The foot-rule problem with numerical analysis

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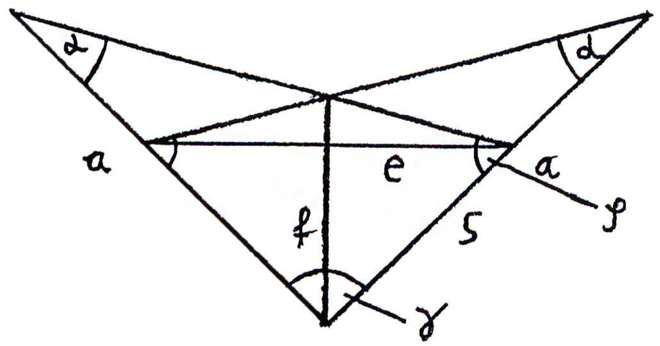
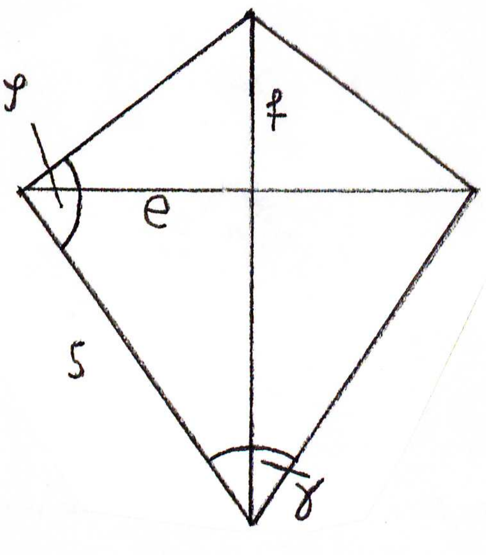
We view the figure in the foot-rule:



The kite is interesting that is spaned to four sides of a foot-rule.



We need the figures. We search the maximum area of the kite with known a and α. γ is the changable variable.

Sine law:

$$\frac{f}{\sin\alpha} = \frac{a}{\sin(180° - \frac{\gamma}{2} - \alpha)} \qquad \Rightarrow \qquad f = \frac{a\sin\alpha}{\sin(180° - \frac{\gamma}{2} - \alpha)}$$

$$\frac{s}{\sin\alpha} = \frac{a}{\sin(180° - \gamma - \alpha)} \qquad \text{and} \qquad \frac{e}{2} = s \cdot \sin\frac{\gamma}{2}$$

$$\Rightarrow \qquad \frac{e}{2} = \frac{a\sin\alpha\sin\frac{\gamma}{2}}{\sin(180° - \gamma - \alpha)}$$

With the area's formula $F = \frac{ef}{2}$ we obtain:

$$F = \frac{a^2\sin^2\alpha\sin\frac{\gamma}{2}}{\sin(180° - \frac{\gamma}{2} - \alpha)\cdot\sin(180° - \gamma - \alpha)} \qquad \alpha = \text{const.}$$

with $\sin(180° - \beta) = \sin\beta$ it becomes:

$$F(\gamma) = \frac{a^2\sin^2\alpha\sin\frac{\gamma}{2}}{\sin(\frac{\gamma}{2} + \alpha)\cdot\sin(\gamma + \alpha)} \tag{1}$$

This area formula must be derived to γ. We calculate the denominator's derivation with the product rule and the chain rule:

$$\left(\sin\left(\frac{\gamma}{2} + \alpha\right)\cdot\sin(\gamma + \alpha)\right)' = \frac{1}{2}\cdot\cos\left(\frac{\gamma}{2} + \alpha\right)\sin(\gamma+\alpha) + \sin\left(\frac{\gamma}{2} + \alpha\right)\cos(\gamma+\alpha)$$

Counter: $\qquad (\sin\frac{\gamma}{2})' = \frac{1}{2}\cdot\cos\frac{\gamma}{2}$

Quotient rule:

$$\frac{F'(\gamma)}{a^2\sin^2\alpha} = \frac{1}{\sin^2(\frac{\gamma}{2} + \alpha)\cdot\sin^2(\gamma + \alpha)}\cdot\left(\frac{1}{2}\cdot\cos\frac{\gamma}{2}\cdot\sin\left(\frac{\gamma}{2} + \alpha\right)\cdot\sin(\gamma + \alpha)\right.$$
$$\left. - \left(\frac{1}{2}\cdot\cos\left(\frac{\gamma}{2} + \alpha\right)\cdot\sin(\gamma + \alpha) + \sin\left(\frac{\gamma}{2} + \alpha\right)\cdot\cos(\gamma + \alpha)\right)\cdot\sin\frac{\gamma}{2}\right)$$

The necessary condition of local extrema is $F'(\gamma) = 0$.

$$\frac{1}{2}\cdot\cos\frac{\gamma}{2}\sin\left(\frac{\gamma}{2} + \alpha\right)\sin(\gamma + \alpha) - \left(\frac{1}{2}\cdot\cos\left(\frac{\gamma}{2} + \alpha\right)\sin(\gamma + \alpha)\right.$$
$$\left. + \sin\left(\frac{\gamma}{2} + \alpha\right)\cos(\gamma + \alpha)\right)\cdot\sin\frac{\gamma}{2} = 0$$

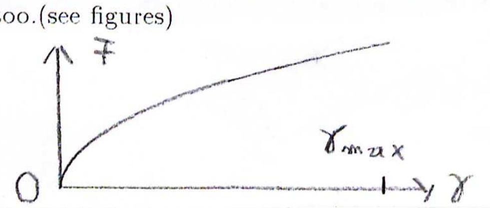
It is valid $\frac{\sin\beta}{\cos\beta} = \tan\beta$. We devide trough $\cos\left(\frac{\gamma}{2}+\alpha\right)\cdot\cos(\gamma+\alpha)$. Then it follows:

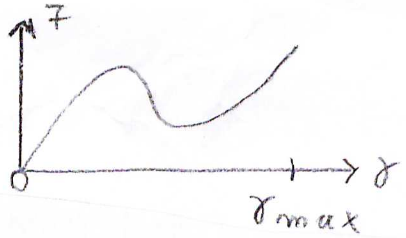
$$\frac{1}{2}\cdot\cos\frac{\gamma}{2}\tan\left(\frac{\gamma}{2}+\alpha\right)\tan(\gamma+\alpha)-\left(\frac{1}{2}\cdot\tan(\gamma+\alpha)+\tan\left(\frac{\gamma}{2}+\alpha\right)\right)\cdot\sin\frac{\gamma}{2}=0$$

divided through $\sin\frac{\gamma}{2}\cdot\tan\left(\frac{\gamma}{2}+\alpha\right)\cdot\tan(\gamma+\alpha)$:

$$\frac{1}{2\cdot\tan\frac{\gamma}{2}}-\frac{1}{2\cdot\tan\left(\frac{\gamma}{2}+\alpha\right)}-\frac{1}{\tan(\gamma+\alpha)}=0 \qquad (2)$$

We must determine γ in dependence of α. This can be done by replacing the tangent-expressions through sine and cosine. In this case the application of the Newton's method is difficult. To decide local maximum or local minimum, we must calculate the 2.derivation in the concrete case. Saddle points are possible, too.(see figures)





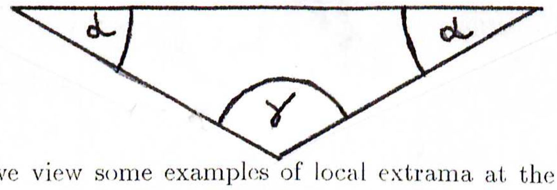
For a kite it must be:

$$360^\circ-\gamma-2\varphi<180^\circ \qquad \text{und} \qquad \varphi=180^\circ-\alpha-\gamma$$

$$\Rightarrow \quad 360^\circ-\gamma-2\cdot(180^\circ-\alpha-\gamma)<180^\circ \quad \Leftrightarrow \quad 2\alpha+\gamma<180^\circ$$

$$\Rightarrow \quad \gamma<180^\circ-2\alpha \quad \Rightarrow \quad \alpha<\frac{180^\circ-\gamma}{2}$$

In case $2\alpha+\gamma=180^\circ$ there is a isosceles triangle (a special kite).



Now we view some examples of local extrama at the foot-rule problem. The result is the following table with α and γ in degree. We choose $a=1$:

α	$\gamma; F(\gamma)$ (local maximum)	$\gamma; F(\gamma)$ (local minimum)
1	1.4154;0.0029955	87.9819;0.0002993
2	2.8383;0.0059966	85.9249;0.0011754
3	4.2761;0.0090093	83.8241;0.0025954
5	7.2309;0.0150936	79.4674;0.0069313
7	10.3603;0.0213007	74.8478;0.0130263
9	13.7813;0.0276928	69.8611;0.0205844
10	15.6584;0.0309822	67.1763;0.0248137
12	19.9567;0.0378132	61.2179;0.0339582
14	25.6819;0.0451162	53.7734:0.0436804
15	30.0000;0.0490381	48.5865;0.0485897
15.5	33.4365;0.0511068	44.7085;0.0510586
15.6	34.4555;0.0515335	43.6009;0.0514777
15.7	35.8097;0.0519664	42.1579;0.0519476
15.75	36.8016;0.0521860	41.1217;0.0521800
15.76	37.0639;0.0522302	40.8505;0.0522262
15.77	37.3704;0.0522746	40.5350;0.0522723
15.78	37.7554;0.0523192	40.1412;0.0523182
15.79	38.3550;0.0523640	39.5290;0.0523639

For $\alpha \geq 15.8°$ there are no local extrema. It must be $\gamma < 180° - 2\alpha$.

References

[1] Harald Schröer "Special extreme value problems and extremum principles" 2002 Wissenschaft & Technik Verlag Berlin