The extended "Euclid"

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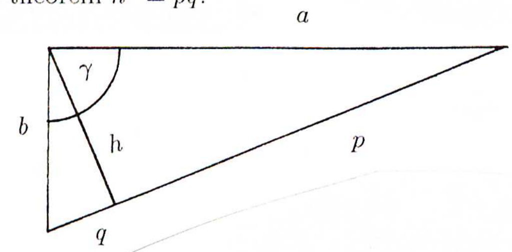
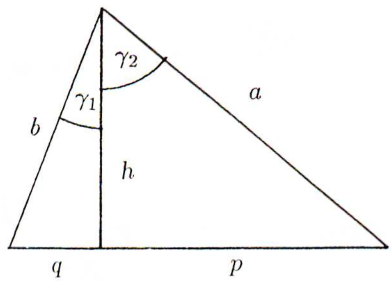
We know in a right-angled triangle Euclid's theorem $a^2 = cp$ and $b^2 = cq$ and the altitude theorem $h^2 = pq$.



$q + p = c$

Now we view a general triangle:



$\gamma_1 + \gamma_2 = \gamma$

$q + p = c$

The question is whether there are relations of the kind $a^2 = f(c, p, \gamma)$, $b^2 = f(c, q, \gamma)$ and $h^2 = f_1(p, q, \gamma)$ in the general triangle? f and f_1 are determined functions.

It is valid at the general triangle:

$$\tan\gamma_1 = \frac{q}{h} \qquad \tan\gamma_2 = \frac{p}{h} \qquad \gamma = \gamma_1 + \gamma_2$$

We use the addition theorem of tangent:

$$\tan(\gamma_1 + \gamma_2) = \frac{\tan\gamma_1 + \tan\gamma_2}{1 - \tan\gamma_1 \tan\gamma_2}$$

Insertion for $\tan\gamma_1$ and $\tan\gamma_2$ leads to:

$$\tan\gamma = \frac{h \cdot (p+q)}{h^2 - pq} \tag{1}$$

We solve this equation to h:

$$h^2 \tan\gamma - h \cdot (p+q) = pq \tan\gamma$$

$$\Rightarrow \qquad h^2 - \frac{p+q}{\tan\gamma} \cdot h = pq$$

Thus we have a quadratic equation. With the solution formula for quadratic equations we get:

$$h = +\sqrt{pq + \frac{(p+q)^2}{4\tan^2\gamma}} + \frac{p+q}{2\tan\gamma} \tag{2}$$

Equation (2) is an extension of the altitude theorem. For $\gamma = 90°$ we yield $h = \sqrt{pq}$ thus the known altitude theorem.

We use the equations $h^2 = a^2 - p^2$ and $c = p + q$, to replace h and q through a and c:

$$\sqrt{a^2 - p^2} = \sqrt{p \cdot (c-p) + \frac{c^2}{4\tan^2\gamma}} + \frac{c}{2\tan\gamma}$$

or:

$$a^2 = \left(\sqrt{p \cdot (c-p) + \frac{c^2}{4\tan^2\gamma}} + \frac{c}{2\tan\gamma}\right)^2 + p^2$$

cyclic permutation:

$$b^2 = \left(\sqrt{q \cdot (c-q) + \frac{c^2}{4\tan^2\gamma}} + \frac{c}{2\tan\gamma}\right)^2 + q^2$$

Both of the last equations are extensions of Euclid's theorem. For $\gamma = 90°$ it follows in fact $a^2 = cp$ and $b^2 = cq$.

Now we transform the equation (1) to p and q:

$$h^2 \tan\gamma - pq \tan\gamma = hp + hq$$

It follows:

$$p = \frac{h^2 \tan\gamma - hq}{h + q\tan\gamma} = \frac{h \cdot (h\tan\gamma - q)}{q\tan\gamma + h}$$

$$q = \frac{h^2 \tan\gamma - hp}{h + p\tan\gamma} = \frac{h \cdot (h\tan\gamma - p)}{p\tan\gamma + h}$$

These last equations are transformations of the extended altitude theorem to p and q.

References

[1] Harald Schröer "Trigonometric problem cases well solved" Wissenschaft & Technik Verlag Berlin 2001