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Parity Properties of Equations, Related to Fermat Last Theorem[1]

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A possibility of elementary proof of Fermat Last Theorem (FLT) on the basis of parity considerations is considered. FLT was formulated by Fermat in 1637, and proved by A. Wiles in 1995. Here, a simpler approach is considered. The idea is to subdivide the initial equation $x^n + y^n = z^n$ into several equations. Then, each one is considered separately, using methods suitable for a particular equation. Proving FLT means to prove that each such sub-equation has no solution in natural numbers. Once this is accomplished, it would mean that the original FLT equation also has no solution in natural numbers.

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1. Introduction

Famous problems stimulate many studies, small fractions of which result in useful outcomes, even though the final goal is not reached. FLT still attracts people, for different reasons. One of them is that the known solution [1] is too complicated, so that maybe some enthusiasts intuitively feel that there should be a simpler approach. Is this intuition right or misleading, is still unknown. Here, one more such an attempt is presented. The idea is to subdivide the original equation into more manageable sub-equations, and to handle each of them separately, using mostly parity considerations.

2. FLT sub-equations

Let us consider the FLT equation.

$$x^a + y^a = z^a \qquad (1)$$

We assume that the power *a* is a natural number $a \geq 3$. The rest of parameters belong to the set of integer numbers **Z**. We will assume that variables *x*, *y*, *z* in (1) have no common divisor. Indeed, if they do have such a divisor *d*, both parts of equation can be divided by d^a, so that the new variables $x_1 = x/d$, $y_1 = y/d$, $z_1 = z/d$ will have no common divisor. We will call such a solution, without a common divisor, a *primitive solution*. From the formulas above, it is clear that any non-primitive solution can be reduced to a primitive solution by dividing by the greatest common divisor. The reverse is also true, that is any non-primitive solution can be obtained from a primitive solution by multiplying the primitive solution by a certain number. So, it is suffice to consider primitive solutions only.

Values *x*, *y*, *z* in (1) cannot be all even. Indeed, if this is so, this means that the solution is not primitive. By dividing it by the greatest common divisor, it can be reduced to a primitive solution. Obviously, *x*, *y*, *z* cannot be all odd. So, the only possible combinations left are when *x* and *y* are both odd, then *z* is even, or when one of the variables, *x* or *y*, is even, and the other is odd. In this case, *z* is odd.

Thus, equation (1) can be subdivided into the following cases, which cover all permissible permutations of equation's parameters.

1. $a = 2n$; $x = 2k+1$; $y = 2p+1$. Then, *z* is even, $z = 2m$.
2. $a = 2n+1$; $x = 2p+1$; $y = 2m$. Then, *z* is odd, $z = 2k+1$.
3. $a = 2n+1$; $x = 2k+1$; $y = 2p+1$. Then, *z* is even, $z = 2m$.
4. $a = 2n$; $x = 2p+1$; $y = 2m$. Then, *z* is odd, $z = 2k+1$.

3. Case 1

Let us assume that (1) has a solution.

$$(2k+1)^{2n} + (2p+1)^{2n} = (2m)^{2n} \tag{2}$$

Binomial expansion of the left part of (2) is as follows.

$$\left[\sum_{i=0}^{2n-2} C_i^{2n}(2k)^{2n-i} + 2n(2k) + 1\right] + \left[\sum_{i=0}^{2n-2} C_i^{2n}(2p)^{2n-i} + 2n(2p) + 1\right] = (2m)^{2n} \tag{3}$$

Transforming (3), one obtains.

$$\sum_{i=0}^{2n-2} C_i^{2n}\left[(2k)^{2n-i} + (2p)^{2n-i}\right] + 4n(k+p) + 2 = (2m)^{2n} \tag{4}$$

The lowest power of terms $2k$ and $2p$ in the sum is $2n-(2n-2)=2$. In other words, all summands in the sum are even, having a factor of two in a degree of two or greater. The second term has a factor of four. Let us divide both parts of (4) by two. We obtain.

$$\sum_{i=0}^{2n-2} C_i^{2n}\left[k(2k)^{2n-i-1} + p(2p)^{2n-i-1}\right] + 2n(k+p) + 1 = m(2m)^{2n-1} \tag{5}$$

The first two summands in the left part of (5) are even. So, the left part presents the sum of two even terms and of an odd term (which is the number one). Thus, the left part is odd.

Since we consider the values of $2n \ge 4$, the power of $2m$ in the right part is $(2n-1) \ge 3$, so that it is even. Therefore, (5) presents an equality of the odd and even numbers, which is impossible. Thus, the initial assumption that (2) has a solution is invalid, and (2) has no solution in natural numbers. In fact, (2) has no solution in integer numbers too, since the parity of the right and left parts of (5) does not depend on the algebraic signs of variables.

4. Cases 2 and 3

For the case 2, the power $a = 2n+1$; $x = 2p+1$; $y = 2m$. Then, z is odd, $z = 2k+1$.

$$(2p+1)^{2n+1} + (2m)^{2n+1} = (2k+1)^{2n+1}$$

It can be rewritten as follows

$$(2k+1)^{2n+1} - (2p+1)^{2n+1} = (2m)^{2n+1} \tag{6}$$

4.1. Presentation of pairs of odd numbers with a factor of four

Let us consider the presentation of pairs of odd numbers, expressed with a factor of four. The values of k and p can be odd or even. Generally, k and p are integers, that is they may have any algebraic sign. However, for the introduction, let us consider non-negative values. For a factor of four, the values of $(2k+1)$ and $(2p+1)$ are presented accordingly as $(4t+1)$, $(4t+3)$ and $(4s+1)$, $(4s+3)$, (Table 1, cells *a1- a4*), where *t* and *s* generally are integers, but at the moment let assume that they are non-negative. Then, cells *b1- b4* show all possible permutations of pairs of positive numbers $(2k+1)$ and $(2p+1)$, expressed with a factor of four.

Pairs of odd numbers with negative algebraic signs are shown in cells *c1-c4*. Odd numbers with mixed negative and positive signs are the in rows '*d*' and '*e*'. This covers all possible combinations of pairs of odd numbers, expressed with a factor of four, when one accounts for algebraic signs and assumes that *t* and *s* are non-negative. Fig. 1a illustrates this consideration.

Table 1. Pairs of odd numbers, expressed with a factor of four.

	0	1	2	3	4
a	*k* *p*	*2t* *2s+1*	*2t+1* *2s*	*2t* *2s*	*2t+1* *2s+1*
b	*2k+1* *2p+1*	***4t+1*** ***4s+3***	***4t+3*** ***4s+1***	***4t+1*** ***4s+1***	***4t+3*** ***4s+3***
c	-(*2k+1*) -(*2p+1*)	-(*4t+1*) -(*4s+3*)	-(*4t+3*) -(*4s+1*)	-(*4t+1*) -(*4s+1*)	-(*4t+3*) -(*4s+3*)
d	(*2k+1*) -(*2p+1*)	*4t+1* -(*4s+3*)	*4t+3* -(*4s+1*)	*4t+1* -(*4s+1*)	*4t+3* -(*4s+3*)
e	-(*2k+1*) (*2p+1*)	-(*4t+1*) *4s+3*	-(*4t+3*) *4s+1*	-(*4t+1*) *4s+1*	-(*4t+3*) *4s+3*

Let us consider another not so obvious presentation of odd numbers and their pairs with a factor of four. This time, we assume that *t* and *s* in Table 1 are integers. Such a presentation is also a symmetrical one, and also covers all possible combinations of pairs of integer odd numbers, expressed with a factor of four. However, that is not a straightforward symmetrical presentation, as it was the previous case, if not presented properly. For instance, for *t* = 2, *s* = 1, we have for the pair from cell *b1* the following: *4t+1=9, 4s+3=7*. Using symmetrical values of *t*=-2, *s*=-1 for negative values, we find $4t+1=-7$, $4s+3=-1$, which is, indeed, an asymmetrical presentation. In order to "map" the variables to the sites of their negative counterparts with the same modules, the values of t_1 =-3, s_1=-2 and the pair [(*4t+3*), (*4s+1*)] from cell *c2* should be used, which gives the required values of (-9) and (-7).) Fig. 1b provides a graphical illustration. Subsections 4.3 and 4.4 present more detail in this regard.

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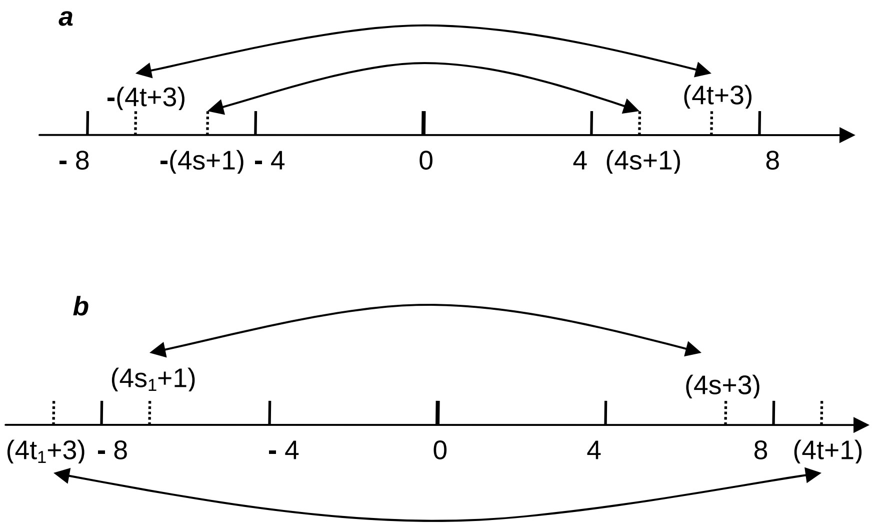

Fig. 1. Symmetrical subsets of odd integer numbers, expressed with a factor of four, for two symmetrical presentations of pairs of odd numbers.

Similarly, one can show that the pairs in cells *b3*, *b4* also produce a symmetrical presentation of pairs of odd numbers, expressed with a factor four, which will be shown in subsection 4.4.

So, in order to reproduce all possible combinations of symmetrical (relative to zero) pairs of odd numbers with a factor of four, one can use only pairs in the row '*b*', but with integer values of *t* and *s*. (The values in this row are in bold.)

4.2. Solution properties of matching equations

We will use a notion of subset of integer numbers $\mathbf{Z_1}$, symmetrical relative to zero. It means the following: If the number *x* in this subset is taken with the opposite algebraic sign, then there is a number (-*x*), belonging to the same subset. If the symmetry is required for a combinations of numbers (*x*, *y*) from the subset $\mathbf{Z_1}$, then the symmetry assumes existence of numbers (-*x*, -*y*) in the same subset. It follows from the said that the combination (-*x*, *y*) has a symmetrical combination (*x*, -*y*). Similarly, one can consider symmetry of combination of three numbers.

We will also introduce a notion of matching equation. It is needed to tie solution properties of equations $x^{2n+1}+y^{2n+1}=z^{2n+1}$ and $x_1^{2n+1}-y_1^{2n+1}=z_1^{2n+1}$ through a certain set of integer numbers, so that if the first equation has no solution in this set, then the second equation also has no solution in the same set. We need to find, which properties such a set should have for that.

Equation $x^{2n+1}+y^{2n+1}=z^{2n+1}$ matches equation $u^{2n+1}-v^{2n+1}=w^{2n+1}$, when these equations differ only by the algebraic signs before the second terms, while the domains of definitions of variables (u, v, w) and (x, y, z) are the same, and symmetric relative to zero, and the variables' positions are related: ($x \leftrightarrow u$, $y \leftrightarrow v$, $z \leftrightarrow w$). For instance, variables *x* and *y* can be defined on the domain of odd integer numbers, while *z* is defined on the domain of even integer numbers. Then, the variables *u* and *v* should be also defined on the domain of odd integer numbers, while the variable *w* should be defined on the domain of even numbers. Note that the locations of *x* and *y* are interchangeable in the first equation, so that it can also be written as $y^{2n+1} + x^{2n+1} = z^{2n+1}$. The matching equation, strictly speaking, then should be $v^{2n+1} - u^{2n+1} = w^{2n+1}$. However, since the domains of definition of *x* and *y* are the same, the equation $u^{2n+1} - v^{2n+1} = w^{2n+1}$ is also a matching equation for $y^{2n+1} + x^{2n+1} = z^{2n+1}$.

Now, we will prove the following Lemma 1.

Lemma 1: *If equation* $x^{2n+1} + y^{2n+1} = z^{2n+1}$ *has no integer solutions for a subset of integer numbers* $\boldsymbol{Z_1}$*,* symmetrical relative to zero, (x, y, z) are integers, such that $(-\infty < x, y, z < \infty)$*, then the matching equation* $u^{2n+1} - v^{2n+1} = w^{2n+1}$ *also has no integer solution in the same subset* $\boldsymbol{Z_1}$ for $(-\infty < u, v, w < \infty)$. *The reverse is also true, that is if the equation* $x^{2n+1} - y^{2n+1} = z^{2n+1}$ *has no solution in the symmetrical subset* $\boldsymbol{Z_1}$*, with* $(-\infty < x, y, z < \infty)$*, then the matching equation* $u^{2n+1} + v^{2n+1} = w^{2n+1}$ *also has no integer solution in the subset* $\boldsymbol{Z_1}$*,* $(-\infty < u, v, w < \infty)$.

Proof: Suppose the equation (*a*) $x^{2n+1} + y^{2n+1} = z^{2n+1}$ has no solution in the subset of integer numbers $\mathbf{Z_1}$, and $(-\infty < x, y, z < \infty)$. Then, we can rewrite it as (*b*) $x^{2n+1} - (-y)^{2n+1} = z^{2n+1}$. Since equation (*b*) is just another presentation of (*a*), it also has no integer solution in $\mathbf{Z_1}$. Let us denote $y_1 = -y$. Since the subset $\mathbf{Z_1}$ is symmetrical relative to zero, y_1 also belongs to subset $\mathbf{Z_1}$. Then, (*b*) can be rewritten as a matching equation (*c*) $x^{2n+1} - y_1^{2n+1} = z^{2n+1}$. The domain of definition of y_1 is the same as for *y*, that is $(-\infty < y_1 < \infty)$, and consequently $(-\infty < x, y_1, z < \infty)$. So, the domains of definition of equations (*a*) and (*c*) are the *same*. The ranges of terms in (*a*) and (*c*) are the same too, since the terms are the same odd functions, so that for the symmetrical domains of definition the terms produce the same ranges and the same sets of values.

The transformations from equation (*a*) to equation (*c*) are equivalent in that regard that they cannot change the solution properties, that is they can neither add nor remove solutions; also, the transformations preserve the original domain of definition of equation's variables. The absence of solutions for (*a*) in the subset $\mathbf{Z_1}$ means that it has no solution for the combinations (x_0, y_0, z_0)

and $(x_0, -y_0, z_0)$, since both combinations belong to the equation's domain, in which it has no solution. Accordingly, for the equation (*c*), taking into account that $y_1 = -y$, that would mean that (*c*) has no solution for the combinations $(x_0, -y_0, z_0)$ and (x_0, y_0, z_0). Therefore, (*c*) has no solution exactly for the same combinations (just listed in reverse order). So, (*c*) has no solution for all combinations of integers from subset $\mathbf{Z_1}$, for which (*a*) has no solution. Thus, we can make the next step and disconnect the sets of variables for equations (*a*) and (*c*). Indeed, the only constraint we found so far, which is required for the matching equation (*c*) to have no solution, is that the values of its variables to be taken from subset $\mathbf{Z_1}$. Accordingly, this means that for any set (u, v, w) defined on $\mathbf{Z_1}$ and preserving positional correspondence of variables with (x, y, z), equation $u^{2n+1} - v^{2n+1} = w^{2n+1}$ has no solution.

Let us repeat the main points, which led to this conclusion: (i) Both equations are defined on the same set of combinations of integers from subset $\mathbf{Z_1}$, with each combination having its symmetrical (relative to zero) counterpart for *y* and *v* values; (ii) Both equations, in fact, are symmetrical presentations of each other, which is due to the symmetry of subset $\mathbf{Z_1}$ and to the same equations' terms, represented by odd functions, which, in turn, provides symmetrical ranges of equations' terms for the corresponding variables; (iii) Equation (*a*) has no solution on subset $\mathbf{Z_1}$, and so its symmetrical presentation, equation $u^{2n+1} - v^{2n+1} = w^{2n+1}$, cannot have solution too, because it is defined on the same symmetrical set of integers, thus reproducing all combinations of numbers, which equation (*a*) could exercise on subset $\mathbf{Z_1}$.

We considered the change of algebraic sign for variable *y*. In fact, such a change of algebraic sign can be done for other variables too, which have symmetric domains of definition, and which are arguments of the terms, represented by odd functions. The important thing is that both equations (*a*) and (*c*) are defined on the same *symmetrical* (relative to zero) subset of integer numbers (in our case, ranging from minus infinity to plus infinity) and have the same equations' terms, represented by odd functions. This secures that equations (*a*) and (*c*) have the same solution properties. This proves the first part of Lemma 1.

The proof of the second part of Lemma is very similar. We assume that equation (*d*) $x^{2n+1} - y^{2n+1} = z^{2n+1}$ has no solution in the subset of integer numbers $\mathbf{Z_1}$, symmetrical relative to zero, and $(-\infty < x, y, z < \infty)$. Then, we can rewrite (*d*) as (*e*) $x^{2n+1} + (-y)^{2n+1} = z^{2n+1}$. Let us denote $y_1 = -y$. Since the subset $\mathbf{Z_1}$ is symmetrical relative to zero, y_1 also belongs to the same subset. Then, (*e*) can be rewritten as a matching equation (*f*): $x^{2n+1} + y_1^{2n+1} = z^{2n+1}$. The domain of definition of y_1 is the same as for *y*, that is $(-\infty < y_1 < \infty)$, so that $(-\infty < x, y_1, z < \infty)$. Consequently, domains of definition of equations (*d*) and (*f*) are *the same*. The transformations from equation (*d*) to equation (*f*) are equivalent; they do not change the solution properties, so that both equations have no solution, since (*d*) has no solution. Then, one can use the combination of variables (u,v,w) from $\mathbf{Z_1}$ for the equation (*g*) $u^{2n+1} + v^{2n+1} = w^{2n+1}$, instead of combination (x, y_1, z), provided the positional correspondence of the variables and the symmetry of theirs domains of definition are preserved. This new matching equation (*g*), in the same way as equation (*d*), has no solution in the subset $\mathbf{Z_1}$, because both equations are defined on the same sets of combinations of numbers from the symmetrical subset $\mathbf{Z_1}$, and produce the same sets of values of equations' terms (since the terms in (*d*) and (*g*) are the same odd functions). In this regard, equation (*g*) is a mirror reflection of equation (*d*), that is the set (x_0, y_0, z_0) for equation (*d*) is substituted by the set $(x_0, -y_0, z_0)$ for equation (*g*), and the set $(x_0, -y_0, z_0)$ for (*d*) is substituted by the set (x_0, y_0, z_0) for (*g*), so that both equations are defined on the same combinations of numbers. Similarly, the sets of values defined by the equations' terms represent the mirror reflection of each other, since they are represented by the same odd functions. Such a one-to-one "mapping" of each equation into the other, meaning both their variables, terms and domains of definition, means that both equations have no solution on the subset $\mathbf{Z_1}$, since (*d*) has no solution on this subset.

The previous note about changing the sign of another variable, instead of *y*, is also applicable to equation (*f*). This proves the second part of Lemma 1, and the whole Lemma.

It follows from the proof of Lemma 1 that only the variable, whose algebraic sign is changing, should have an odd power (or, to be an odd function in general). For instance, if we change only *y* in equations (*a*) or (*d*), substituting *y* for (-*y*), while the other terms remain the same, it is suffice for the subset $\mathbf{Z_1}$ to be symmetrical only for this variable. In such an arrangement, the unchanging during the transformations terms may be represented by other functions or parameters, not necessarily by an odd function. However, such terms should not change their values, when the algebraic signs of their arguments change (if the domains of definition of the arguments allow for such a change).

This consideration is valid for one and more terms, satisfying the conditions of their oddness and the algebraic symmetry of domains of definition of their arguments. Therefore, we can formulate the following Corollary for two terms (the case of one term is a particular case).

Corollary 1*: If equation $x^{2n+1} + y^{2n+1} = f(z)$ has no integer solutions for a subset of integer numbers* ***Z_1****, in which domains of definition of integers x and y are symmetrical relative to zero, and $(-\infty < x, y < \infty)$, then the matching equation $u^{2n+1} - v^{2n+1} = f(z)$ also has no integer solution in the same subset* ***Z_1*** *for $(-\infty < u, v < \infty)$. The reverse is also true, that is if the equation $x^{2n+1} - y^{2n+1} = f(z)$ has no solution in the subset* ***Z_1****, with symmetrical domains of definition of x and y $(-\infty < x, y < \infty)$, then the matching equation $u^{2n+1} + v^{2n+1} = f(z)$ also has no integer solution in the subset* ***Z_1****, $(-\infty < u, v < \infty)$.*

Proof: It repeats the proof of Lemma 1, still accounting for the fact that the substitutions *y* for $(-y)$, and *x* for (-*x*), do not change symmetrical (relative to zero) domains of definition of these variables, so that the new variables still belong to subset **Z_1**, for which both the original and matching equations have no solution. Indeed, (i) the variables of the matching equation positionally correspond to substituted variables of the original equation; (ii) the domains of definition of variables in the matching equation correspond to domains of definition of variables in the original equation; (iii) ranges of the affected terms of the original and matching equations are the same, since they are represented by the same odd functions in both equations, and so change algebraic signs according to the change of signs of variables. Thus, we have equivalent presentations of the same equation. Since the original equation has no solution in **Z_1**, this means that its symmetrical presentation, the matching equation, has no solution in this subset too.

The essence of Lemma 1 and Corollary 1 is rather obvious. Suppose, some equation has one or more terms, represented by odd functions (like a variable in odd power). Also, the equation has no solution on a subset of integers, symmetrical for the variables, which are arguments of odd terms. Then, the substitution of such a variable by its algebraic opposite does not change the solution properties of the equation, because the domain of definition of this variable, and the range of corresponding term, remain the same. In fact, the only thing we do, is changing the algebraic sign of a variable - which has a symmetrical domain of definition anyway, - for the term, represented by an odd function, but such a substitution cannot change the solution properties, since, for these conditions, the matching equation is an equivalent presentation of the original equation.

4.3. Finding domain where equation (6) has no solution

As a difference of two numbers in odd powers, equation (6) can be transformed to

$$2(k-p)\sum_{i=0}^{2n}(2k+1)^{2n-i}(2p+1)^{i}=(2m)^{2n+1} \tag{7}$$

Dividing both parts of (7) by two, one obtains

$$(k-p)\sum_{i=0}^{2n}(2k+1)^{2n-i}(2p+1)^{i}=m(2m)^{2n} \tag{8}$$

Here, the sum represents the sum of an odd number of odd numbers, so that it is odd. If the first factor $(k-p)$ is odd, then the left part in (8) is odd, while the right part is even, which means that (8) has no solution in this case. The expression $(k-p)$ is odd when one of the terms is odd and the other is even (cells *a1*, *a2* in Table 1). Both *k* and *p* are integers. The change of their algebraic signs does not change parity of the left part of (8). Accordingly, *t* and *s* are integers. Corresponding pairs of odd numbers with a factor of four are in cells (*b1*, *b2*), that is $[4t+1, 4s+3]$ and $[4t+3, 4s+1]$. (We will denote these two pair combinations as subset $\mathbf{Z_a}$.)

The assertion that the change of algebraic signs of *k* and *p* do not change the parity of the left part of (8) is important. It follows from the fact that (7) accounts for the algebraic signs of $(2p+1)$ and $(2k+1)$. Furthermore, this also can be shown using the presentation of these values through modules and binomial expansions. For instance, let us assume that $p<0$. Then, we can write $(2p+1)=(1-2|p|)$. Then, the binomial expansion will be as follows.

$$(1-|2p|)^{2n+1}=\sum_{i=0}^{2n+1}(-1)^{i}C_{i}^{2n+1}|2p|^{i}=1-(2n+1)|2p|+\sum_{i=2}^{2n+1}(-1)^{i}C_{i}^{2n+1}|2p|^{i}$$

Indeed, as we can see, the first two terms are not affected by the change of the algebraic sign of *p* in such a way, that this could change the parity of the left part of (8). The same is true for the negative value of $(2k+1)$. Numbers 'one' in both binomial expansions preserve the algebraic signs, so that they are still mutually canceled, while the difference $(k-p)$ becomes $(|k|-|p|)$, whose parity is the same as of $(k-p)$.

Let us show that subset $\mathbf{Z_a}$ is symmetrical relative to zero for variables $(2p+1)$ and $(2k+1)$, when they are defined as the above pairs, that as the pairs in cells (*b1*, *b2*) (*t* and *s* are integers). The pair of variables $[4t+1, 4s+3]$ with positive *t* and *s* has a matching pair of variables $[4t_1+3, 4s_1+1]$ with negative values (having the same absolute values, but the opposite algebraic signs), when $t_1=-t-1$, $s_1=-s-1$. Indeed, in this case, we obtain the numbers with the same absolute values but with the opposite algebraic signs.

$4t_1+3=4(-t-1)+3=-4t-1=-(4t+1)$

$4s_1+1=4(-s-1)+1=-4s-3=-(4s+3)$.

We can do such substitutions and use the pairs $[4t+1, 4s+3]$ and $[4t+3, 4s+1]$ interchangeably, because, as it was found, equation (8) has no solution in integer numbers for the *both* pairs. It follows from the obtained result that if a pair from subset $\mathbf{Z_a}$ has terms with different algebraic signs, there is always a matching pair of terms with the opposite algebraic signs in the subset $\mathbf{Z_a}$, for which (8) has no solution as well.

Since this subset is symmetrical relative to zero, it satisfies conditions of Lemma 1.

Although the fact of symmetry of the obtained subset was not difficult to establish, the consequences of this symmetry are important. It is due to this symmetry (which is a condition for using Lemma 1), that we can say that the equation $(2k+1)^{2n+1}+(2p+1)^{2n+1}=(2m)^{2n+1}$ also has no solution on the subset $\mathbf{Z_a}$.

4.4. Considering case 3

We have $a=2n+1$; $x=2k+1$; $y=2p+1$. Equation (1) becomes as follows.

$$(2k+1)^{2n+1}+(2p+1)^{2n+1}=(2m)^{2n+1} \tag{9}$$

The left part of equation (9), as a sum of two numbers in odd powers, can be presented as

$$2(k+p+1)\sum_{i=0}^{2n}(-1)^i(2k+1)^{2n-i}(2p+1)^i=(2m)^{2n+1} \tag{10}$$

Dividing both parts of (10) by two, one obtains

$$(k+p+1)\sum_{i=0}^{2n}(-1)^i(2k+1)^{2n-i}(2p+1)^i=m(2m)^{2n} \tag{11}$$

The sum in the left part contains an odd number of odd numbers, so that it is odd. If the first term $(k+p+1)$ is odd, then the left part in (11) is odd, while the right part is even. This means that (11) has no solution in integer numbers in this case. The value of $(k+p+1)$ is odd when $(k+p)$ is even, that is when both summands are either odd or even (input parameters in cells *a3* and *a4*, Table 1). Corresponding pairs of odd integer numbers with a factor of four are in the cells (*b3*, *b4*) (which are $[4t+1, 4s+1]$ and $[4t+3, 4s+3]$). We will call these pair combinations as subset $\mathbf{Z_s}$.

The subset $\mathbf{Z_s}$, same as subset $\mathbf{Z_a}$, is symmetrical relative to zero for variables $(2p+1)$ and $(2k+1)$, when they are defined as the found pairs, that is the pairs in cells (*b3, b4*) (*t* and *s* are integers). The pair of variables $[4t+1, 4s+1]$ with positive t and s has a matching pair of variables $[4t_1+3, 4s_1+3]$ with negative values, when $t_1=-t-1$, $s_1=-s-1$. In this case, we obtain the numbers with the same absolute values but the opposite algebraic signs.

$$4t_1+3=4(-t-1)+3=-4t-1=-(4t+1)$$

$$4s_1+3=4(-s-1)+3=-4s-1=-(4s+1)$$

Vice versa, for the pair $[4t+3, 4s+3]$, we have a matching pair $[4t_1+1, 4s_1+1]$, when $t_1=-t-1$, $s_1=-s-1$.

$$4t_1+1=4(-t-1)+1=-4t-3=-(4t+3)$$

$$4s_1+1=4(-s-1)+1=-4s-3=-(4s+3)$$

So, the united combinations of pairs of odd numbers, that is the union of subsets $\mathbf{Z_a}$ and $\mathbf{Z_s}$, for which equations (6) and (9) have no solutions, comprise the entire row '*b*' in Table 1 (which represents all possible combinations of odd numbers expressed with a factor of four). Equation (9) has no solution for the pair subset $\mathbf{Z_s}$. Then, according to Lemma 1, the matching equation (6) also has no solution for this subset. At the same time, (6) has no solution for the subset $\mathbf{Z_a}$. The union of these subsets covers all pair combinations of odd numbers in row '*b*' of Table 1, so that (6) has no solution in integer numbers for all combinations of odd numbers. Since row '*b*' in Table 1 includes all possible combinations of odd numbers, it means that (6) has no solution for any combination of odd numbers.

On the other hand, we found that (6) has no solution for the pair combinations from the subset $\mathbf{Z_a}$. Since (9) is a matching equation for (6), according to Lemma 1, this means that (9) also has no solution for the subset $\mathbf{Z_a}$, since (6) has no solution in this subset. Besides, we found that (9) also has no solution for the subset $\mathbf{Z_s}$. These two subsets comprise the row '*b*' in Table 1, which includes all possible combinations of pairs of odd numbers with a factor of four. Thus, (9) has no solution in integer numbers. This addresses the cases 2 and 3. Thus, (6) and (9) have no integer solution.

5. Case 4

In this case, $a=2n$; $x=2p+1$; $y=2m$, $z=2k+1$. Equation (1) can be presented in two forms.

$$(2k+1)^{2n}-(2p+1)^{2n}=(2m)^{2n} \quad (12)$$

$$(2p+1)^{2n}+(2m)^{2n}=(2k+1)^{2n} \quad (13)$$

Because of the even power $a = 2n$, we may consider (12) and (13) as defined on the set of integer numbers.

Let us consider (13). It can be rewritten as follows.

$$[(2p+1)^n]^2 + [(2m)^n]^2 = [(2k+1)^n]^2 \qquad (14)$$

We will use Theorem 1 (p. 38) from Chapter 2 in [2]. The Theorem says the following: *All the primitive solutions of the equation* $x^2 + y^2 = z^2$ *for which y is even number are given by the formulae* $x = M^2 - N^2$, $y = 2MN$, $z = M^2 + N^2$, *where M, N are taken to be pairs of relatively prime numbers, one of them even and the other odd and M greater than N.*

All solutions of (14) are defined as follows.

$$(2p+1)^n = (M^2 - N^2)L;\ (2m)^n = 2MNL;\ (2k+1)^n = (M^2 + N^2)L \qquad (15)$$

Here, in accordance with the aforementioned Theorem 1, *M* and *N* are pairs of relatively prime natural numbers, one of them even and the other is odd, and *M* > *N*. Substituting (15) into (14), we can see that by dividing both parts by L^2, it can be reduced to an equation, whose terms have no common divisor. So, if a solution of such an equation exists, it can be reduced to a primitive solution, and vice versa - any non-primitive solution can be obtained from a primitive solution. Thus, it is suffice to consider only primitive solutions.

Let us consider the first expression from (15).

$$(2p+1)^n = (M^2 - N^2) \qquad (16)$$

According to the aforementioned Theorem, *M* and *N* have different parity. Suppose *M* is even and *N* is odd. Then, we can rewrite (16) as

$$(2p+1)^n = (2d)^2 - (2c+1)^2 \qquad (17)$$

When *M* is odd and *N* is even, (16) transforms into

$$(2p+1)^n = (2c+1)^2 - (2d)^2 \qquad (18)$$

5.1. The case of odd n

Let us consider an odd value $n = 2q+1$. Performing transformations and presenting the left part of (17) as a binomial expansion, we obtain

$$\sum_{i=0}^{2q-1} C_i^{2q+1} (2p)^{2q+1-i} + (2q+1)(2p) + 1 = 4(d^2 - c^2 - c) - 1 \qquad (19)$$

Transferring (-1) from the right side to the left, and dividing the equation by two, one obtains.

$$\sum_{i=0}^{2q-1} C_i^{2q+1} p(2p)^{2q-i} + (2q+1)p + 1 = 2(d^2 - c^2 - c) \qquad (20)$$

The right part is even. The sum in the left part is even, since all summands are even. The left part is odd when *p* is even, *p* = *2s*, and so $2p+1=(4s+1)$. Then, (20) represents an equality of odd and even numbers. Hence, equations (16) and (17), and consequently (12) and (13) for odd *n*, have no integer solutions for this scenario. The result does not depend on *k* and *m*, so that they can be any integer numbers, the parities of both parts of (20) remain the same.

Thus, we can assume in (17), (19) and (20) that $(-\infty < s < \infty)$, $(-\infty < p < \infty)$, and, consequently, $(-\infty < (4s+1) < \infty)$ and $(-\infty < (2p+1) < \infty)$. When $(2p+1) < 0$, we can multiply both parts of (17) by (-1), and obtain

$$-(2p+1)^{2q+1} = (2c+1)^2 - (2d)^2 \qquad (21)$$

For negative $(2p+1)$ and $(2c+1) > (2d)$, equation (21) is effectively equation (18), when $(2c+1) > (2d)$. Being a particular case of (17) and (20), it also has no solution, Thus, we found that even value of *p*, which produces the value of $2p+1=(4s+1)$ (both *p* and *s* are integers), in fact, covers both equations (17) and (18), that is the cases when *M* is even and *M* is odd. Thus, we conclude that when $2p+1=(4s+1)$, equations (17) and (18), and consequently equations (12) and (13) for odd *n*, have no solution in integer numbers. The values of $(2k+1)$ and *m* in this case do not depend on *p*, and so they can be any values.

Let us consider similarly equation (18). Using binomial expansion, we can represent it as follows.

$$\sum_{i=0}^{2q-1} C_i^{2q+1} (2p)^{2q+1-i} + (2q+1)2p + 1 = 4(c^2 + c - d^2) + 1 \qquad (22)$$

Dividing both parts by two, one obtains.

$$\sum_{i=0}^{2q-1} C_i^{2q+1} p(2p)^{2q-i} + (2q+1)p = 2(d^2 - c^2 - c) \qquad (23)$$

The right part of (23) is even. The left part is odd when *p* is odd, that is *p* = *2s+1*, and so $2p+1=4s+3$. Then, (23) represents equality of odd and even numbers, which is impossible. Hence, equations (23) and (22), and consequently (12) and (13) for odd *n*, have no integer solution in this case. Note that the change of algebraic signs of *p* and *s* does not change the parity of the left part in (23) while *p* remains odd, which means that (23), (22), and consequently (12) and (13) for odd *n*, do not have solutions for $2p+1=4s+3, (-\infty < (4s+3) < \infty)$.

For negative $(2p+1)$, (18) can be rewritten as follows

$$-(2p+1)^n = (2d)^2 - (2c+1)^2 \qquad (24)$$

This makes the left part of (24) positive. When $(2d) > (2c+1)$, (24) effectively becomes equation (17), which also has no solution for odd *p*, being a particular case of (23) and (18) for the considered scenario. So, as previously for even *p*, we found that the odd value of *p*, producing the value of $2p+1=(4s+3)$ (*p* and *s* are integers), covers both equations (17) and (18), that is the cases when *M* is even and *N* is odd, and vice versa. Thus, we conclude that when $2p+1=(4s+3)$, equations (17) and (18), and consequently equations (12) and (13) for odd *n*, have no solution in integer numbers. The values of *k* and *m* in this case also do not depend of *p*, and can be any integers.

Since the found values $(4s+1)$ and $(4s+3)$ represent all odd numbers $(-\infty < s < \infty)$, we conclude that when *n* is odd, equations (12) and (13) have no solution for any odd number $2p+1$, $(-\infty < p < \infty)$. Since the value $(2k+1)$ can be any odd number for both $2p+1=4s+1$ and $2p+1=4s+3$, this covers all possible combinations of odd numbers. Therefore, for odd *n*, equations (12) and (13) have no solution for any combination of odd numbers $2p+1$ and $2k+1$.

Although this result proves that (12) and (13) have no solution in integer numbers, for completeness, we consider the values of *k* too, similarly to (16) - (18), that is the equation $(2k+1)^n = (M^2+N^2)$. Using the aforementioned Theorem 1 from [2], we can write

$$(2k+1)^n = (2d)^2 + (2c+1)^2 \qquad (25)$$

Let us present the left part as a binomial extension.

$$\sum_{i=0}^{2q-1} C_i^{2q+1}(2k)^{2q+1-i} + (2q+1)2k + 1 = 4(c^2+c+d^2)+1 \qquad (26)$$

Dividing both parts by two, we obtain.

$$\sum_{i=0}^{2q-1} C_i^{2q+1}k(2k)^{2q-i} + (2q+1)k = 2(d^2-c^2-c) \qquad (27)$$

The right part is even. Equation (27) has no solution when the left part is odd, which happens when *k* is odd, that is when $k=2t+1$, $2k+1=4t+3$. Note that the parity of the left part in (27) does not depend on the algebraic signs of *k* and *t*. Thus, (25), and consequently (12) and (13) for odd *n*, have no solution for all $2k+1=4t+3$. The values of *p* and *m* can be any integers, allowable in (12) and (13).

Assuming $2k+1<0$, one can rewrite (25) as

$$-(2k+1)^n = (2d)^2 + (2c+1)^2$$

Using binomial expansion, one obtains that this equation has no solution for even *k*. This produces the value $2k+1=4t+1$, which, combined with the above result $2k+1=4t+3$, produces the set of all odd numbers. Since the value of *k* in this case does not depend on *p* and *m*, this result covers all possible combinations of odd numbers $2k+1$ and $2p+1$ in (12) and (13), for odd *n*. So, these equations have no solution in integer numbers for odd *n*.

The next possible venue for the same task could be considering the above independent equations as a system.

$$(2p+1)^{2q+1} = M^2 - N^2$$

$$(2k+1)^{2q+1} = M^2 + N^2 \qquad (28)$$

Summing up and subtracting these equations, we obtain an equivalent system of independent equations (formally, the independence of equations can be proved considering the matrix rank).

$$(2k+1)^{2q+1} + (2p+1)^{2q+1} = 2M^2$$

$$(2k+1)^{2q+1} - (2p+1)^{2q+1} = 2N^2 \qquad (29)$$

Here, *M* and *N* can be interchangeably the odd and even numbers $(2c+1)$ and $(2d)$.

Let us assume $M=(2c+1)$. Then, the first equation in (29) can be transformed as follows.

$$2(k+p+1)\sum_{i=0}^{i=2q}(-1)^i(2k+1)^{2q-i}(2p+1)^i = 2(2c+1)^2 \qquad (30)$$

Dividing both parts of (30) by two, one obtains.

$$(k+p+1)\sum_{i=0}^{i=2q}(-1)^i(2k+1)^{2q-i}(2p+1)^i = (2c+1)^2 \qquad (31)$$

The sum is odd as the sum of an odd number of odd numbers. The right part is odd. The left part is even when $(k+p+1)$ is even, that is when $(k+p)$ is odd. In this case, we have an equality of odd and even numbers, so that (31) has no solution.

Considering the second equation in (29), we obtain

$$(k-p)\sum_{i=0}^{i=2q}(2k+1)^{2q-i}(2p+1)^i = (2d)^2 \qquad (32)$$

Equation (32) has no solution when $(k-p)$ is odd. Note that the only restriction we obtained so far is that both values, $(k+p)$ and $(k-p)$, should be odd, while *k* and *p* may have any algebraic signs. The appropriate values of *k* and *p* are in the cells (*a1*, *a2*) in Table 1, with the corresponding pairs of odd numbers in cells (*b1*, *b2*), which are $[4t+1, 4s+3]$ and $[4t+3, 4s+1]$ - same as in subsection 4.3. Let us denote the obtained subset of pairs, for which equations in (29) have no solution, as $\mathbf{Z_1}$. Similarly to pairs of odd numbers, considered in subsection 4.3, this subset is symmetrical relative to zero, so that the conditions of Corollary 1 are fulfilled for both equations in (29), and their matching equations

$$(2k+1)^{2q+1} - (2p+1)^{2q+1} = 2M^2$$

$$(2k+1)^{2q+1} + (2p+1)^{2q+1} = 2N^2 \qquad (33)$$

According to Corollary 1, the matching equations do not have solutions on the same subset $\mathbf{Z_1}$ of pairs of odd numbers too.

On the other hand, the matching equations (33) do not have solution when $(k-p)$ in the first equation, and $(k+p)$ in the second equations, are even. Indeed, in this case, the equations can be transformed as follows.

$$(k-p)\sum_{i=0}^{i=2q}(2k+1)^{2q-i}(2p+1)^i = (2c+1)^2$$

$$(k+p+1)\sum_{i=0}^{i=2q}(-1)^i(2k+1)^{2q-i}(2p+1)^i = (2d)^2 \qquad (34)$$

Then, it follows from (34) that these equations have no solution when $(k-p)$ and $(k+p)$ are even. The appropriate values of *k* and *p* are in cells (*a3*, *a4*) in Table 1, with the appropriate pairs of odd numbers in cells (*b3*, *b4*), that is $[4t+1, 4s+1]$. The found subset of pairs, let us call it $\mathbf{Z_2}$, is symmetrical, which was shown in subsection 4.4 for these pairs. So, the conditions for Corollary 1 are fulfilled. This, in turn, means that the matching equations (29) also have no solution for the pairs from subset $\mathbf{Z_2}$. The obtained subset $\mathbf{Z_2}$ complements the earlier found subset $\mathbf{Z_1}$ for odd values of $(k-p)$ and $(k+p)$, expressed with a factor of four. So, the entire row '*b*' in Table 1 is covered. It means that equations (29), and consequently (12) and (13) for odd *n*, have no solution for $M=(2c+1)$ and $N=2d$.

Similarly to (30) - (34), we can consider the case of even *M* and odd *N*, that is when

$$(2k+1)^{2q+1} + (2p+1)^{2q+1} = 2(2d)^2$$

$$(2k+1)^{2q+1} - (2p+1)^{2q+1} = 2(2c+1)^2 \qquad (35)$$

Equations (35) can be transformed as follows.

$$(k+p+1)\sum_{i=0}^{i=2q}(-1)^i(2k+1)^{2q-i}(2p+1)^i=(2d)^2$$

$$(k-p)\sum_{i=0}^{i=2q}(2k+1)^{2q-i}(2p+1)^i=(2c+1)^2 \quad (36)$$

These equations have no solution when $(k+p)$ in the first equation, and $(k-p)$ in the second, are even. These values correspond to input values in cells (*a3*, *a4*) in Table 1, with pairs of odd numbers in the cells (*b3*, *b4*), that is the pairs $[4t+1, 4s+1]$ and $[4t+3, 4s+3]$. In the previous notations, we obtained subset $\mathbf{Z_2}$. It was shown in subsection 4.4 that the subset of pairs of odd numbers $\mathbf{Z_2}$ is symmetrical (relative to zero), so that Corollary 1 is applicable. The matching equations, which also have no solution for the obtained subset, are as follows.

$$(2k+1)^{2q+1}-(2p+1)^{2q+1}=2(2d)^2$$

$$(2k+1)^{2q+1}+(2p+1)^{2q+1}=2(2c+1)^2 \quad (37)$$

On the other hand, the matching equations (37) do not have solution for the odd $(k-p)$ in the first equation, and the odd $(k+p)$ in the second one (this can be shown similar to (36)). These values correspond to inputs in cells (*a1*, *a2*), with the corresponding pairs in cells (*b1, b2*), that is the pairs $[4t+1, 4s+3]$ and $[4t+3, 4s+1]$. In the previous notations, this is subset $\mathbf{Z_1}$. It was shown in subsection 4.3 that subset $\mathbf{Z_1}$ is a symmetrical one (relative to zero), so that the conditions for Corollary 1 are fulfilled, which means that the matching equations (35) also have no solution in $\mathbf{Z_1}$. Two found subsets cover the entire Table 1. This proves that equations (29) have no solution for all allowable values of *M* and *N*, which also means that (12) and (13) have no solution in integer numbers for odd *n*.

5.2. Even n

Let us consider even $n=2q$. Then, (12) can be presented as follows.

$$[(2k+1)^q]^4-[(2p+1)^q]^4=[(2m)^{2q}]^2 \quad (38)$$

According to Corollary 1 (p. 52) from Chapter 2 in [2], equation (38) has no solutions in natural numbers (because of the even power, this also means that (38) has no solution in integer numbers). The corollary is read as follows: *There are no natural numbers a, b, c such that* $a^4-b^4=c^2$.

Since $(2k+1)^q$, $(2p+1)^q$ and $(2m)^{2q}$ cannot be natural numbers, $(2k+1)$, $(2p+1)$ and $(2m)$ cannot be natural numbers too. Indeed, if one assumes that these are natural numbers, then their powers has to be natural numbers too, which contradicts to the aforementioned Corollary.

So, equations (12), (13) have no solution for even *n*.

The same result can be obtained using the property of Pythagorean triangles that there is no Pythagorean triangle, whose sides are squares. Indeed, we can rewrite (38) as follows.

$$[(2p+1)^q]^4 + [(2m)^q]^4 = [(2k+1)^q]^4 \qquad (39)$$

Corollary 2 on p. 53, Chapter 2 in [2] says: *There are no natural numbers x, y, z satisfying the equation* $x^4 + y^4 = z^4$. This means that (38), and consequently (12) and (13) for even *n*, have no solution in natural numbers.

Thus, we proved that (12) and (13) have no solution in integer numbers for odd and even *n*.

6. Conclusion

We considered all four possible cases for FLT equation from section two, and proved that the equations do not have solution in integer numbers. Since these cases represent a complete set of possible combinations of variables for equation (1), this means that (1) has no solution in integer numbers, and in natural numbers in particular, which proves FLT.

References

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[2] Sierpinski W. Elementary theory of numbers. PWN - Polish Scientific Publishers, Warszawa (1988)

Appendix A

Expressing pairs of odd numbers with a factor greater than four

We considered pairs of odd numbers with a factor of four. The idea can be extended to pairs of odd numbers with a factor of eight, sixteen, etc. Table A1 shows all pairs of positive odd numbers, corresponding to pairs $[4t+1, 4s+1]$ and $[4t+3, 4s+3]$, expressed with a factor of eight. Consider the equation

$$(4t+1)^{2n+1} - (4s+1)^{2n+1} = (2m)^{2n+1} \qquad (A1)$$

It can be transformed to

$$4(t-s)\sum_{i=0}^{2n}(4t+1)^{2n-i}(4s+1)^{i} = (2m)^{2n+1} \qquad (A2)$$

Dividing both parts of (A2) by four, one obtains

$$(t-s)\sum_{i=0}^{2n}(4t+1)^{2n-i}(4s+1)^{i} = m^{2}(2m)^{2n-1} \qquad (A3)$$

Table A1. Pairs of positive odd numbers, corresponding to $[4t+1, 4s+1]$ and $[4t+3, 4s+3]$, expressed with a factor of eight.

	0	1	2	3	4
a	*t* *s*	*2c* *2b+1*	*2c+1* *2b*	*2c* *2b*	*2c+1* *2b+1*
b	*4t+1* *4s+1*	***8c+1*** ***8b+5***	***8c+5*** ***8b+1***	*8c+1* *8b+1*	*8c+5* *8b+5*
c	*4t+3* *4s+3*	***8c+3*** ***8b+7***	***8c+7*** ***8b+3***	*8c+3* *8b+3*	*8c+7* *8b+7*

In (A3), the sum in the left part represents the sum of an odd number of odd numbers, so that it is odd. If the first term $(t-s)$ is odd, then the left part in (A3) is odd, while the right part is even, which means that (A3) has no solution in natural numbers for this case. The value of $(t-s)$ is odd when one of the terms is odd, and the other is even. In Table A1, these scenarios correspond to input parameters *t* and *s* in the cells *a1* and *a2*. Corresponding pairs of odd numbers with a factor of eight are accordingly in the cells *b1* and *b2* (that is [*8c+1*, *8b+5*] and [*8c+5*, *8b+1*]).

Similarly, we can consider the equation

$$(4t+3)^{2n+1} - (4s+3)^{2n+1} = (2m)^{2n+1} \qquad (A4)$$

It can be transformed to

$$4(t-s)\sum_{i=0}^{2n}(4t+3)^{2n-i}(4s+3)^{i} = (2m)^{2n+1} \qquad (A5)$$

Dividing both parts of (A5) by four, one obtains

$$(t-s)\sum_{i=0}^{2n}(4t+3)^{2n-i}(4s+3)^{i} = m^{2}(2m)^{2n-1} \qquad (A6)$$

Similarly to (A3), the sum in (A6) is odd. So, when the factor $(t-s)$ is odd, the equations (A6), has no solution. Corresponding combinations of input parameters *t* and *s* are in cells *a1*, *a2*, with the appropriate pairs of odd numbers with a factor of eight in cells *c1*, *c2*, that is the pairs [*8c+3*, *8b+7*] and [*8c+7*, *8b+3*]. All found pairs, for which (A1) and (A4) have no solution, are in bold in Table A1. (Extending the obtained subsets of natural numbers to integer ones, and proving the symmetry of such subsets, would allow application of Lemma 1 to matching equations, which will prove that equations have no solution for the rest of pairs in Table A1.)

We can continue to use a similar approach for the rest of pairs in columns 3 and 4, that is composing an equation similar to (A1) for the pairs of odd numbers in cells *b3*, *b4*, *c3*, *c4*. For instance, for the pair [*8c+1*, *8b+1*], it would be

$$(8c+1)^{2n+1}-(8b+1)^{2n+1}=(2m)^{2n+1} \quad \text{(A7)}$$

This way, we will find pairs of odd numbers, for which this equation has no solution. However, this time, we would have to divide equations by eight, not by four, as in the case of (A2) and (A5), and consequently n has to be greater than one, in order for the right part to remain even. Then, for the remaining half of combinations, this time composed of terms of type $(16v+e)$, where e is an integer, we can again compose equations, similar to (A7), and find a new set of pairs for which it has no solution, which again will be half of combinations. However, the equations have to be divided by 16 this time.

Such a procedure can continue, increasing the fraction of combinations, for which the original equations (A1) or (A6), or similar equations, have no solution, although the following iterations will need greater value of n.