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ORIGINAL ARTICLE

On Neutrosophic Soft Linear Spaces

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Abstract The concept of neutrosophic soft linear space (NSLS) is introduced and its several related properties and structural characteristics are investigated in this paper with suitable examples. Then cartesian product of neutrosophic soft linear spaces, neutrosophic soft subspace and neutrosophic soft vector are defined and illustrated by examples. Finally some related basic theorems have been established, too.

Keywords Neutrosophic soft linear space (NSLS) · Cartesian product of NSLSs · Neutrosophic soft subspace · Neutrosophic soft vector

1. Introduction

After the pioneering work of Zadeh [1] and Attanosov [2], a progressive developments have been made by Bag and Samanta [3], Samanta and Jebril [4], Park [5], Katsaras [6], Kramosil and Michalek [7] and others in the field of normed linear spaces, metric spaces and topological spaces. Researchers in economics, sociology, medical science and many other several fields deal daily with the complexities of modeling uncertain data. Classical methods are not always successful because the

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uncertain appearing in these domains may be of various types. While probability theory, theory of fuzzy set, intuitionistic fuzzy set and other mathematical tools are well known and often useful approaches to describe uncertainty but each of these theories has it's different difficulties as pointed out by Molodtsov [8]. In 1999, Molodtsov [8] introduced a novel concept of soft set theory which is free from the parametrization inadequacy syndrome of different theories dealing with uncertainty. This makes the theory very convenient and easy to apply in practice. Then, many authors have designed their research works on several algebraic structures using this noble concept for instance, Maji et al. [9-12], Aktas and Cagman [13], Dinda et al. [14], Basu et al. [15, 16], Aygunoglu and Aygun [17], Yaqoob et al. [18], Varol et al. [19], Zhang [20], Das et al. [21], Beaula and Priyanga [22], Tanay and Kandemir [23] and many others.

After introduction of Neutrosophic set (NS) by Smarandache [24] which is a generalisation of classical set, fuzzy set, intuitionistic fuzzy set, Maji [25] has introduced a combined concept Neutrosophic soft set (NSS). Consequently, several mathematicians have produced their research works in different mathematical structures for instance, Broumi [26], Bera and Mahapatra [27, 43-46], Broumi et al. [28-39], Pramanik [40, 41], Cetkin et al. [47-49]. Later, this concept has been modified by Deli and Broumi [42].

This paper presents the notion of neutrosophic soft linear space along with investigation of some related properties and theorems. Section 2 gives some preliminary useful definitions related to it. In Section 3, neutrosophic soft linear space is defined along with some properties. Section 4 and Section 5 deal with the cartesian product of NSLSs and neutrosophic soft subspace, respectively. The concept of neutrosophic soft vector and neutrosophic soft scalar along with their properties are introduced in Section 6. Finally, the conclusion of our work has been stated in Section 7.

2. Preliminaries

We recall some basic definitions related to fuzzy set, soft set, neutrosophic set for the sake of completeness which are found in the literature [8, 14, 24, 25, 42, 45, 46].

Definition 2.1 *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is continuous t- norm if* $*$ *satisfies the following conditions :*
(i) $*$ *is commutative and associative.*
(ii) $*$ *is continuous.*
(iii) $a * 1 = 1 * a = a,\ \forall a \in [0,1]$.
(iv) $a * b \leq c * d$ *if* $a \leq c,\ b \leq d$ *with* $a, b, c, d \in [0,1]$.

Some examples of continuous *t*-norm are $a * b = ab$, $a * b = \min\{a, b\}$, $a * b = \max\{a + b - 1, 0\}$.

Definition 2.2 *A binary operation* $\diamond : [0,1] \times [0,1] \to [0,1]$ *is continuous t-conorm (s-norm) if* $\diamond$ *satisfies the following conditions :*
(i) $\diamond$ *is commutative and associative.*
(ii) $\diamond$ *is continuous.*
(iii) $a \diamond 0 = 0 \diamond a = a,\ \forall a \in [0,1]$.
(iv) $a \diamond b \leq c \diamond d$ *if* $a \leq c,\ b \leq d$ *with* $a, b, c, d \in [0,1]$.

A few examples of continuous s-norm are $a \diamond b = a + b - ab$, $a \diamond b = \max\{a, b\}$, $a \diamond b = \min\{a + b, 1\}$. $*$ is called an idempotent t-norm and $\diamond$ is called an idempotent s-norm, if $a * a = a$ and $a \diamond a = a \ \ \forall a \in [0, 1]$. The only idempotent t-norm and idempotent s-norm are min operator and max operator, respectively.

Definition 2.3 *Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A in X is characterized by a truth-membership function T_A, an indeterminacy-membership function I_A and a falsity-membership function F_A. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]^-0, 1^+[$. That is $T_A, I_A, F_A : X \to]^-0, 1^+[$. There is no restriction on the sum of $T_A(x), I_A(x)$, $F_A(x)$ and so, $^-0 \leq supT_A(x) + supI_A(x) + supF_A(x) \leq 3^+$.*

Definition 2.4 *Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denote the power set of U. Then for $A \subseteq E$, a pair (F, A) is called a soft set over U, where $F : A \to P(U)$ is a mapping.*

Definition 2.5 *Let U be an initial universe set and E be a set of parameters. Let $NS(U)$ denote the set of all NSs of U. Then for $A \subseteq E$, a pair (F, A) is called an NSS over U, where $F : A \to NS(U)$ is a mapping.*

This concept has been redefined by Deli and Broumi [42] as given below.

Definition 2.6 *Let U be an initial universe set and E be a set of parameters. Let $NS(U)$ denote the set of all NSs of U. Then, a neutrosophic soft set N over U is a set defined by a set valued function f_N representing a mapping $f_N : E \to NS(U)$ where f_N is called approximate function of the neutrosophic soft set N. In other words, the neutrosophic soft set is a parameterized family of some elements of the set $NS(U)$ and therefore it can be written as a set of ordered pairs,*

$$N = \{(e, \{< x, T_{f_N(e)}(x), I_{f_N(e)}(x), F_{f_N(e)}(x) > | x \in U\}) | e \in E\},$$

where $T_{f_N(e)}(x), I_{f_N(e)}(x), F_{f_N(e)}(x) \in [0, 1]$, respectively are called the truth-membership, indeterminacy-membership, falsity-membership function of $f_N(e)$. Since supremum of each T, I, F is 1 so the inequality $0 \leq T_{f_N(e)}(x) + I_{f_N(e)}(x) + F_{f_N(e)}(x) \leq 3$ is obvious.

Example 2.1 Let $U = \{h_1, h_2, h_3\}$ be a set of houses and $E = \{e_1$(beautiful), e_2(wooden), e_3(costly)$\}$ be a set of parameters with respect to which the nature of houses are described. Let

$$f_N(e_1) = \{< h_1, (0.5, 0.6, 0.3) >, < h_2, (0.4, 0.7, 0.6) >, < h_3, (0.6, 0.2, 0.3) >\},$$
$$f_N(e_2) = \{< h_1, (0.6, 0.3, 0.5) >, < h_2, (0.7, 0.4, 0.3) >, < h_3, (0.8, 0.1, 0.2) >\},$$
$$f_N(e_3) = \{< h_1, (0.7, 0.4, 0.3) >, < h_2, (0.6, 0.7, 0.2) >, < h_3, (0.7, 0.2, 0.5) >\}.$$

Then $N = \{[e_1, f_N(e_1)], [e_2, f_N(e_2)], [e_3, f_N(e_3)]\}$ is an NSS over (U, E). The tabular representation of the NSS N is given in Table 1.

Definition 2.7 *The complement of a neutrosophic soft set N is denoted by N^c and is defined by :*

$$N^c = \{(e, \{< x, F_{f_N(e)}(x), 1 - I_{f_N(e)}(x), T_{f_N(e)}(x) > | x \in U\}) | e \in E\}.$$

Definition 2.8 *Let N_1 and N_2 be two NSSs over the common universe (U, E). Then N_1 is said to be the neutrosophic soft subset of N_2 if*

$$T_{f_{N_1}(e)}(x) \leq T_{f_{N_2}(e)}(x), I_{f_{N_1}(e)}(x) \geq I_{f_{N_2}(e)}(x), F_{f_{N_1}(e)}(x) \geq F_{f_{N_2}(e)}(x), \forall e \in E, \forall x \in U.$$

We write $N_1 \subseteq N_2$ and then N_2 is the neutrosophic soft superset of N_1.

Table 1 : Tabular form of NSS N.

	$f_N(e_1)$	$f_N(e_2)$	$f_N(e_3)$
h_1	(0.5,0.6,0.3)	(0.6,0.3,0.5)	(0.7,0.4,0.3)
h_2	(0.4,0.7,0.6)	(0.7,0.4,0.3)	(0.6,0.7,0.2)
h_3	(0.6,0.2,0.3)	(0.8,0.1,0.2)	(0.7,0.2,0.5)

Definition 2.9 *Let N_1 and N_2 be two NSSs over the common universe (U, E). Then their union is denoted by $N_1 \cup N_2 = N_3$ and is defined as :*

$$N_3 = \{(e, \{< x, T_{f_{N_3}(e)}(x), I_{f_{N_3}(e)}(x), F_{f_{N_3}(e)}(x) > | x \in U\}) | e \in E\},$$

where $T_{f_{N_3}(e)}(x) = T_{f_{N_1}(e)}(x) \diamond T_{f_{N_2}(e)}(x), I_{f_{N_3}(e)}(x) = I_{f_{N_1}(e)}(x) * I_{f_{N_2}(e)}(x)$ *and* $F_{f_{N_3}(e)}(x) = F_{f_{N_1}(e)}(x) * F_{f_{N_2}(e)}(x)$.

Their intersection is denoted by $N_1 \cap N_2 = N_4$ and is defined as :

$$N_4 = \{(e, \{< x, T_{f_{N_4}(e)}(x), I_{f_{N_4}(e)}(x), F_{f_{N_4}(e)}(x) > | x \in U\}) | e \in E\},$$

where $T_{f_{N_4}(e)}(x) = T_{f_{N_1}(e)}(x) * T_{f_{N_2}(e)}(x), I_{f_{N_4}(e)}(x) = I_{f_{N_1}(e)}(x) \diamond I_{f_{N_2}(e)}(x)$ *and* $F_{f_{N_4}(e)}(x) = F_{f_{N_1}(e)}(x) \diamond F_{f_{N_2}(e)}(x)$.

Definition 2.10 *Let N_1 and N_2 be two NSSs over the common universe (U, E). Then their 'AND' operation is denoted by $N_1 \wedge N_2 = N_5$ and is defined as :*

$$N_5 = \{[(a, b), \{< x, T_{f_{N_5}(a,b)}(x), I_{f_{N_5}(a,b)}(x), F_{f_{N_5}(a,b)}(x) > | x \in U\}] | (a, b) \in E \times E\},$$

where $T_{f_{N_5}(a,b)}(x) = T_{f_{N_1}(a)}(x) * T_{f_{N_2}(b)}(x), I_{f_{N_5}(a,b)}(x) = I_{f_{N_1}(a)}(x) \diamond I_{f_{N_2}(b)}(x)$ *and* $F_{f_{N_5}(a,b)}(x) = F_{f_{N_1}(a)}(x) \diamond F_{f_{N_2}(b)}(x)$.

Their 'OR' operation is denoted by $N_1 \vee N_2 = N_6$ and is defined as :

$$N_6 = \{[(a, b), \{< x, T_{f_{N_6}(a,b)}(x), I_{f_{N_6}(a,b)}(x), F_{f_{N_6}(a,b)}(x) > | x \in U\}] | (a, b) \in E \times E\},$$

where $T_{f_{N_6}(a,b)}(x) = T_{f_{N_1}(a)}(x) \diamond T_{f_{N_2}(b)}(x), I_{f_{N_6}(a,b)}(x) = I_{f_{N_1}(a)}(x) * I_{f_{N_2}(b)}(x)$ *and* $F_{f_{N_6}(a,b)}(x) = F_{f_{N_1}(a)}(x) * F_{f_{N_2}(b)}(x)$.

Definition 2.11 *A neutrosophic soft set N over (U, E) is said to be null neutrosophic soft set denoted by ϕ_u if*

$$T_{f_N(e)}(x) = 0, I_{f_N(e)}(x) = 1, F_{f_N(e)}(x) = 1, \ \forall e \in E, \ \forall x \in U.$$

A neutrosophic soft set N over (U, E) is said to be absolute neutrosophic soft set denoted by 1_u if

$$T_{f_N(e)}(x) = 1, I_{f_N(e)}(x) = 0, F_{f_N(e)}(x) = 0,\ \forall e \in E,\ \forall x \in U.$$

Clearly, $\phi_u^c = 1_u$ and $1_u^c = \phi_u$.

Definition 2.12 *A neutrosophic soft point in an NSS N over (U, E) is defined as an element $(e, f_N(e))$ of N, for $e \in E$ and is denoted by e_N, if $f_N(e) \notin \phi_u$ and $f_N(e') \in \phi_u$, $\forall e' \in E - \{e\}$.*

The complement of a neutrosophic soft point e_N is another neutrosophic soft point e_N^c such that $f_N^c(e) = (f_N(e))^c$.

A neutrosophic soft point $e_N \in M$, M being an NSS if for $e \in E$, $f_N(e) \leq f_M(e)$ i.e.,

$$T_{f_N(e)}(x) \leq T_{f_M(e)}(x),\ I_{f_N(e)}(x) \geq I_{f_M(e)}(x),\ F_{f_N(e)}(x) \geq F_{f_M(e)}(x),\ \forall x \in U.$$

Example 2.2 Let $U = \{x_1, x_2, x_3\}$ and $E = \{e_1, e_2\}$. Then,

$$e_{1N} = \{< x_1, (0.6, 0.4, 0.8) >, < x_2, (0.8, 0.3, 0.5) >, < x_3, (0.3, 0.7, 0.6) >\}$$

is a neutrosophic soft point whose complement is

$$e_{1N}^c = \{< x_1, (0.8, 0.6, 0.6) >, < x_2, (0.5, 0.7, 0.8) >, < x_3, (0.6, 0.3, 0.3) >\}.$$

For another NSS M defined on same (U, E), let

$$f_M(e_1) = \{< x_1, (0.7, 0.4, 0.7) >, < x_2, (0.8, 0.2, 0.4) >, < x_3, (0.5, 0.6, 0.5) >\}.$$

Then $f_N(e_1) \leq f_M(e_1)$ i.e., $e_{1N} \in M$.

3. Neutrosophic Soft Linear Spaces

In this section, we have defined NSLS with suitable examples and have studied some basic properties related to it.

Unless otherwise stated, $V(K)$ is a vector space over the field K and E is treated as the parametric set through out this paper, $e \in E$ an arbitrary parameter.

Definition 3.1 *A neutrosophic set $B = \{< x, (T_B(x), I_B(x), F_B(x)) > | x \in V\}$ on a vector space $V(K)$ is called a neutrosophic sub-vector space of $V(K)$ if*

$$\text{(i)} \begin{cases} T_B(x + y) \geq T_B(x) * T_B(y), \\ I_B(x + y) \leq I_B(x) \diamond I_B(y), \\ F_B(x + y) \leq F_B(x) \diamond F_B(y),\ \forall x, y \in V. \end{cases} \quad \text{(ii)} \begin{cases} T_B(\lambda x) \geq T_B(x), \\ I_B(\lambda x) \leq I_B(x), \\ F_B(\lambda x) \leq F_B(x),\ \forall x \in V,\ \forall \lambda \in K. \end{cases}$$

An NSS N on $V(K)$ is called a neutrosophic soft vector space / linear space (NSLS) if $f_N(e)$ is a neutrosophic sub-vector space on $V(K)$ for each $e \in E$.

Example 3.1 Let $E = \{e_1, e_2, e_3, ...e_n\}$ be the parametric set and $\mathbf{R}^n(\mathbf{R})$ be the n-dimensional Euclidean space. Let us define a mapping $f_N : E \to NS(\mathbf{R}^n)$, for any $t \in \mathbf{R}^n$, as following :

$$T_{f_N(e_i)}(t) = \begin{cases} 1/2, & \text{if } i\text{-th co-ordinate of } t \text{ is zero,} \\ 0, & \text{otherwise.} \end{cases}$$

$$I_{f_N(e_i)}(t) = \begin{cases} 0, & \text{if } i\text{-th co-ordinate of } t \text{ is zero,} \\ 1/4, & \text{otherwise.} \end{cases}$$

$$F_{f_N(e_i)}(t) = \begin{cases} 0, & \text{if } i\text{-th co-ordinate of } t \text{ is zero,} \\ 1/10, & \text{otherwise.} \end{cases}$$

The t-norm $(*)$ and s-norm $(\diamond)$ are defined as $a * b = \min\{a, b\}, a \diamond b = \max\{a, b\}$. Then, N forms an NSS as well as NSLS over $\mathbf{R}^n(\mathbf{R})$ with respect to parametric set E.

For convenience, we take an attempt for the parameter e_1 and the Euclidean space $\mathbf{R}^3(\mathbf{R})$. Then the following four cases arise to choose $x, y \in \mathbf{R}^3$.

Case 1 : if $x = (0, 2, 4)$ and $y = (0, 3, 2)$, then $x + y = (0, 5, 6)$.
Case 2 : if $x = (0, 1, 3)$ and $y = (3, 0, 2)$, then $x + y = (3, 1, 5)$.
Case 3 : if $x = (1, 3, 2)$ and $y = (5, 6, 1)$, then $x + y = (6, 9, 3)$.
Case 4 : if $x = (5, 2, 1)$ and $y = (-5, 3, 4)$, then $x + y = (0, 5, 5)$.

From these four cases, the first set of conditions can be easily verified and then the second set, too.

Example 3.2 Consider a real vector space $\mathbf{C} = \{a + ib : a, b \in \mathbf{R}, i = \sqrt{-1}\}$ and the parametric set $E = \{\alpha, \beta, \gamma\}$. We divide the elements of C into four classes e.g.,

$(C1)$ $\{ib : b \in \mathbf{R} - \{0\}\}$ when real part is zero,
$(C2)$ $\{a : a \in \mathbf{R} - \{0\}\}$ when imaginary part is zero,
$(C3)$ $\{a + ib : a, b \in \mathbf{R} - \{0\}\}$ when both parts are nonzero,
$(C4)$ $\{0 + i0\}$, the null vector.

If $x \in (C1)$ and $y \in (C2)$, then $x + y \in (C3)$. We write $(C1) + (C2) = (C3)$. Then the addition operation table on $\mathbf{C}$ can be put as in Table 2.

We now define an NSS N over $(\mathbf{C}, E)$ as given by the Table 3. Corresponding t-norm $(*)$ and s-norm $(\diamond)$ are defined as $a * b = \max\{a + b - 1, 0\}$, $a \diamond b = \min\{a + b, 1\}$; Then, N forms an NSLS over $(\mathbf{C}(\mathbf{R}), E)$.

Example 3.3 Let $E = \mathbf{N}$ (the set of natural numbers) be the parametric set and $\mathbf{R}^3 = \{a = (x, y, z) | x, y, z \in \mathbf{R}\}$ be a real vector space. Define a mapping $f_M : \mathbf{N} \to NS(\mathbf{R}^3)$, for any $a \in \mathbf{R}^3$ and $n \in \mathbf{N}$, as following :

$$T_{f_M(n)}(a) = \frac{1}{n},\ I_{f_M(n)}(a) = \frac{1}{2n},\ F_{f_M(n)}(a) = 1 - \frac{1}{n},\ \forall a \in \mathbf{R}^3.$$

The t-norm $(*)$ and s-norm$(\diamond)$ are defined as $a * b = \min\{a, b\}, a \diamond b = \max\{a, b\}$. Then, M is a neutrosophic soft linear space over $(\mathbf{R}^3(\mathbf{R}), \mathbf{N})$.

Corollary 3.1 *Let N be an NSLS over $(V(K), E)$. Then for $x \in V$ and $\lambda (\neq 0) \in K$, $T_{f_N(e)}(\lambda x) = T_{f_N(e)}(x)$, $I_{f_N(e)}(\lambda x) = I_{f_N(e)}(x)$, $F_{f_N(e)}(\lambda x) = F_{f_N(e)}(x)$ hold.*

Proof

$$T_{f_N(e)}(x) = T_{f_N(e)}[\lambda^{-1}(\lambda x)] \geq T_{f_N(e)}(\lambda x),$$
$$I_{f_N(e)}(x) = I_{f_N(e)}[\lambda^{-1}(\lambda x)] \leq I_{f_N(e)}(\lambda x),$$
$$F_{f_N(e)}(x) = F_{f_N(e)}[\lambda^{-1}(\lambda x)] \leq F_{f_N(e)}(\lambda x).$$

Now, from the 2nd set of conditions in definition of NSLS, the result follows.

Table 2 : Table for addition operation on C.

+	$(C1)$	$(C2)$	$(C3)$	$(C4)$
$(C1)$	$(C1)$	$(C3)$	$(C3)$	$(C1)$
$(C2)$	$(C3)$	$(C2)$	$(C3)$	$(C2)$
$(C3)$	$(C3)$	$(C3)$	$(C3)$	$(C3)$
$(C4)$	$(C1)$	$(C2)$	$(C3)$	$(C4)$

Table 3 : Tabular form of neutrosophic soft set N.

	$f_N(\alpha)$	$f_N(\beta)$	$f_N(\gamma)$
$(C1)$	$(0.69, 0.31, 0.32)$	$(0.68, 0.21, 0.76)$	$(0.72, 0.21, 0.16)$
$(C2)$	$(0.62, 0.32, 0.42)$	$(0.62, 0.31, 0.79)$	$(0.84, 0.16, 0.25)$
$(C3)$	$(0.58, 0.41, 0.66)$	$(0.59, 0.42, 0.80)$	$(0.69, 0.31, 0.39)$
$(C4)$	$(0.71, 0.27, 0.53)$	$(0.67, 0.43, 0.84)$	$(0.79, 0.19, 0.41)$

Proposition 3.1 *Let N be an NSLS over $(V(K), E)$. Then for each $x \in V$, followings hold.*

(i) $T_{f_N(e)}(-x) = T_{f_N(e)}(x)$, $I_{f_N(e)}(-x) = I_{f_N(e)}(x)$, $F_{f_N(e)}(-x) = F_{f_N(e)}(x)$.

(ii) $T_{f_N(e)}(\theta) \geq T_{f_N(e)}(x)$, $I_{f_N(e)}(\theta) \leq I_{f_N(e)}(x)$, $F_{f_N(e)}(\theta) \leq F_{f_N(e)}(x)$.
if $a * b$ =min$\{a, b\}$ and $a \diamond b$ =max$\{a, b\}$. (θ being the null vector of V)

Proof (i) For $\lambda = -1$, the result directly follows from above corollary.
(ii) For the null vector $\theta \in V$,

$$T_{f_N(e)}(\theta) = T_{f_N(e)}(x + (-x)) \geq T_{f_N(e)}(x) * T_{f_N(e)}(-x) = T_{f_N(e)}(x) * T_{f_N(e)}(x) = T_{f_N(e)}(x),$$
$$I_{f_N(e)}(\theta) = I_{f_N(e)}(x + (-x)) \leq I_{f_N(e)}(x) \diamond I_{f_N(e)}(-x) = I_{f_N(e)}(x) \diamond I_{f_N(e)}(x) = I_{f_N(e)}(x),$$
$$F_{f_N(e)}(\theta) = F_{f_N(e)}(x + (-x)) \leq F_{f_N(e)}(x) \diamond F_{f_N(e)}(-x) = F_{f_N(e)}(x) \diamond F_{f_N(e)}(x) = F_{f_N(e)}(x).$$

Hence, the proposition is proved.

Proposition 3.2 *An NSS N on $V(K)$ is called an NSLS space with respect to the set E iff followings hold on the assumption that $a * b$ =min$\{a, b\}$ and $a \diamond b$ =max$\{a, b\}$.*

$$\begin{cases} T_{f_N(e)}(\lambda x + \mu y) \geq T_{f_N(e)}(x) * T_{f_N(e)}(y), \\ I_{f_N(e)}(\lambda x + \mu y) \leq I_{f_N(e)}(x) \diamond I_{f_N(e)}(y), \\ F_{f_N(e)}(\lambda x + \mu y) \leq F_{f_N(e)}(x) \diamond F_{f_N(e)}(y), \ \forall\, x, y \in V, \ \forall \lambda, \mu \in F, \ \forall e \in E. \end{cases}$$

Proof First suppose N be an NSLS on $V(K)$ with respect to the set E. Then,

$$T_{f_N(e)}(\lambda x+\mu y)\geq T_{f_N(e)}(\lambda x)*T_{f_N(e)}(\mu y)\geq T_{f_N(e)}(x)*T_{f_N(e)}(y),$$
$$I_{f_N(e)}(\lambda x+\mu y)\leq I_{f_N(e)}(\lambda x)\diamond I_{f_N(e)}(\mu y)\leq I_{f_N(e)}(x)\diamond I_{f_N(e)}(y),$$
$$F_{f_N(e)}(\lambda x+\mu y)\leq F_{f_N(e)}(\lambda x)\diamond F_{f_N(e)}(\mu y)\leq F_{f_N(e)}(x)\diamond F_{f_N(e)}(y).$$

Conversely, by Proposition 3.1,

$$T_{f_N(e)}(\lambda x)=T_{f_N(e)}(\theta+\lambda x)\geq T_{f_N(e)}(\theta)*T_{f_N(e)}(x)\geq T_{f_N(e)}(x)*T_{f_N(e)}(x)=T_{f_N(e)}(x),$$
$$I_{f_N(e)}(\lambda x)=I_{f_N(e)}(\theta+\lambda x)\leq I_{f_N(e)}(\theta)\diamond I_{f_N(e)}(x)\leq I_{f_N(e)}(x)\diamond I_{f_N(e)}(x)=I_{f_N(e)}(x),$$
$$F_{f_N(e)}(\lambda x)=F_{f_N(e)}(\theta+\lambda x)\leq F_{f_N(e)}(\theta)\diamond F_{f_N(e)}(x)\leq F_{f_N(e)}(x)\diamond F_{f_N(e)}(x)=F_{f_N(e)}(x).$$

$$T_{f_N(e)}(x+y)=T_{f_N(e)}(x+(-1)(-y))\geq T_{f_N(e)}(x)*T_{f_N(e)}(-y)\geq T_{f_N(e)}(x)*T_{f_N(e)}(y),$$
$$I_{f_N(e)}(x+y)=I_{f_N(e)}(x+(-1)(-y))\leq I_{f_N(e)}(x)\diamond I_{f_N(e)}(-y)\leq I_{f_N(e)}(x)\diamond I_{f_N(e)}(y),$$
$$F_{f_N(e)}(x+y)=F_{f_N(e)}(x+(-1)(-y))\leq F_{f_N(e)}(x)\diamond F_{f_N(e)}(-y)\leq F_{f_N(e)}(x)\diamond F_{f_N(e)}(y).$$

Hence, the proof is completed.

Theorem 3.1 *Let M and N be two an NSLSs over $(V(K),E)$. Then, $M\cap N$ is also an NSLS over $(V(K),E)$.*

Proof Let $M\cap N=P$. Now, for $x,y\in V$,

$$\begin{aligned}T_{f_P(e)}(x+y)&=T_{f_M(e)}(x+y)*T_{f_N(e)}(x+y)\\&\geq[T_{f_M(e)}(x)*T_{f_M(e)}(y)]*[T_{f_N(e)}(x)*T_{f_N(e)}(y)]\\&=[T_{f_M(e)}(x)*T_{f_M(e)}(y)]*[T_{f_N(e)}(y)*T_{f_N(e)}(x)]\ (\text{as } * \text{ is commutative})\\&=T_{f_M(e)}(x)*[T_{f_M(e)}(y)*T_{f_N(e)}(y)]*T_{f_N(e)}(x)\ (\text{as } * \text{ is associative})\\&=T_{f_M(e)}(x)*T_{f_P(e)}(y)*T_{f_N(e)}(x)\\&=T_{f_M(e)}(x)*T_{f_N(e)}(x)*T_{f_P(e)}(y)\ (\text{as } * \text{ is commutative})\\&=T_{f_P(e)}(x)*T_{f_P(e)}(y).\end{aligned}$$

Hence, $T_{f_P(e)}(x+y)\geq T_{f_P(e)}(x)*T_{f_P(e)}(y)$.

Also, $T_{f_P(e)}(\lambda x)=T_{f_M(e)}(\lambda x)*T_{f_N(e)}(\lambda x)\geq T_{f_M(e)}(x)*T_{f_N(e)}(x)=T_{f_P(e)}(x)$. Thus, $T_{f_P(e)}(\lambda x)\geq T_{f_P(e)}(x)$ for $\lambda\in K$.

Next,

$$\begin{aligned}I_{f_P(e)}(x+y)&=I_{f_M(e)}(x+y)\diamond I_{f_N(e)}(x+y)\\&\leq[I_{f_M(e)}(x)\diamond I_{f_M(e)}(y)]\diamond[I_{f_N(e)}(x)\diamond I_{f_N(e)}(y)]\\&=[I_{f_M(e)}(x)\diamond I_{f_M(e)}(y)]\diamond[I_{f_N(e)}(y)\diamond I_{f_N(e)}(x)]\ (\text{as } \diamond \text{ is commutative})\\&=I_{f_M(e)}(x)\diamond[I_{f_M(e)}(y)\diamond I_{f_N(e)}(y)]\diamond I_{f_N(e)}(x)\ (\text{as } \diamond \text{ is associative})\\&=I_{f_N(e)}(x)\diamond I_{f_P(e)}(y)\diamond I_{f_N(e)}(x)\\&=I_{f_M(e)}(x)\diamond I_{f_N(e)}(x)\diamond I_{f_P(e)}(y)\ (\text{as } \diamond \text{ is commutative})\\&=I_{f_P(e)}(x)\diamond I_{f_P(e)}(y).\end{aligned}$$

Thus, $I_{f_P(e)}(x+y) \leq I_{f_P(e)}(x) \diamond I_{f_P(e)}(y)$ and

$$I_{f_P(e)}(\lambda x) = I_{f_M(e)}(\lambda x) \diamond I_{f_N(e)}(\lambda x) \leq I_{f_M(e)}(x) \diamond I_{f_N(e)}(x) = I_{f_P(e)}(x),$$

i.e., $I_{f_P(e)}(\lambda x) \leq I_{f_P(e)}(x)$ for $\lambda \in F$.

Similarly, $F_{f_P(e)}(x+y) \leq F_{f_P(e)}(x) \diamond F_{f_P(e)}(y)$ and $F_{f_P(e)}(\lambda x) \leq F_{f_P(e)}(x)$.

This ends the theorem.

The theorem is also true for a family of NSLSs over $(V(K), E)$.

Remark 3.1 *For two NSLSs M and N over $(V(K), E)$, $M \cup N$ is not generally an NSLS over $(V(K), E)$. It is possible if any one is contained in other.*

For instance, let us consider two NSLSs M and N over the real vector space $V = \mathbf{R}^3$ and the parametric set $E = \{e_i | i = 1, 2, 3\}$ as following :

$$T_{f_M(e_i)}(x) = \begin{cases} 1/2, & \text{if } i\text{-th co-ordinate of } x \in \mathbf{R}^3 \text{ is nonzero only,} \\ 0, & \text{otherwise.} \end{cases}$$

$$I_{f_M(e_i)}(x) = \begin{cases} 0, & \text{if } i\text{-th co-ordinate of } x \in \mathbf{R}^3 \text{ is nonzero only,} \\ 1/4, & \text{otherwise.} \end{cases}$$

$$F_{f_M(e_i)}(x) = \begin{cases} 2/5, & \text{if } i\text{-th co-ordinate of } x \in \mathbf{R}^3 \text{ is nonzero only,} \\ 1, & \text{otherwise.} \end{cases}$$

$$T_{f_N(e_i)}(x) = \begin{cases} 2/5, & \text{if } i\text{-th co-ordinate of } x \in \mathbf{R}^3 \text{ is zero only,} \\ 1/10, & \text{otherwise.} \end{cases}$$

$$I_{f_N(e_i)}(x) = \begin{cases} 0, & \text{if } i\text{-th co-ordinate of } x \in \mathbf{R}^3 \text{ is zero only,} \\ 1/5, & \text{otherwise.} \end{cases}$$

$$F_{f_N(e_i)}(x) = \begin{cases} 1/6, & \text{if } i\text{-th co-ordinate of } x \in \mathbf{R}^3 \text{ is zero only,} \\ 1/3, & \text{otherwise.} \end{cases}$$

Corresponding t-norm $(*)$ and s-norm$(\diamond)$ are defined as $a*b = \min\{a, b\}$, $a \diamond b = \max\{a, b\}$. Let $M \cup N = P$. Then for $x = (1, 0, 0)$, $y = (0, 1, 1) \in \mathbf{R}^3$ and $e_1 \in E$,

$$T_{f_P(e_1)}(x+y) = T_{f_M(e_1)}(1,1,1) \diamond T_{f_N(e_1)}(1,1,1) = \max\{0, \frac{1}{10}\} = \frac{1}{10},$$

$$\begin{aligned} T_{f_P(e_1)}(x) * T_{f_P(e_1)}(y) &= \{T_{f_M(e_1)}(x) \diamond T_{f_N(e_1)}(x)\} * \{T_{f_M(e_1)}(y) \diamond T_{f_N(e_1)}(y)\} \\ &= \min[\max\{\frac{1}{2}, \frac{1}{10}\}, \max\{0, \frac{2}{5}\}] = \min\{\frac{1}{2}, \frac{2}{5}\} = \frac{2}{5}. \end{aligned}$$

Hence, $T_{f_P(e_1)}(x+y) < T_{f_P(e_1)}(x) * T_{f_P(e_1)}(y)$, i.e., $M \cup N$ is not an NSLS here.

Now, if we define N over $(\mathbf{R}^3, E)$ as following :

$$T_{f_N(e_i)}(x) = \begin{cases} 1/6, & \text{if } i\text{-th co-ordinate of } x \in \mathbf{R}^3 \text{ is zero only,} \\ 0, & \text{otherwise.} \end{cases}$$

$$I_{f_N(e_i)}(x) = \begin{cases} 3/4, & \text{if } i\text{-th co-ordinate of } x \in \mathbf{R}^3 \text{ is zero only,} \\ 1/2, & \text{otherwise.} \end{cases}$$

$$F_{f_N(e_i)}(x) = \begin{cases} 7/10, & \text{if } i\text{-th co-ordinate of } x \in \mathbf{R}^3 \text{ is zero only,} \\ 1, & \text{otherwise.} \end{cases}$$

Then, it can be easily verified that $N \subseteq M$ and $M \cup N$ is an NSLS over $(\mathbf{R}^3(\mathbf{R}), E)$.

Theorem 3.2 *Let M and N be two NSLSs over $(V(K), E)$. Then, $M \wedge N$ is also an NSLS over $(V(K), E \times E)$.*

Proof Let $M \wedge N = Q$. Then for $x, y \in V$ and $(a, b) \in E \times E$,

$$\begin{aligned}
T_{f_{Q(a,b)}}(x+y) &= T_{f_{M(a)}}(x+y) * T_{f_{N(b)}}(x+y) \\
&\geq [T_{f_{M(a)}}(x) * T_{f_{M(a)}}(y)] * [T_{f_{N(b)}}(x) * T_{f_{N(b)}}(y)] \\
&= [T_{f_{M(a)}}(x) * T_{f_{M(a)}}(y)] * [T_{f_{N(b)}}(y) * T_{f_{N(b)}}(x)] \text{ (as } * \text{ is commutative)} \\
&= T_{f_{M(a)}}(x) * [T_{f_{M(a)}}(y) * T_{f_{N(b)}}(y)] * T_{f_{N(b)}}(x) \text{ (as } * \text{ is associative)} \\
&= T_{f_{M(a)}}(x) * T_{f_{Q(a,b)}}(y) * T_{f_{N(b)}}(x) \\
&= T_{f_{M(a)}}(x) * T_{f_{N(b)}}(x) * T_{f_{Q(a,b)}}(y) \text{ (as } * \text{ is commutative)} \\
&= T_{f_{Q(a,b)}}(x) * T_{f_{Q(a,b)}}(y).
\end{aligned}$$

Hence, $T_{f_{Q(a,b)}}(x+y) \geq T_{f_{Q(a,b)}}(x) * T_{f_{Q(a,b)}}(y)$ and for $\lambda \in F$,
$T_{f_{Q(a,b)}}(\lambda x) = T_{f_{M(a)}}(\lambda x) * T_{f_{N(b)}}(\lambda x) \geq T_{f_{M(a)}}(x) * T_{f_{N(b)}}(x) = T_{f_{Q(a,b)}}(x)$.
Similarly, $I_{f_{Q(a,b)}}(x+y) \leq I_{f_{Q(a,b)}}(x) \diamond I_{f_{Q(a,b)}}(y)$, $I_{f_{Q(a,b)}}(\lambda x) \leq I_{f_{Q(a,b)}}(x)$ and
$F_{f_{Q(a,b)}}(x+y) \leq F_{f_{Q(a,b)}}(x) \diamond F_{f_{Q(a,b)}}(y)$, $F_{f_{Q(a,b)}}(\lambda x) \leq F_{f_{Q(a,b)}}(x)$.
This completes the proof.

The theorem is true for a family of NSLSs over $(V(K), E)$.

4. Cartesian Product of Neutrosophic Soft Linear Spaces

Here, the concept of cartesian product of NSLSs has been introduced along with a basic theorem and it's verification by an example.

Definition 4.1 *Let M and N be two NSLSs over $(V(K), E)$ and $(W(K), E)$, respectively. Then their cartesian product is $M \times N = S$ where $f_S(a, b) = f_M(a) \times f_N(b)$ for $(a, b) \in E \times E$. Analytically,*
$f_S(a, b) = \{< (x, y), T_{f_{S(a,b)}}(x, y), I_{f_{S(a,b)}}(x, y), F_{f_{S(a,b)}}(x, y) > |(x, y) \in V \times W\}$ *with*

$$\begin{cases}
T_{f_{S(a,b)}}(x, y) = T_{f_{M(a)}}(x) * T_{f_{N(b)}}(y), \\
I_{f_{S(a,b)}}(x, y) = I_{f_{M(a)}}(x) \diamond I_{f_{N(b)}}(y), \\
F_{f_{S(a,b)}}(x, y) = F_{f_{M(a)}}(x) \diamond F_{f_{N(b)}}(y).
\end{cases}$$

This definition can be extended for more than two NSLSs.

Theorem 4.1 *Let M and N be two NSLSs over $(V(K), E)$ and $(W(K), E)$, respectively. Then their cartesian product $M \times N$ is an NSLS over $([V \times W](K), E \times E)$.*

Proof Let $M \times N = S$ where $f_S(a, b) = f_M(a) \times f_N(b)$ for $(a, b) \in E \times E$. Then for

$(x_1, y_1), (x_2, y_2) \in V \times W$,

$$\begin{aligned}
T_{f_S(a,b)}[(x_1, y_1) + (x_2, y_2)] &= T_{f_S(a,b)}(x_1 + x_2, y_1 + y_2) \\
&= T_{f_M(a)}(x_1 + x_2) * T_{f_N(b)}(y_1 + y_2) \\
&\geq [T_{f_M(a)}(x_1) * T_{f_M(a)}(x_2)] * [T_{f_N(b)}(y_1) * T_{f_N(b)}(y_2)] \\
&= [T_{f_M(a)}(x_1) * T_{f_N(b)}(y_1)] * [T_{f_M(a)}(x_2) * T_{f_N(b)}(y_2)] \\
&= T_{f_S(a,b)}(x_1, y_1) * T_{f_S(a,b)}(x_2, y_2).
\end{aligned}$$

$$\begin{aligned}
I_{f_S(a,b)}[(x_1, y_1) + (x_2, y_2)] &= I_{f_S(a,b)}(x_1 + x_2, y_1 + y_2) \\
&= I_{f_M(a)}(x_1 + x_2) \diamond I_{f_N(b)}(y_1 + y_2) \\
&\leq [I_{f_M(a)}(x_1) \diamond I_{f_M(a)}(x_2)] \diamond [I_{f_N(b)}(y_1) \diamond I_{f_N(b)}(y_2)] \\
&= [I_{f_M(a)}(x_1) \diamond I_{f_N(b)}(y_1)] \diamond [I_{f_M(a)}(x_2) \diamond I_{f_N(b)}(y_2)] \\
&= I_{f_S(a,b)}(x_1, y_1) \diamond I_{f_S(a,b)}(x_2, y_2).
\end{aligned}$$

Similarly, $F_{f_S(a,b)}[(x_1, y_1) + (x_2, y_2)] \leq F_{f_S(a,b)}(x_1, y_1) \diamond F_{f_S(a,b)}(x_2, y_2)$.

Next,

$$\begin{aligned}
T_{f_S(a,b)}[\lambda(x_1, y_1)] &= T_{f_S(a,b)}(\lambda x_1, \lambda y_1) \\
&= T_{f_M(a)}(\lambda x_1) * T_{f_N(b)}(\lambda y_1) \\
&\geq T_{f_M(a)}(x_1) * T_{f_N(b)}(y_1) \\
&= T_{f_S(a,b)}(x_1, y_1).
\end{aligned}$$

Similarly,

$$I_{f_S(a,b)}[\lambda(x_1, y_1)] \leq I_{f_S(a,b)}(x_1, y_1) \text{ and } F_{f_S(a,b)}[\lambda(x_1, y_1)] \leq F_{f_S(a,b)}(x_1, y_1).$$

Hence, the theorem is proved.

Example 4.1 Consider the real vector space $V = \mathbf{R}^2$ and the parametric set $E = \{e_1, e_2\}$. We divide $\mathbf{R}^2$ in four halves *viz.* A : the origin, B : X-axis/A, C : Y-axis/A, D : $\mathbf{R}^2/\{A, B, C\}$. If $b \in B$ and $c \in C$, then $b + c \in D$ and so on.

We now define two NSSs M and N over $(\mathbf{R}^2, E)$ as given by Table 4 and Table 5.

Table 4 : Tabular form of NSS M.

	$f_N(e_1)$	$f_N(e_2)$
A	$(0.71, 0.27, 0.53)$	$(0.67, 0.43, 0.84)$
B	$(0.62, 0.32, 0.42)$	$(0.62, 0.31, 0.79)$
C	$(0.69, 0.31, 0.32)$	$(0.68, 0.21, 0.76)$
D	$(0.58, 0.41, 0.51)$	$(0.59, 0.42, 0.80)$

The t-norm $(*)$ and s-norm$(\diamond)$ are defined as : $a * b = ab$, $a \diamond b = a + b - ab$.

Then, M and N both form NSLSs over $(\mathbf{R}^2(\mathbf{R}), E)$. Their cartesian product $M \times N = S$ over $([\mathbf{R}^2 \times \mathbf{R}^2](\mathbf{R}), E \times E)$ is given in Table 6 (T, I, F being round off upto two decimal places).

Table 5 : Tabular form of NSS N.

	$f_N(e_1)$	$f_N(e_2)$
A	(0.79, 0.19, 0.41)	(0.42, 0.38, 0.61)
B	(0.84, 0.16, 0.25)	(0.32, 0.47, 0.49)
C	(0.72, 0.21, 0.16)	(0.25, 0.53, 0.51)
D	(0.69, 0.31, 0.29)	(0.59, 0.68, 0.73)

Table 6 : Tabular representation of $S = M \times N$.

	$f_S(e_1, e_1)$	$f_S(e_1, e_2)$	$f_S(e_2, e_1)$	$f_S(e_2, e_2)$
$A \times A$	(0.56, 0.41, 0.72)	(0.30, 0.55, 0.82)	(0.53, 0.54, 0.91)	(0.28, 0.65, 0.94)
$A \times B$	(0.60, 0.39, 0.65)	(0.23, 0.61, 0.76)	(0.56, 0.52, 0.88)	(0.21, 0.70, 0.92)
$A \times C$	(0.51, 0.42, 0.61)	(0.18, 0.66, 0.77)	(0.48, 0.55, 0.87)	(0.17, 0.73, 0.92)
$A \times D$	(0.49, 0.50, 0.67)	(0.42, 0.77, 0.87)	(0.46, 0.61, 0.89)	(0.40, 0.82, 0.96)
$B \times A$	(0.49, 0.45, 0.66)	(0.26, 0.58, 0.77)	(0.49, 0.44, 0.88)	(0.26, 0.57, 0.92)
$B \times B$	(0.52, 0.43, 0.57)	(0.20, 0.64, 0.70)	(0.52, 0.42, 0.84)	(0.20, 0.63, 0.89)
$B \times C$	(0.45, 0.46, 0.51)	(0.16, 0.68, 0.72)	(0.45, 0.45, 0.82)	(0.16, 0.68, 0.90)
$B \times D$	(0.43, 0.53, 0.59)	(0.37, 0.78, 0.84)	(0.42, 0.52, 0.85)	(0.37, 0.78, 0.94)
$C \times A$	(0.55, 0.44, 0.60)	(0.29, 0.57, 0.73)	(0.54, 0.36, 0.86)	(0.29, 0.51, 0.91)
$C \times B$	(0.58, 0.42, 0.49)	(0.22, 0.63, 0.65)	(0.57, 0.34, 0.82)	(0.22, 0.58, 0.88)
$C \times C$	(0.50, 0.45, 0.43)	(0.17, 0.68, 0.67)	(0.49, 0.38, 0.80)	(0.17, 0.63, 0.88)
$C \times D$	(0.48, 0.52, 0.52)	(0.40, 0.78, 0.82)	(0.47, 0.45, 0.83)	(0.40, 0.75, 0.94)
$D \times A$	(0.46, 0.52, 0.71)	(0.24, 0.63, 0.81)	(0.47, 0.53, 0.88)	(0.25, 0.64, 0.92)
$D \times B$	(0.48, 0.50, 0.63)	(0.19, 0.69, 0.75)	(0.50, 0.51, 0.85)	(0.19, 0.69, 0.90)
$D \times C$	(0.42, 0.53, 0.59)	(0.15, 0.72, 0.76)	(0.42, 0.54, 0.83)	(0.15, 0.73, 0.90)
$D \times D$	(0.40, 0.59, 0.65)	(0.34, 0.81, 0.87)	(0.41, 0.60, 0.86)	(0.35, 0.81, 0.95)

Clearly, S forms an NSLS over $([\mathbf{R}^3 \times \mathbf{R}^3](\mathbf{R}), E \times E)$. For the sake of convenience, one discussion is provided here.

Let $x_1 = (1, 0), x_2 = (-1, 0) \in B$ and $y_1 = (0, 1), y_2 = (0, -1) \in C$. Then,

$$(x_1, y_1) + (y_1, y_1), (x_1, y_1) + (y_2, y_1), (x_1, y_2) + (y_1, y_2), (x_1, y_2) + (y_2, y_2),$$
$$(x_2, y_1) + (y_1, y_1), (x_2, y_1) + (y_2, y_1), (x_2, y_2) + (y_1, y_2), (x_2, y_2) + (y_2, y_2) \in D \times C$$

and

$$(x_1, y_1) + (y_1, y_2), (x_1, y_1) + (y_2, y_2), (x_1, y_2) + (y_1, y_1), (x_1, y_2) + (y_2, y_1),$$
$$(x_2, y_1) + (y_1, y_2), (x_2, y_1) + (y_2, y_2), (x_2, y_2) + (y_1, y_1), (x_2, y_2) + (y_2, y_1) \in D \times A.$$

Thus $(B \times C) + (C \times C) = D \times A$ or $D \times C$. In any case, all the inequalities of Definition 3.1 are obvious.

5. Neutrosophic Soft Subspace

In this section, we have defined neutrosophic soft subspace with suitable examples. Then, we have discussed some related basic properties.

Definition 5.1 *Let N_1 and N_2 be two NSLSs over $(V(K), E)$. Then N_1 is neutrosophic soft subspace of N_2 if $N_1 \subseteq N_2$, i.e., $T_{f_{N_1}(e)}(x) \le T_{f_{N_2}(e)}(x), I_{f_{N_1}(e)}(x) \ge I_{f_{N_2}(e)}(x), F_{f_{N_1}(e)}(x) \ge F_{f_{N_2}(e)}(x)$, $\forall x \in V$, $\forall e \in E$.*

Example 5.1 Let us consider two NSLSs M and N over the real vector space $V = \mathbf{R}^3$ and the parametric set $E = \{e\}$ as following :

$$T_{f_{M(e)}}(x) = \begin{cases} 1/4, & \text{if } x \in \{(a, b, c) \in \mathbf{R}^3 : a + b + c = 0\}, \\ 0, & \text{otherwise.} \end{cases}$$

$$I_{f_{M(e)}}(x) = \begin{cases} 1/10, & \text{if } x \in \{(a, b, c) \in \mathbf{R}^3 : a + b + c = 0\}, \\ 1/2, & \text{otherwise.} \end{cases}$$

$$F_{f_{M(e)}}(x) = \begin{cases} 0, & \text{if } x \in \{(a, b, c) \in \mathbf{R}^3 : a + b + c = 0\}, \\ 1/6, & \text{otherwise.} \end{cases}$$

$$T_{f_{N(e)}}(x) = \begin{cases} 1/2, & \text{if } x \in \{(a, b, c) \in \mathbf{R}^3 : a + b + c = 0\}, \\ 2/7, & \text{otherwise.} \end{cases}$$

$$I_{f_{N(e)}}(x) = \begin{cases} 0, & \text{if } x \in \{(a, b, c) \in \mathbf{R}^3 : a + b + c = 0\}, \\ 1/3, & \text{otherwise.} \end{cases}$$

$$F_{f_{N(e)}}(x) = \begin{cases} 0, & \text{if } x \in \{(a, b, c) \in \mathbf{R}^3 : a + b + c = 0\}, \\ 1/9, & \text{otherwise.} \end{cases}$$

The t-norm $(*)$ and s-norm $(\diamond)$ are defined as $a * b$ =max$\{a + b - 1, 0\}$ and $a \diamond b$ =min$\{a + b, 1\}$. Then M is a neutrosophic soft subspaces of N over $(\mathbf{R}^3(\mathbf{R}), E)$.

Example 5.2 We consider the Example 3.2 and define another NSLS M over $(\mathbf{C}(\mathbf{R}), E)$ given by the Table 7.

Obviously, M is a neutrosophic soft subspace of N over $(\mathbf{C}(\mathbf{R}), E)$.

Corollary 5.1 *Let N be an NSLS over $(V(K), E)$. Then for arbitrary but fixed $\lambda \in K$, $\lambda N = \{(e, \lambda f_N(e)) | e \in E\}$ is also a neutrosophic soft linear space over $(V(K), E)$ where $\lambda f_N(e) = \{< \lambda x, T_{f_{N(e)}}(\lambda x), I_{f_{N(e)}}(\lambda x), F_{f_{N(e)}}(\lambda x) > | x \in V\}$. Moreover λN is a neutrosophic soft subspace of N.*

Proof Clearly $\lambda x \in V$ for $x \in V$, $\lambda \in K$. Since N be an NSLS over $(V(K), E)$, so by construction of λN,

$$\begin{cases} T_{f_{N(e)}}(\lambda x + \lambda y) \ge T_{f_{N(e)}}(\lambda x) * T_{f_{N(e)}}(\lambda y), \\ I_{f_{N(e)}}(\lambda x + \lambda y) \le I_{f_{N(e)}}(\lambda x) \diamond I_{f_{N(e)}}(\lambda y), \\ F_{f_{N(e)}}(\lambda x + \lambda y) \le F_{f_{N(e)}}(\lambda x) \diamond F_{f_{N(e)}}(\lambda y), \ \forall\, \lambda x, \lambda y \in V, \ \forall e \in E. \end{cases}$$

$$\begin{cases} T_{f_{N(e)}}(\mu(\lambda x)) \ge T_{f_{N(e)}}(\lambda x), \\ I_{f_{N(e)}}(\mu(\lambda x)) \le I_{f_{N(e)}}(\lambda x), \\ F_{f_{N(e)}}(\mu(\lambda x)) \le F_{f_{N(e)}}(\lambda x), \ \forall\, \lambda x \in V, \ \mu \in K, e \in E. \end{cases}$$

Hence, λN is a neutrosophic soft linear space over $(V(K), E)$. Next,

$$T_{f_N(e)}(x) = T_{f_N(e)}(\lambda^{-1}(\lambda x)) \geq T_{f_N(e)}(\lambda x),$$
$$I_{f_N(e)}(x) = I_{f_N(e)}(\lambda^{-1}(\lambda x)) \leq I_{f_N(e)}(\lambda x),$$
$$F_{f_N(e)}(x) = F_{f_N(e)}(\lambda^{-1}(\lambda x)) \leq F_{f_N(e)}(\lambda x),\ \forall\, \lambda(\neq 0) \in K, x \in V, e \in E.$$

Thus, λN is a neutrosophic soft subspace of N and this ends the proof.

Table 7 : Tabular form of NSLS M.

	$f_M(\alpha)$	$f_M(\beta)$	$f_M(\gamma)$
$(C1)$	$(0.59, 0.38, 0.62)$	$(0.63, 0.51, 0.79)$	$(0.70, 0.31, 0.32)$
$(C2)$	$(0.41, 0.49, 0.64)$	$(0.56, 0.63, 0.89)$	$(0.67, 0.41, 0.39)$
$(C3)$	$(0.56, 0.43, 0.68)$	$(0.45, 0.52, 0.88)$	$(0.60, 0.36, 0.48)$
$(C4)$	$(0.49, 0.50, 0.70)$	$(0.60, 0.49, 0.91)$	$(0.48, 0.52, 0.54)$

Corollary 5.2 *Let N be an NSLS over $(V(K), E)$. Then for arbitrary but fixed $\lambda, \mu \in K$, $\lambda N+\mu N = \{(e, (\lambda f_N+\mu f_N)(e))|e \in E\}$ is again an NSLS over $(V(K), E)$ where $(\lambda f_N+\mu f_N)(e) = \{< (\lambda x+\mu y), T_{f_N(e)}(\lambda x+\mu y), I_{f_N(e)}(\lambda x+\mu y), F_{f_N(e)}(\lambda x+\mu y) > |x, y \in V\}$. Moreover $(\lambda N + \mu N)$ is a neutrosophic soft subspace of N.*

Proof Since $V(K)$ is a vector space, so $x + y$, $\lambda x + \mu y \in V$ for $x, y \in V$ and $\lambda, \mu \in F$. Hence the proof is similar to the above corollary.

Corollary 5.3 *Let $f_N(e)$, $e \in E$ be a neutrosophic subspace on $V(K)$ where N is an NSLS over $(V(K), E)$. Then $\{\lambda f_N(e)|\lambda \in K\}$ is also a neutrosophic subspace on $V(K)$ where $\lambda f_N(e) = \{< \lambda x, T_{f_N(e)}(\lambda x), I_{f_N(e)}(\lambda x), F_{f_N(e)}(\lambda x) > |x \in V\}$.*

Proof It is obvious.

For instance, if $V = \{x, y, z\}$ and $K = \{\lambda, \mu\}$, then $\lambda x, \lambda y, \lambda z, \mu x, \mu y, \mu z \in V$ and so $\lambda x + \lambda x, \lambda x + \lambda y, \lambda x + \mu x, \lambda x + \mu y, \cdots \in V$. Now since $f_N(e)$, $e \in E$ is a neutrosophic subspace on $V(K)$, so all the inequalities hold good.

Theorem 5.1 *Let N be an NSLS over $(V(K), E)$ and N_1, N_2 be two neutrosophic soft subspaces of N. If T, I, F of neutrosophic soft linear space N obey the disciplines of idempotent t-Norm and idempotent s-norm, then,*
(i) *$N_1 \cap N_2$ is a neutrosophic soft subspace of N.*
(ii) *$N_1 \wedge N_2$ is a neutrosophic soft subspace of $N \wedge N$.*

Proof The intersection($\cap$), AND($\wedge$) of two NSLSs is also so by Theorems 3.1 and 3.2. Now to complete this theorem, we only verify the criteria of neutrosophic soft subspace in each case.

(i) Let $N_3 = N_1 \cap N_2$. For $x \in V$, $e \in E$,

$$T_{f_{N_3}(e)}(x) = T_{f_{N_1}(e)}(x) * T_{f_{N_2}(e)}(x) \leq T_{f_N(e)}(x) * T_{f_N(e)}(x) = T_{f_N(e)}(x),$$
$$I_{f_{N_3}(e)}(x) = I_{f_{N_1}(e)}(x) \diamond I_{f_{N_2}(e)}(x) \geq I_{f_N(e)}(x) \diamond I_{f_N(e)}(x) = I_{f_N(e)}(x),$$
$$F_{f_{N_3}(e)}(x) = F_{f_{N_1}(e)}(x) \diamond F_{f_{N_2}(e)}(x) \geq F_{f_N(e)}(x) \diamond F_{f_N(e)}(x) = F_{f_N(e)}(x).$$

(ii) Let $N_3 = N_1 \wedge N_2$ and $x \in V$, $(a,b) \in E \times E$. Then,

$$T_{f_{N_3}(a,b)}(x) = T_{f_{N_1}(a)}(x) * T_{f_{N_2}(b)}(x) \leq T_{f_N(a)}(x) * T_{f_N(b)}(x) = T_{f_N(a,b)}(x),$$
$$I_{f_{N_3}(a,b)}(x) = I_{f_{N_1}(a)}(x) \diamond I_{f_{N_2}(b)}(x) \geq I_{f_N(a)}(x) \diamond I_{f_N(b)}(x) = I_{f_N(a,b)}(x),$$
$$F_{f_{N_3}(a,b)}(x) = F_{f_{N_1}(a)}(x) \diamond F_{f_{N_2}(b)}(x) \geq F_{f_N(a)}(x) \diamond F_{f_N(b)}(x) = F_{f_N(a,b)}(x).$$

The theorems are also true for a family of neutrosophic soft subspaces of N.

6. Vectors in Neutrosophic Soft Linear Space

Here, first we have defined neutrosophic soft field and then neutrosophic soft vector, neutrosophic soft scalar, vector addition, scalar multiplication and have studied some related basic properties.

Definition 6.1 *A neutrosophic set $A = \{< x, T_A(x), I_A(x), F_A(x) > | x \in K\}$ over a field $(K, +, \cdot)$ is called a neutrosophic subfield of $(K, +, \cdot)$ if the followings hold.*

$$\text{(i)} \begin{cases} T_A(x+y) \geq T_A(x) * T_A(y), \\ I_A(x+y) \leq I_A(x) \diamond I_A(y), \\ F_A(x+y) \leq F_A(x) \diamond F_A(y), \ \forall\, x, y \in K. \end{cases}$$

$$\text{(ii)} \begin{cases} T_A(-x) \geq T_A(x), \\ I_A(-x) \leq I_A(x), \\ F_A(-x) \leq F_A(x), \ \forall\, x \in K. \end{cases}$$

$$\text{(iii)} \begin{cases} T_A(x.y) \geq T_A(x) * T_A(y), \\ I_A(x.y) \leq I_A(x) \diamond I_A(y), \\ F_A(x.y) \leq F_A(x) \diamond F_A(y), \ \forall\, x, y \in K. \end{cases}$$

$$\text{(iv)} \begin{cases} T_A(x^{-1}) \geq T_A(x), \\ I_A(x^{-1}) \leq I_A(x), \\ F_A(x^{-1}) \leq F_A(x), \ \forall\, x(\neq 0) \in K. \end{cases}$$

An NSS N over a field $(K, +, \cdot)$ is called a neutrosophic soft field if $f_N(e)$ is a neutrosophic subfield of $(K, +, \cdot)$ for each $e \in E$.

Example 6.1 Let $E = \mathbf{N}$ (the set of natural numbers) be the parametric set and $K = (\mathbf{R}, +, \cdot)$ be the field of all rational number. Define a mapping $f_M : \mathbf{N} \to NS(\mathbf{R})$ where, for any $n \in \mathbf{N}$ and $x \in \mathbf{R}$,

$$T_{f_M(n)}(x) = \begin{cases} 0, & \text{if } x \text{ is rational}, \\ 1/n, & \text{if } x \text{ is irrational}. \end{cases}$$

$$I_{f_M(n)}(x) = \begin{cases} 1/2n, & \text{if } x \text{ is rational}, \\ 0, & \text{if } x \text{ is irrational}. \end{cases}$$

$$F_{f_M(n)}(x) = \begin{cases} 1 - 1/n, & \text{if } x \text{ is rational}, \\ 0, & \text{if } x \text{ is irrational}. \end{cases}$$

The t-norm $(*)$ and s-norm $(\diamond)$ are defined as $a*b = \min\{a, b\}$, $a \diamond b = \max\{a, b\}$. Then, M forms an NSS as well as a neutrosophic soft field over $[(\mathbf{R}, +, \cdot), \mathbf{N}]$.

Example 6.2 Let us consider the field $\mathbf{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$ and $E = \{e_1, e_2, e_3, e_4\}$ be the set of parameters. We define $f_N(e_1), f_N(e_2), f_N(e_3), f_N(e_4)$ as given by the following Table 8.

Table 8 : Tabular form of neutrosophic soft field N.

	$f_N(e_1)$	$f_N(e_2)$	$f_N(e_3)$	$f_N(e_4)$
$\overline{0}$	$(0.65, 0.34, 0.14)$	$(0.88, 0.12, 0.72)$	$(0.72, 0.21, 0.16)$	$(0.69, 0.31, 0.32)$
$\overline{1}$	$(0.71, 0.22, 0.78)$	$(0.71, 0.19, 0.44)$	$(0.84, 0.16, 0.25)$	$(0.62, 0.32, 0.42)$
$\overline{2}$	$(0.75, 0.25, 0.52)$	$(0.83, 0.11, 0.28)$	$(0.69, 0.31, 0.39)$	$(0.58, 0.41, 0.66)$

Corresponding t-norm $(*)$ and s-norm $(\diamond)$ are defined as $a * b = \max\{a + b - 1, 0\}$, $a \diamond b = \min\{a + b, 1\}$. Then, N forms a neutrosophic soft field over $[(\mathbf{Z}_3, +, \cdot), E]$.

Definition 6.2 *A neutrosophic soft scalar in a neutrosophic soft field N over (K, E), K being scalar field, is defined by a neutrosophic soft element $(e, f_N(e))$ of N for $e \in E$ and is denoted by $\hat{e}_N$.*

A neutrosophic soft scalar $\hat{e}_N \in M$, M being a neutrosophic soft field over (K, E) if $f_N(e) \leq f_M(e)$, i.e.,

$$T_{f_N(e)}(x) \leq T_{f_M(e)}(x),\ I_{f_N(e)}(x) \geq I_{f_M(e)}(x),\ F_{f_N(e)}(x) \geq F_{f_M(e)}(x),\ \forall x \in V.$$

Example 6.3 In the Example 6.2, there are four neutrosophic soft elements $(e_1, f_N(e_1))$, $(e_2, f_N(e_2)), (e_3, f_N(e_3)), (e_4, f_N(e_4))$ of the neutrosophic soft field N over (K, E). Hence, the neutrosophic soft scalars in N are $\hat{e}_{1N}, \hat{e}_{2N}, \hat{e}_{3N}, \hat{e}_{4N}$.

Definition 6.3 *A neutrosophic soft vector in an NSLS N over $(V(K), E)$ is defined by a neutrosophic soft element $(e, f_N(e))$ of N for $e \in E$ and is denoted by $\vec{e}_N$.*

A neutrosophic soft vector $\vec{e}_N \in M$, M being a neutrosophic soft linear space over $(V(K), E)$ if $f_N(e) \leq f_M(e)$.

Example 6.4 Consider the Example 3.1. Take the points $x_i \in \mathbf{R}^n$, $1 \leq i \leq n$ (whose i-th co-ordinate is zero) and $y \in \mathbf{R}^n$ (otherwise). Then,

$$\begin{aligned} f_N(e_1) &= \{< x_1, (1/2, 0, 0) >, < y, (0, 1/4, 1/10) >\}, \\ f_N(e_2) &= \{< x_2, (1/2, 0, 0) >, < y, (0, 1/4, 1/10) >\}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_N(e_n) &= \{< x_n, (1/2, 0, 0) >, < y, (0, 1/4, 1/10) >\}. \end{aligned}$$

are the neutrosophic soft elements in the neutrosophic soft linear space N. Hence, the neutrosophic soft vectors in N are $\vec{e}_{1N}, \vec{e}_{2N}, \cdots, \vec{e}_{nN}$.

Definition 6.4 *A neutrosophic soft vector $\vec{e}_N$ in an NSLS N over $(V(K), E)$ is called a null neutrosophic soft vector denoted by Θ if $e_N \in \phi_v$. Otherwise it is non-null neutrosophic soft vector. The zero element of the neutrosophic soft field N over (K, E) is denoted by $\hat{\phi}_{e_N}$.*

A neutrosophic soft vector $\vec{e}_N$ in an NSLS N over $(V(K), E)$ is called an absolute *neutrosophic soft vector denoted by* Υ *if* $e_N \in 1_v$. *The absolute element of the neutrosophic soft field N over* (K, E) *is denoted by* $\hat{1}_{e_N}$.

If e_N *be a neutrosophic soft element of an NSS N over* (U, E), *then*

$$-e_N = \{< x, (F_{f_N(e)}(x), 1 - I_{f_N(e)}(x), T_{f_N(e)}(x)) > | x \in U\}.$$

Obviously, $-\Theta = \Upsilon$, $-\Upsilon = \Theta$ and $-\hat{\phi}_{e_N} = \hat{1}_{e_N}$, $-\hat{1}_{e_N} = \hat{\phi}_{e_N}$.

Definition 6.5 *Let* $\vec{e}_{1N}, \vec{e}_{2N}$ *be two neutrosophic soft vectors in an NSLS N over* $(V(K), E)$ *and* $\hat{e}_M$ *be a neutrosophic soft scalar in a neutrosophic soft field M over* (K, E). *Then the vector addition* $\vec{e}_{1N} \oplus \vec{e}_{2N}$ *and the scalar multiplication* $\hat{e}_M \odot \vec{e}_{1N}$ *are respectively defined as :*

$$\{< x, T_{f_N(e_1)}(x) * T_{f_N(e_2)}(x), I_{f_N(e_1)}(x) \diamond I_{f_N(e_2)}(x), F_{f_N(e_1)}(x) \diamond F_{f_N(e_2)}(x) > | x \in V\},$$
$$\{< (\mu, x), T_{f_M(e)}(\mu) * T_{f_N(e_1)}(x), I_{f_M(e)}(\mu) \diamond I_{f_N(e_1)}(x), F_{f_M(e)}(\mu) \diamond F_{f_N(e_1)}(x) > | (\mu, x) \in K \times V\}.$$

Example 6.5 We consider the Example 6.4. Then,

$$\vec{e}_{1N} \oplus \vec{e}_{2N} = \{< x_1, (0, 1/4, 1/10) >, < x_2, (0, 1/4, 1/10) >, < y, (0, 1/4, 1/10) >\}.$$

The t-norm ($*$) and s-norm ($\diamond$) are defined as $a * b = \min\{a, b\}$, $a \diamond b = \max\{a, b\}$.

Example 6.6 In the Example 6.2, $(\mathbf{Z}_3(\mathbf{Z}_3), +, \cdot)$ is a vector space and so N is an NSLS defined by same table also over $[(\mathbf{Z}_3(\mathbf{Z}_3), +, \cdot), E]$. Then,

$$\vec{e}_{1N} \oplus \vec{e}_{2N} = \{< \overline{0}, (0.53, 0.46, 0.86) >, < \overline{1}, (0.42, 0.41, 1) >, < \overline{2}, (0.58, 0.36, 0.80) >\},$$
$$\begin{aligned}\hat{e}_{1N} \odot \vec{e}_{2N} = \{&< (\overline{0}, \overline{0}), (.53, .46, .86) >, < (\overline{0}, \overline{1}), (.36, .53, .58) >, < (\overline{0}, \overline{2}), (.48, .45, .42) \\ &>< (\overline{1}, \overline{0}), (.59, .34, 1) >, < (\overline{1}, \overline{1}), (.42, .41, 1) >, < (\overline{1}, \overline{2}), (.54, .33, 1) > \\ &< (\overline{2}, \overline{0}), (.63, .37, 1) >, < (\overline{2}, \overline{1}), (.46, .44, .96) >, < (\overline{2}, \overline{2}), (.58, .36, .8) >\}.\end{aligned}$$

Corresponding t-norm ($*$) and s-norm ($\diamond$) are defined as $a * b = \max\{a + b - 1, 0\}$, $a \diamond b = \min\{a + b, 1\}$.

Theorem 6.1 *Let N be an NSLS over* $(V(K), E)$ *and M be a neutrosophic soft field over* (K, E). *Then, the followings hold.*

(i) $\Theta \oplus \vec{e}_N = \Theta,\ \forall \vec{e}_N \in N$.
(ii) $\Upsilon \oplus \vec{e}_N = \vec{e}_N,\ \forall \vec{e}_N \in N$.
(iii) $\hat{\phi}_M \odot \vec{e}_N = \Theta,\ \forall \vec{e}_N \in N$.
(iv) $\hat{e}_M \odot \Theta = \Theta,\ \forall \tilde{e}_M \in M$.
(v) $\hat{1}_M \odot \vec{e}_N = \vec{e}_N,\ \forall \vec{e}_N \in N$.
(vi) $\hat{e}_M \odot \Upsilon = \vec{e}_M,\ \forall \tilde{e}_M \in M$.
(vii) $-\hat{1}_M \odot \vec{e}_N = -\Upsilon,\ \forall \vec{e}_N \in N$.

Proof For all the proofs, we shall use the Definition 6.5.

(i) $\Theta \oplus \vec{e}_N$

$= \{< x, (0 * T_{f_N(e)}(x), 1 \diamond I_{f_N(e)}(x), 1 \diamond F_{f_N(e)}(x)) > | \ x \in V\}$

$= \{< x, (0, 1, 1) > |x \in V\}$

$= \Theta.$

(ii) $\Upsilon \oplus \vec{e}_N$

$= \{< x, (1 * T_{f_N(e)}(x), 0 \diamond I_{f_N(e)}(x), 0 \diamond F_{f_N(e)}(x)) > | \ x \in V\}$

$= \{< x, (T_{f_N(e)}(x), I_{f_N(e)}(x), F_{f_N(e)}(x)) > | \ x \in V\}$

$= \vec{e}_N.$

(iii) $\hat{\phi}_M \odot \vec{e}_N$

$= \{< (\mu, x), (0 * T_{f_N(e)}(x), 1 \diamond I_{f_N(e)}(x), 1 \diamond F_{f_N(e)}(x)) > | \ (\mu, x) \in K \times V\}$

$= \{< (\mu, x), (0, 1, 1) > | \ (\mu, x) \in K \times V\}$

$= \Theta.$

(iv) $\hat{e}_M \odot \Theta$

$= \{< (\mu, x), (T_{f_M(e)}(\mu) * 0, I_{f_M(e)}(\mu) \diamond 1, F_{f_M(e)}(\mu) \diamond 1) > | \ (\mu, x) \in K \times V\}$

$= \{< (\mu, x), (0, 1, 1) > | \ (\mu, x) \in K \times V\}$

$= \Theta.$

(v) $\hat{1}_M \odot \vec{e}_N$

$= \{< (\mu, x), (1 * T_{f_N(e)}(x), 0 \diamond I_{f_N(e)}(x), 0 \diamond F_{f_N(e)}(x)) > | \ (\mu, x) \in K \times V\}$

$= \{< (\mu, x), (T_{f_N(e)}(x), I_{f_N(e)}(x), F_{f_N(e)}(x)) > | \ (\mu, x) \in K \times V\}$

$= \vec{e}_N.$

(vi) $\hat{e}_M \odot \Upsilon$

$= \{< (\mu, x), (T_{f_M(e)}(\mu) * 1, I_{f_M(e)}(\mu) \diamond 0, F_{f_M(e)}(\mu) \diamond 0) > | \ (\mu, x) \in K \times V\}$

$= \{< (\mu, x), (T_{f_M(e)}(\mu), I_{f_M(e)}(\mu), F_{f_M(e)}(\mu)) > | \ (\mu, x) \in K \times V\}$

$= \vec{e}_M.$

(vii) $-\tilde{1}_M = \{< \mu, (0, 1, 1) > | \ \mu \in K\}$ and so,

$-\hat{1}_M \odot \vec{e}_N$

$= \{< (\mu, x), (0 * T_{f_N(e)}(\mu), 1 \diamond I_{f_N(e)}(\mu), 1 \diamond F_{f_N(e)}(\mu)) > | \ (\mu, x) \in K \times V\}$

$= \{< (\mu, x), (0, 1, 1) > | \ (\mu, x) \in K \times V\}$

$= -\Upsilon = \Theta.$

Remark 6.1 However, $\hat{e}_M \odot \vec{e}_N = \Theta$ does not necessarily imply that either $\hat{e}_M = \hat{\phi}_{e_M}$ or $\vec{e}_N = \Theta$.

For example, if $\hat{e}_M = \{< \mu, \ (0, 0.2, 1) >, < \nu, \ (0, 0.4, 1) >\}$ and $\vec{e}_N = \{< x, (0.5, 1, 0.1) >, \ < y, (0, 1, 0.6) >\}$, then $\hat{e}_M \odot \vec{e}_N = \{< (\mu, x), (0, 1, 1) >, \ < (\mu, y), (0, 1, 1) >, < (\nu, x), (0, 1, 1) >, < (\nu, y), (0, 1, 1) >\} = \Theta$ with respect to $a * b = \min\{a, b\}$, $a \diamond$

b =max$\{a, b\}$ but neither $\hat{e}_M = \hat{\phi}_{e_M}$ nor $\vec{e}_N = \Theta$.

Definition 6.6 *Let $\vec{e}_{1N}, \vec{e}_{2N}, \cdots, \vec{e}_{nN}$ be a finite number of neutrosophic soft vectors in an NSLS N over $(V(K), E)$. Then for a finite number of neutrosophic soft scalars $\hat{e}_{1M}, \hat{e}_{2M}, \cdots, \hat{e}_{nM}$ in a neutrosophic soft field M over (K, E), the expression $(\hat{e}_{1M} \odot \vec{e}_{1N}) \oplus (\hat{e}_{2M} \odot \vec{e}_{2N}) \oplus \cdots \oplus (\hat{e}_{nM} \odot \vec{e}_{nN})$ is called a linear combination of the respective neutrosophic soft vectors.*

Example 6.7 In the Example 6.2, $(\mathbf{Z}_3(\mathbf{Z}_3), +, \cdot)$ is a vector space and so N is an NSLS defined by same table also over $[(\mathbf{Z}_3(\mathbf{Z}_3), +, \cdot), E]$. Here, the neutrosophic soft vectors and the neutrosophic soft scalars are $\{\vec{e}_{1N}, \vec{e}_{2N}, \vec{e}_{3N}, \vec{e}_{4N}\}$ and $\{\hat{e}_{1N}, \hat{e}_{2N}, \hat{e}_{3N}, \hat{e}_{4N}\}$, respectively. Then,

Table 9 : Tabular form of scalar multiplication on N.

	$\hat{e}_{1N} \odot \vec{e}_{1N}$	$\hat{e}_{2N} \odot \vec{e}_{2N}$	$\hat{e}_{3N} \odot \vec{e}_{3N}$	$\hat{e}_{4N} \odot \vec{e}_{4N}$
$(\bar{0}, \bar{0})$	(0.65, 0.34, 0.14)	(0.88, 0.12, 0.72)	(0.72, 0.21, 0.16)	(0.69, 0.31, 0.32)
$(\bar{0}, \bar{1})$	(0.65, 0.34, 0.78)	(0.71, 0.19, 0.72)	(0.72, 0.21, 0.25)	(0.62, 0.32, 0.42)
$(\bar{0}, \bar{2})$	(0.65, 0.34, 0.52)	(0.83, 0.12, 0.72)	(0.69, 0.31, 0.39)	(0.58, 0.41, 0.66)
$(\bar{1}, \bar{0})$	(0.65, 0.34, 0.78)	(0.71, 0.19, 0.72)	(0.72, 0.21, 0.25)	(0.62, 0.32, 0.42)
$(\bar{1}, \bar{1})$	(0.71, 0.22, 0.78)	(0.71, 0.19, 0.44)	(0.84, 0.16, 0.25)	(0.62, 0.32, 0.42)
$(\bar{1}, \bar{2})$	(0.75, 0.25, 0.78)	(0.71, 0.19, 0.44)	(0.69, 0.31, 0.39)	(0.58, 0.41, 0.66)
$(\bar{2}, \bar{0})$	(0.65, 0.34, 0.52)	(0.83, 0.12, 0.72)	(0.69, 0.31, 0.39)	(0.58, 0.41, 0.66)
$(\bar{2}, \bar{1})$	(0.71, 0.25, 0.78)	(0.71, 0.19, 0.44)	(0.69, 0.31, 0.39)	(0.58, 0.41, 0.66)
$(\bar{2}, \bar{2})$	(0.75, 0.25, 0.52)	(0.83, 0.11, 0.28)	(0.69, 0.31, 0.39)	(0.58, 0.41, 0.66)

Hence,

$$
\begin{aligned}
&(\hat{e}_{1N} \odot \vec{e}_{1N}) \oplus (\hat{e}_{2N} \odot \vec{e}_{2N}) \oplus (\hat{e}_{3N} \odot \vec{e}_{3N}) \oplus (\hat{e}_{4N} \odot \vec{e}_{4N}) \\
&= \{< (\bar{0}, \bar{0}), (.65, .34, .72) >, < (\bar{0}, \bar{1}), (.62, .34, .78) >, < (\bar{0}, \bar{2}), (.58, .41, .72) >, \\
&\quad < (\bar{1}, \bar{0}), (.62, .34, .78) >, < (\bar{1}, \bar{1}), (.62, .32, .78) >, < (\bar{1}, \bar{2}), (.58, .41, .78) >, \\
&\quad < (\bar{2}, \bar{0}), (.58, .41, .72) >, < (\bar{2}, \bar{1}), (.58, .41, .78) >, < (\bar{2}, \bar{2}), (.58, .41, .66) >\}.
\end{aligned}
$$

The t-norm $(*)$ and s-norm $(\diamond)$ are defined as $a * b$ =min$\{a, b\}, a \diamond b$ =max$\{a, b\}$.

Example 6.8 We consider the NSLS defined in Example 3.3 and the neutrosophic soft field described in Example 6.1. Suppose, $\{\vec{e}_{1M}, \vec{e}_{2M}, \vec{e}_{3M}, \vec{e}_{4M}\}$ is a finite set of neutrosophic soft vectors and $\{\hat{e}_{1M}, \hat{e}_{2M}, \hat{e}_{3M}, \hat{e}_{4M}\}$ is a finite set of neutrosophic soft scalars. In the following tables (Tables 10, 11 and 12), $a \in \mathbf{R}^3$ and q, r denote rational and irrational number, respectively.

Hence,

$$
\begin{aligned}
&(\hat{e}_{1M} \odot \vec{e}_{1M}) \oplus (\hat{e}_{2M} \odot \vec{e}_{2M}) \oplus (\hat{e}_{3M} \odot \vec{e}_{3M}) \oplus (\hat{e}_{4M} \odot \vec{e}_{4M}) \\
&= \{< (q, a), (0, 1/2, 3/4) >, < (r, a), (1/4, 1/2, 3/4) >\}.
\end{aligned}
$$

The t-norm $(*)$ and s-norm $(\diamond)$ are defined as $a * b$ =min$\{a, b\}, a \diamond b$ =max$\{a, b\}$.

Table 10 : Tabular form of NSLS M.

	$\vec{e}_{1M}$	$\vec{e}_{2M}$	$\vec{e}_{3M}$	$\vec{e}_{4M}$
a	$(1, 1/2, 0)$	$(1/2, 1/4, 1/2)$	$(1/3, 1/6, 2/3)$	$(1/4, 1/8, 3/4)$

Table 11 : Tabular form of neutrosophic soft field M.

	$\hat{e}_{1M}$	$\hat{e}_{2M}$	$\hat{e}_{3M}$	$\hat{e}_{4M}$
q	$(0, 1/2, 0)$	$(0, 1/4, 1/2)$	$(0, 1/6, 2/3)$	$(0, 1/8, 3/4)$
r	$(1, 0, 0)$	$(1/2, 0, 0)$	$(1/3, 0, 0)$	$(1/4, 0, 0)$

Table 12 : Tabular form of scalar multiplication on M.

	$\hat{e}_{1M} \odot \vec{e}_{1M}$	$\hat{e}_{2M} \odot \vec{e}_{2M}$	$\hat{e}_{3M} \odot \vec{e}_{3M}$	$\hat{e}_{4M} \odot \vec{e}_{4M}$
(q, a)	$(0, 1/2, 0)$	$(0, 1/4, 1/2)$	$(0, 1/6, 2/3)$	$(0, 1/8, 3/4)$
(r, a)	$(1, 1/2, 0)$	$(1/2, 1/4, 1/2)$	$(1/3, 1/6, 2/3)$	$(1/4, 1/8, 3/4)$

Proposition 6.1 *Let N be an NSLS over $(V(K), E)$ and M be a neutrosophic soft field over (K, E). Then, the followings hold.*
(i) $\vec{e}_{1N} \oplus \vec{e}_{2N} \in N \wedge N,\ \forall \vec{e}_{1N}, \vec{e}_{2N} \in N$.
(ii) $\hat{e}_{1M} \odot \vec{e}_{2N} \in N,\ \forall \hat{e}_{1M} \in M,\ \forall \vec{e}_{2N} \in N$.
(iii) $(\hat{e}_{1M} \odot \vec{e}_{1N}) \oplus (\hat{e}_{2M} \odot \vec{e}_{2N}) \in N \wedge N,\ \forall \hat{e}_{1M}, \hat{e}_{2M} \in M,\ \forall \vec{e}_{1N}, \vec{e}_{2N} \in N$.

Proof For all the proofs, we shall use the Definition 6.5.
(i) $\vec{e}_{1N} \oplus \vec{e}_{2N} = f_N(e_1) \cap f_N(e_2) \in N \wedge N$.
(ii) $\forall \mu \in K$ and $\forall x \in V$, we have $\mu x \in V$. Now, for $e_1, e_2 \in E$, $\hat{e}_{1M} \odot \vec{e}_{2N} = f_M(e_1) \cap f_N(e_2) = f_N(e_2) \in N$, by the sense of Definition 6.3.
(iii) It directly follows from above two cases.

Remark 6.2 *Clearly, $\hat{e}_M \odot \vec{e}_{1N}$ is a neutrosophic soft vector in the NSLS N over $([K \times V](K), E)$ and $(\hat{e}_{1M} \odot \vec{e}_{1N}) \oplus (\hat{e}_{2M} \odot \vec{e}_{2N})$ is also a neutrosophic soft vector in the NSLS $N \wedge N$ over $([K \times V](K), E \times E)$.*

Definition 6.7 *A finite set of neutrosophic soft vectors $\{\vec{e}_{1N}, \vec{e}_{2N}, \cdots, \vec{e}_{nN}\}$ in an NSLS N over $(V(K), E)$ is called linearly dependent if there exists neutrosophic soft scalars $\{\hat{e}_{1M}, \hat{e}_{2M}, \cdots, \hat{e}_{nM}\}$ not all zero elements (i.e., not all $\hat{\phi}_{e_M}$) in a neutrosophic soft field M over (K, E) such that $(\hat{e}_{1M} \odot \vec{e}_{1N}) \oplus (\hat{e}_{2M} \odot \vec{e}_{2N}) \oplus \cdots \oplus (\hat{e}_{nM} \odot \vec{e}_{nN}) = \Theta$, the null neutrosophic soft vector of $(N \wedge N \wedge \cdots \wedge N)$, n times.*

For all non-null vectors, if the above identity implies $\hat{e}_{1M} = \hat{e}_{2M} = \cdots = \hat{e}_{nM} = \hat{\phi}_{e_M}$, then the set of neutrosophic soft vectors is called linearly independent in N.

Example 6.9 We consider the NSLS N over $[(\mathbf{Z}_2(\mathbf{Z}_2), +, \cdot), E]$ and the neutrosophic soft field M over $[(\mathbf{Z}_2, +, \cdot), E]$ for $E = \{e_1, e_2, e_3\}$ defined as in the following tables (Tables 13, 14 and 15).

Table 13 : Tabular form of NSLS N.

	$\vec{e}_{1N}$	$\vec{e}_{2N}$	$\vec{e}_{3N}$
$\overline{0}$	$(0.4, 0.5, 0.7)$	$(0.7, 0.2, 0.3)$	$(0.8, 0.6, 0.3)$
$\overline{1}$	$(0.6, 0.3, 0.8)$	$(0.5, 0.3, 0.4)$	$(0.4, 0.5, 0.5)$

Table 14 : Tabular form of neutrosophic soft field M.

	$\hat{e}_{1M}$	$\hat{e}_{2M}$	$\hat{e}_{3M}$
$\overline{0}$	$(0.7, 0.4, 0.3)$	$(0.6, 0.5, 0.4)$	$(0.3, 0.5, 0.4)$
$\overline{1}$	$(0.5, 0.7, 0.2)$	$(0.7, 0.3, 0.2)$	$(0.4, 0.6, 0.7)$

Table 15 : Tabular form of scalar multiplication on N.

	$\hat{e}_{1M} \odot \vec{e}_{1N}$	$\hat{e}_{3M} \odot \vec{e}_{2N}$	$\hat{e}_{2M} \odot \vec{e}_{3N}$
$(\overline{0}, \overline{0})$	$(0.1, 0.9, 1)$	$(0, 0.7, 0.7)$	$(0.4, 1, 0.7)$
$(\overline{0}, \overline{1})$	$(0.3, 0.7, 1)$	$(0, 0.8, 0.8)$	$(0, 1, 0.9)$
$(\overline{1}, \overline{0})$	$(0, 1, 0.9)$	$(0.1, 0.8, 1)$	$(0.5, 0.9, 0.5)$
$(\overline{1}, \overline{1})$	$(0.1, 1, 1)$	$(0, 0.9, 1)$	$(0.1, 0.8, 0.7)$

Now,

$$\begin{aligned}
&(\hat{e}_{1M} \odot \vec{e}_{1N}) \oplus (\hat{e}_{3M} \odot \vec{e}_{2N}) \oplus (\hat{e}_{2M} \odot \vec{e}_{3N}) \\
&= \{< (\overline{0}, \overline{0}), (0, 1, 1) >, < (\overline{0}, \overline{1}), (0, 1, 1) >, < (\overline{1}, \overline{0}), (0, 1, 1) >, < (\overline{1}, \overline{1}), (0, 1, 1) >\} \\
&= \Theta,
\end{aligned}$$

but none of the neutrosophic soft scalars is $\hat{\phi}_{e_M}$ i.e., $\{\vec{e}_{1N}, \vec{e}_{2N}, \vec{e}_{3N}\}$ is linearly dependent in N. Corresponding t-Norm $(*)$ and s-Norm $(\diamond)$ are defined as $a * b = \max\{a + b - 1, 0\}, a \diamond b = \min\{a + b, 1\}$.

Example 6.10 The absolute neutrosophic soft vector Υ and the null neutrosophic soft vector Θ defined in any NSLS over $(V(K), E)$ are linearly independent and dependent vector, respectively.

Definition 6.8 *If $\hat{e}_{1M}, \hat{e}_{1P}, \hat{e}_{2Q}$ be three neutrosophic soft scalars over (K, E), then $\hat{e}_{1M} \oplus \hat{e}_{1P}$ and $\hat{e}_{1M} \odot \hat{e}_{2Q}$ are defined, respectively, as :*

$$\begin{aligned}
&\{< \mu, [T_{f_{M(e_1)}}(\mu) * T_{f_{P(e_1)}}(\mu), I_{f_{M(e_1)}}(\mu) \diamond I_{f_{P(e_1)}}(\mu), F_{f_{M(e_1)}}(\mu) \diamond F_{f_{P(e_1)}}(\mu)] > | \mu \in K\}, \\
&\{< \mu, [T_{f_{M(e_1)}}(\mu) * T_{f_{Q(e_2)}}(\mu), I_{f_{M(e_1)}}(\mu) \diamond I_{f_{Q(e_2)}}(\mu), F_{f_{M(e_1)}}(\mu) \diamond F_{f_{Q(e_2)}}(\mu)] > | \mu \in K\}.
\end{aligned}$$

Clearly, $\hat{e}_{1M} \oplus \hat{e}_{1P}$ and $\hat{e}_{1M} \odot \hat{e}_{2Q}$ are also neutrosophic soft scalars belonging to $M \cap P$ and $M \wedge Q$, respectively.

Theorem 6.2 *Let $S = \{\vec{e}_{1N}, \vec{e}_{2N}, \cdots, \vec{e}_{nN}\}$ be a finite set of neutrosophic soft vectors in an NSLS N over $(V(K), E)$. Then the collection of all linear combinations of the vectors in S forms a subspace of the NSLS $(N \wedge N \wedge \cdots \wedge N)$, n times, over $[K \times V](K)$ in the sense of classical set theory.*

Proof Let $W = \{(\hat{e}_{1M} \odot \vec{e}_{1N}) \oplus (\hat{e}_{2M} \odot \vec{e}_{2N}) \oplus \cdots \oplus (\hat{e}_{nM} \odot \vec{e}_{nN}) | \hat{e}_{1M}, \hat{e}_{2M}, \cdots, \hat{e}_{nM}$ are neutrosophic soft scalars over $(K, E)\}$. Suppose,

$$\begin{aligned}
\vec{e}_{N_1} &= (\hat{e}_{1M} \odot \vec{e}_{1N}) \oplus (\hat{e}_{2M} \odot \vec{e}_{2N}) \oplus \cdots \oplus (\hat{e}_{nM} \odot \vec{e}_{nN}), \\
\vec{e}_{N_1} &= (\hat{e}_{1M} \odot \vec{e}_{1N}) \oplus (\hat{e}_{2M} \odot \vec{e}_{2N}) \oplus \cdots \oplus (\hat{e}_{nM} \odot \vec{e}_{nN}).
\end{aligned}$$

Then,

$$\begin{aligned}
&\vec{e}_{N_1} \oplus \vec{e}_{N_2} \\
&= [(\hat{e}_{1M} \oplus \hat{e}_{1P}) \odot \vec{e}_{1N}] \oplus \cdots \oplus [(\hat{e}_{nM} \oplus \hat{e}_{nP}) \odot \vec{e}_{nN}] \\
&= (\hat{e}_{1T} \odot \vec{e}_{1N}) \oplus \cdots \oplus (\hat{e}_{nT} \odot \vec{e}_{nN}), \text{ for } \hat{e}_{iT} = \hat{e}_{iM} \oplus \hat{e}_{iP}, 1 \leq i \leq n
\end{aligned}$$

and for a neutrosophic soft scalar $\hat{e}_Q$ over (K, E),

$$\begin{aligned}
&\hat{e}_Q \odot \vec{e}_{N_1} \\
&= \hat{e}_Q \odot [(\hat{e}_{1M} \odot \vec{e}_{1N}) \oplus (\hat{e}_{2M} \odot \vec{e}_{2N}) \oplus \cdots \oplus (\hat{e}_{nM} \odot \vec{e}_{nN})] \\
&= [(\hat{e}_Q \odot \hat{e}_{1M}) \odot \vec{e}_{1N}] \oplus \cdots \oplus [(\hat{e}_Q \odot \hat{e}_{nM}) \odot \vec{e}_{nN}].
\end{aligned}$$

Clearly, $\vec{e}_{N_1} \oplus \vec{e}_{N_2}$, $\hat{e}_Q \odot \vec{e}_{N_1} \in W$ and so W is a subspace of $(N \wedge N \wedge \ldots \wedge N)$, n times, over $[K \times V](K)$ in classical sense.

Example 6.11 We take the NSLS N over $[(\mathbf{R}^3(\mathbf{R}), +, \cdot), E]$ described in Example 3.1. Let $\{x_1 = (0, 1, 1), x_2 = (1, 0, 1), x_3 = (1, 1, 0)\} \subset \mathbf{R}^3$ and $E = \{e_1, e_2, e_3\}$. The tabular representation of N is given in Table 16.

Table 16 : Tabular form of NSLS N.

	$\vec{e}_{1N}$	$\vec{e}_{2N}$	$\vec{e}_{3N}$
x_1	$(0.5, 0, 0)$	$(0, 0.25, 0.1)$	$(0, 0.25, 0.1)$
x_2	$(0, 0.25, 0.1)$	$(0.5, 0, 0)$	$(0, 0.25, 0.1)$
x_3	$(0, 0.25, 0.1)$	$(0, 0.25, 0.1)$	$(0.5, 0, 0)$

Next, we consider two neutrosophic soft fields P and T over $(\mathbf{R}, E)$ for $E = \{e_1, e_2, e_3\}$ given in Table 17 and Table 18 respectively. The elements of $\mathbf{R}$ are divided into two kinds i.e., q (rational number) and r (irrational number).

Table 17 : Tabular form of neutrosophic soft field P.

	$\hat{e}_{1P}$	$\hat{e}_{2P}$	$\hat{e}_{3P}$
q	$(0, 0.5, 0)$	$(0, 0.25, 0.5)$	$(0, 0.2, 0.7)$
r	$(1, 0, 0)$	$(0.5, 0, 0)$	$(0.25, 0, 0)$

Table 18 : Tabular form of neutrosophic soft field T.

	$\hat{e}_{1T}$	$\hat{e}_{2T}$	$\hat{e}_{3T}$
q	$(0.4, 0.6, 0.3)$	$(0.2, 0.5, 0.7)$	$(0, 0.6, 0.7)$
r	$(0.6, 0.2, 0)$	$(0.4, 0, 0.3)$	$(0.7, 0.2, 0.3)$

The t-norm $(*)$ and s-norm $(\diamond)$ are taken here as $a * b = \min\{a, b\}$, $a \diamond b = \max\{a, b\}$. Tabular form of scalar multiplication on N by P and N by T are given in Table 19 and

Table 20 respectively.

$$\text{So, } \vec{e}_{N_1} = (\hat{e}_{1P} \odot \vec{e}_{1N}) \oplus (\hat{e}_{2P} \odot \vec{e}_{2N}) \oplus (\hat{e}_{3P} \odot \vec{e}_{3N}) \in N \wedge N \wedge N \text{ over } (K \times V),$$
$$= \{< qx_1, (0, 0.5, 0.7) >, < qx_2, (0, 0.5, 0.7) >, < qx_3, (0, 0.5, 0.7) >,$$
$$< rx_1, (0, 0.25, 0.1) >, < rx_2, (0, 0.25, 0.1) >, < rx_3, (0, 0.25, 0.1) >\}.$$

Table 19 : Tabular form of scalar multiplication on N by P.

	$\hat{e}_{1P} \odot \vec{e}_{1N}$	$\hat{e}_{2P} \odot \vec{e}_{2N}$	$\hat{e}_{3P} \odot \vec{e}_{3N}$
qx_1	$(0, 0.5, 0)$	$(0, 0.25, 0.5)$	$(0, 0.25, 0.7)$
qx_2	$(0, 0.5, 0.1)$	$(0, 0.25, 0.5)$	$(0, 0.25, 0.7)$
qx_3	$(0, 0.5, 0.1)$	$(0, 0.25, 0.5)$	$(0, 0.2, 0.7)$
rx_1	$(0.5, 0, 0)$	$(0, 0.25, 0.1)$	$(0, 0.25, 0.1)$
rx_2	$(0, 0.25, 0.1)$	$(0.5, 0, 0)$	$(0, 0.25, 0.1)$
rx_3	$(0, 0.25, 0.1)$	$(0, 0.25, 0.1)$	$(0.25, 0, 0)$

Table 20 : Tabular form of scalar multiplication on N by T.

	$\hat{e}_{1T} \odot \vec{e}_{1N}$	$\hat{e}_{2T} \odot \vec{e}_{2N}$	$\hat{e}_{3T} \odot \vec{e}_{3N}$
qx_1	$(0.4, 0.6, 0.3)$	$(0, 0.5, 0.7)$	$(0, 0.6, 0.7)$
qx_2	$(0, 0.6, 0.3)$	$(0.2, 0.5, 0.7)$	$(0, 0.6, 0.7)$
qx_3	$(0, 0.6, 0.3)$	$(0.2, 0.5, 0.7)$	$(0, 0.6, 0.7)$
rx_1	$(0.5, 0.2, 0)$	$(0, 0.25, 0.3)$	$(0, 0.25, 0.3)$
rx_2	$(0, 0.25, 0.1)$	$(0.4, 0, 0.3)$	$(0, 0.25, 0.3)$
rx_3	$(0, 0.25, 0.1)$	$(0.4, 0.25, 0.3)$	$(0.5, 0.2, 0.3)$

$$\text{So, } \vec{e}_{N_2} = (\hat{e}_{1T} \odot \vec{e}_{1N}) \oplus (\hat{e}_{2T} \odot \vec{e}_{2N}) \oplus (\hat{e}_{3T} \odot \vec{e}_{3N}) \in N \wedge N \wedge N \text{ over } (K \times V)$$
$$= \{< qx_1, (0, 0.6, 0.7) >, < qx_2, (0, 0.6, 0.7) >, < qx_3, (0, 0.6, 0.7) >,$$
$$< rx_1, (0, 0.25, 0.3) >, < rx_2, (0, 0.25, 0.3) >, < rx_3, (0, 0.25, 0.3) >\}.$$

$$\text{Now, } \vec{e}_{N_1} \oplus \vec{e}_{N_2} = \{< qx_1, (0, 0.6, 0.7) >, < qx_2, (0, 0.6, 0.7) >, < qx_3, (0, 0.6, 0.7) >,$$
$$< rx_1, (0, 0.25, 0.3) >, < rx_2, (0, 0.25, 0.3) >, < rx_3, (0, 0.25, 0.3) >\}.$$

Next, for a neutrosophic soft scalar $\hat{e}_{1Q} = \{< q, (0.3, 0.6, 0.8) >, < r, (0.5, 0.3, 0.4) > \}$ over (K, E),

$$\hat{e}_{1Q} \odot \hat{e}_{1P} = \{< q, (0, 0.6, 0.8) >, < r, (0.5, 0.3, 0.4) >\},$$
$$\hat{e}_{1Q} \odot \hat{e}_{2P} = \{< q, (0, 0.6, 0.8) >, < r, (0.5, 0.3, 0.4) >\},$$
$$\hat{e}_{1Q} \odot \hat{e}_{3P} = \{< q, (0, 0.6, 0.8) >, < r, (0.25, 0.3, 0.4) >\}$$

and so the scalar multiplication on N can be put in a tabular form as in Table 21. Thus,

Table 21 : Tabular form of scalar multiplication on N.

	$(\hat{e}_{1Q} \odot \hat{e}_{1P}) \odot \vec{e}_{1N}$	$(\hat{e}_{1Q} \odot \hat{e}_{2P}) \odot \vec{e}_{2N}$	$(\hat{e}_{1Q} \odot \hat{e}_{3P}) \odot \vec{e}_{3N}$
qx_1	$(0, 0.6, 0.8)$	$(0, 0.6, 0.8)$	$(0, 0.6, 0.8)$
qx_2	$(0, 0.6, 0.8)$	$(0, 0.6, 0.8)$	$(0, 0.6, 0.8)$
qx_3	$(0, 0.6, 0.8)$	$(0, 0.6, 0.8)$	$(0, 0.6, 0.8)$
rx_1	$(0.5, 0.3, 0.4)$	$(0, 0.3, 0.4)$	$(0, 0.3, 0.4)$
rx_2	$(0, 0.3, 0.4)$	$(0.5, 0.3, 0.4)$	$(0, 0.3, 0.4)$
rx_3	$(0, 0.3, 0.4)$	$(0, 0.3, 0.4)$	$(0.25, 0.3, 0.4)$

$$\begin{aligned}
\hat{e}_{1Q} \odot \vec{e}_{N_1} &= \hat{e}_{1Q} \odot [(\hat{e}_{1P} \odot \vec{e}_{1N}) \oplus (\hat{e}_{2P} \odot \vec{e}_{2N}) \oplus (\hat{e}_{3P} \odot \vec{e}_{3N})] \\
&= [(\hat{e}_{1Q} \odot \hat{e}_{1P}) \odot \vec{e}_{1N}] \oplus [(\hat{e}_{1Q} \odot \hat{e}_{2P}) \odot \vec{e}_{2N}] \oplus [(\hat{e}_{1Q} \odot \hat{e}_{3P}) \odot \vec{e}_{3N}] \\
&= \{< qx_1, (0, 0.6, 0.8) >, < qx_2, (0, 0.6, 0.8) >, < qx_3, (0, 0.6, 0.8) >, \\
&\quad < rx_1, (0, 0.3, 0.4) >, < rx_2, (0, 0.3, 0.4) >, < rx_3, (0, 0.3, 0.4) >\}.
\end{aligned}$$

Clearly, $\vec{e}_{N_1} \oplus \vec{e}_{N_2} = \vec{e}_{N_2}$ and $\hat{e}_{1Q} \odot \vec{e}_{N_1} \subset \vec{e}_{N_1}, \vec{e}_{N_2}$ by sense of Definition 2.8.

7. Conclusion

The theoretical point of view of NSLS has been introduced and illustrated by suitable examples in the present paper. Here, we also have defined the cartesian product of NSLSs, neutrosophic soft subspaces and neutrosophic soft vector in NSLS. Some related theorems have been established and verified by suitable examples. We expect the future works on neutrosophic soft vector upon this concept.

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