hence, the χ^+-chromatic mean is $\mu_{\chi^+}(P_n) = \frac{3}{2}$ and the χ^+-chromatic variance is $\sigma^2_{\chi^+}(P_n) = \frac{1}{4}$.

(ii) If n is odd, χ^+-coloring assigns color c_1 to $\frac{n-1}{2n}$ vertices and color c_2 to the remaining $\frac{n+1}{2n}$ vertices. Then the *p.m.f* is

$$f(i) = \begin{cases} \frac{n-1}{2n}; & i = 1, \\ \frac{n+1}{2n}; & i = 2, \\ 0; & \text{elsewhere.} \end{cases}$$

Then, the χ^+-chromatic mean of P_n is given by

$$\mu_{\chi^+}(P_n) = 1 \cdot \frac{n-1}{2n} + 2 \cdot \frac{n+1}{2n} = \frac{3n+1}{2n}$$

and its χ^+-chromatic variance is given by

$$\sigma^2_{\chi^+}(P_n) = 1^2 \cdot \frac{n-1}{2n} + 2^2 \cdot \frac{n+1}{2n} - \left(\frac{3n+1}{2n}\right)^2 = \frac{n^2+1}{4n^2}. \qquad \square$$

The following proposition discusses the χ^+-chromatic mean and variance of a cycle on n vertices.

Proposition 2.14 *The χ^+-chromatic mean of a cycle C_n is*

$$\mu_{\chi^+}(C_n) = \begin{cases} \frac{3}{2}; & \text{if } n \text{ is even,} \\ \frac{5n-3}{2n}; & \text{if } n \text{ is odd,} \end{cases}$$

and the χ^+-chromatic variance of P_n is

$$\sigma^2_{\chi^+}(C_n) = \begin{cases} \frac{1}{4}; & \text{if } n \text{ is even,} \\ \frac{n^2+8n-9}{4n^2}; & \text{if } n \text{ is odd.} \end{cases}$$

Proof Here, we have to consider the following two cases.

(i) If n is even, as mentioned in Proposition 2.13, exactly $\frac{n}{2}$ vertices of C_n have color c_1 and color c_2 each. Then, exactly as explained there, we have, $\mu_{\chi^+}(C_n) = \frac{3}{2}$ and $\sigma^2_{\chi^+}(C_n) = \frac{1}{4}$.

(ii) If n is odd, χ^+-coloring assigns color c_1 to one vertex, color c_2 to $\frac{n-1}{2n}$ vertices and color c_3 to the remaining $\frac{n-1}{2n}$ vertices of the cycle C_n. Then the *p.m.f* is

$$f(i) = \begin{cases} 1; & i = 1, \\ \frac{n-1}{2n}; & i = 2, 3 \\ 0; & \text{elsewhere.} \end{cases}$$

Then, the χ^+-chromatic mean of C_n is

$$\mu_{\chi^+}(C_n) = 1 \cdot \frac{1}{2n} + 2 \cdot \frac{n-1}{2n} + 3 \cdot \frac{n-1}{2n} = \frac{5n-3}{2n}$$

and its χ^+-chromatic variance is

$$\sigma^2_{\chi^+}(C_n) = 1^2 \cdot \frac{1}{n} + 2^2 \cdot \frac{n+1}{2n} + 3^2 \cdot \frac{n-1}{2n} - \left(\frac{5n-3}{2n}\right)^2 = \frac{n^2+8n-9}{4n^2}. \qquad \square$$

The following proposition discusses the χ^+-chromatic mean and variance of a wheel graph on n vertices.

Proposition 2.15 *The χ^+-chromatic mean of a wheel graph W_n is*

$$\mu_{\chi^+}(W_n) = \begin{cases} \frac{5n-3}{2n}; & \text{if } n \text{ is odd,} \\ \frac{3n+1}{2n+2}; & \text{if } n \text{ is even,} \end{cases}$$

and the χ^+-chromatic variance of W_n is

$$\sigma^2_{\chi^+}(W_n) = \begin{cases} \frac{n^2+30n-31}{4n^2}; & \text{if } n \text{ is odd,} \\ \frac{n^2+32n-64}{4n^2}; & \text{if } n \text{ is even.} \end{cases}$$

Proof As mentioned in Proposition 1.10, the wheel graph W_n is 3-colorable, when n is odd and 4-colorable when is even. Then, we have to consider the following cases.

(i) First, assume that n is an odd integer. Then, the outer cycle C_{n-1} of W_n is an even cycle. Hence, we can assign color c_1 to the central vertex of W_n, color c_2 to $\frac{n-1}{2}$ vertices of C_{n-1} and color c_3 to the remaining $\frac{n-1}{2}$ vertices of C_{n-1}. Hence the corresponding $p.m.f$ for W_n is given by

$$f(i) = \begin{cases} \frac{1}{n}; & \text{if} \quad i = 1, \\ \frac{n-1}{2n}; & \text{if} \quad i = 2, 3, \\ 0; & \text{elsewhere.} \end{cases}$$

Hence, the χ^+-chromatic mean is

$$\mu_{\chi^+}(W_n) = 1 \cdot \frac{1}{n} + 2 \cdot \frac{n-1}{2n} + 3 \cdot \frac{n-1}{2n} = \frac{5n-3}{2n}$$

and the χ^+-chromatic variance is

$$\begin{aligned} \sigma^2_{\chi^+}(W_n) &= 1^2 \cdot \frac{1}{n} + (2^2 + 3^2) \cdot \frac{n-1}{2n} - (\mu_{chi}(W_n))^2 \\ &= \left(\frac{13(n-1)}{2n} + \frac{1}{n}\right) - \left(\frac{5n-3}{2n}\right)^2 = \frac{n^2 + 30n - 31}{4n^2}. \end{aligned}$$

(ii) Let n be an even integer. Then, the outer cycle C_{n-1} of W_n is an odd cycle. Hence, we can assign color c_1 to the central vertex of W_n, color c_2 to one vertex of the outer cycle C_{n-1}, color c_3 to $\frac{n-2}{2}$ vertices of C_{n-1} and color c_4 to the remaining $\frac{n-2}{2}$ vertices of C_{n-1}. Therefore, the corresponding $p.m.f$ for W_n is given by

$$f(i) = \begin{cases} \frac{1}{n}; & \text{if} \quad i = 1, 2 \\ \frac{n-2}{2n}; & \text{if} \quad i = 3, 4, \\ 0; & \text{elsewhere.} \end{cases}$$

Hence, the corresponding χ^+-chromatic mean is

$$\mu_{\chi^+}(W_n) = 1 \cdot \frac{1}{n} + 2 \cdot \frac{1}{n} + 3 \cdot \frac{n-2}{2n} + 4 \cdot \frac{n-2}{2n} = \frac{7n-8}{2n}$$

and the χ^+-chromatic variance is

$$\begin{aligned} \sigma^2_{\chi^+}(W_n) &= (1^2 + 2^2) \cdot \frac{1}{n} + (3^2 + 4^2) \cdot \frac{n-2}{2n} - (\mu_\chi(W_n))^2 \\ &= \left(5 \cdot \frac{1}{n} + 25 \cdot \frac{n-2}{2n}\right) - \left(\frac{7n-8}{2n}\right)^2 = \frac{n^2 + 32n - 64}{4n^2}. \end{aligned}$$

□

2.3 Some Interpretations

A *block graph* or *clique tree* G is an undirected graph in which every biconnected component (block) is a clique. By Theorem 2.7, minimum proper coloring of every component of G follows uniform distribution. Hence, we have

Theorem 2.16 *The probability distribution of a block graph G is mixture of discrete uniform distributions.*

An n-partite graph is a graph whose set of vertices can be partitioned in to n subsets such that no two vertices in the same partitions are adjacent. Then, we have the following result.

Theorem 2.17 *Let G be a regular k-partite graph on vertices. Then, any minimal proper coloring of G follows uniform distribution (in each partition).*

proof Any minimal proper coloring of a k-partite graph contains k-colors. Let G be an r-regular k-partite graph. Then, $rk = n$. Then, the $p.m.f$ of G is

$$f(i) = \begin{cases} \frac{1}{k}; & i = 1, 2, 3, \ldots, k, \\ 0; & \text{elsewhere.} \end{cases}$$

which is that of the $DU(k)$ distribution. □

Corollary 2.18 *Let G be a k-partite graph. Then, the χ-chromatic mean (and χ^+-chromatic mean) of G is $\frac{k+1}{2}$ and the χ-chromatic variance (and χ^+-chromatic variance) of G is $\frac{k^2-1}{12}$.*

Proof The proof follows immediately from the fact that the minimal proper coloring of a k-partite graph follows uniform distribution. □

Certain areas where these notions can be made use of are: nodes in communication and traffic networks.

§3. Scope for Further Studies

In this paper, we have extended the notions of mean and variance to the theory of graph coloring and determined their values for certain graphs and graph classes. More problems in this area are still open.

The χ-chromatic mean and variance of many other graph classes are yet to be studied. Determining the sum, mean and variance corresponding to the coloring of certain generalized graphs like generalized Petersen graphs, fullerence graphs etc. are some of the promising open problems. Studies on the sum, mean and variance corresponding to different types of edge colorings, map colorings, total colorings etc. of graphs also offer much for future studies.

We can associate many other parameters to graph coloring and other notions like covering, matching etc. All these facts highlight a wide scope for future studies in this area.

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