A potential relation between the algebraic approach to calculus and rational functions

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Summary

This paper considers: (1) Colignatus (2011), "*Conquest of the Plane*" (COTP), with its new algebraic approach to calculus, and (2) the theory of rational functions (RF).

This paper assumes that most readers will come from a background with RF. They might be interested whether there might be some (re-) design by using some notions from COTP.

The derivative *df* / *dx* = *f '* [*x*] concerns the slope of the function. If *df* is polynomial then we have a rational function *r* = *df* / *dx*. The theory of rational functions at the fundamental level (RF-FL) recognises domains and singularities. Conventionally singularities must be resolved by using limits. For polynomials there is the possibility of factoring, *df* = *f '* [*x*] *dx*. Multiplicative factoring can be proven by use of coefficients only, which leads to Ruffini's Rule.

The major conceptual issue w.r.t. factoring is whether the multiplicative form *df* = *f '* [*x*] *dx* can still be recognised as the slope *df* / *dx* (since a slope is given by the tangent in trigonometry). Ruffini's Rule factors and solves *df* / *dx* by "synthetic division", but to what extent is "synthetic" also proper division, so that "eliminating" the factor *dx* generates a result that can be understood as the slope of the function at that point ?

This conceptual problem is resolved as follows. We better state explicitly that the domain must be manipulated. Let *y* // *x* be the following process or program, called *dynamic division* or *dynamic quotient*, with numerator *y* and denominator *x*:

> *y* // *x* ≡ { *y* / *x*, unless *x* is a variable and then: assume *x* ≠ 0, simplify the expression *y* / *x*, declare the result valid also for the domain extension *x* = 0 }

The algebraic definition of the derivative then follows directly:

$$f'[x] = \{\Delta f \,//\, \Delta x, \text{ then set } \Delta x = 0\}$$

This implies that the expression "*df* / *dx"* only has proper meaning as an operator "d / d*x*" applied to *f,* without proper division. This also means that we finally have a sound interpretation for differentials. These would not be infinitesimals. The differentials *df* and *dx* are better seen as variables, so that, when *f '* [*x*] has been found by other methods of algebraic manipulation, we can define *df* = *f '* [*x*] *dx* for the incline (tangent) to *f.*

This gives:

$$df \,//\, dx = f'[x]\, dx \,//\, dx = f'[x]$$

The group theory approach to rational functions (RF-GT) (the version that we looked at) appears to have limited value, because of the assumption that these "functions" don't have domains. If its results are to be useful, they must be translated, and domains and singularities come into consideration anyhow. The notion of an equivalence class relies on limits and continuity, and the manipulation of the domain is not explicit enough.

An algebraic approach to calculus is possible that relies on algebra and expressions only, and that manipulates the domain to find the slope of the function. The formal continuity given by the expression is sufficient, and there is no need for numerical continuity and limits.

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1. Introduction

1.1. Basic terms

This paper considers:

(1) Colignatus (2011a), *Conquest of the Plane* (COTP), with its new algebraic approach to calculus, including its *Reading Notes* (2011b), and
(2) the theory of rational functions (RF). [1]

There are only few readers who are aware about even the existence of COTP. Perhaps those few might be interested how it relates to RF.

This paper however assumes that most readers will come from a background with RF, and that they might be interested whether there might be some (re-) design by using some notions from COTP.

The step from calculus to RF arises naturally. The derivative concerns the slope of a function. This starts with Δf / Δx, and then searches for the tangent from trigonometry. Obviously Δx is polynomial. If Δf is polynomial then we are looking at RF.

Let us first clarify the basic terms.

1.2. Rational functions (RF)

The theory of rational functions starts with this notion:

> "A rational function is a function $w = R(z)$, where $R(z)$ is a rational expression in z, i.e. an expression obtained from an independent variable z and some finite set of numbers (real or complex) by means of a finite number of arithmetical operations. A rational function can be written (non-uniquely) in the form $R(z) = P(z) / Q(z)$ where P, Q are polynomials, $Q(z) \neq 0$." [2]

A root x of a polynomial $P[x]$ is a solution of $P[x] = 0$. The root factor will have the form of $(x – a)$, for example $x + 1 = x - (-1)$. The term "zero" is used conventionally somewhat confusingly for both the root and the root factor, and thus it is better to use "zero" for 0 only. Factors need not have roots, for example $x^2 + 1 = 0$ only has complex solutions.

It appears that there are two versions of this theory:

(1) RF-FL: There is a *fundamental level* version for matricola for non-math majors and perhaps highschool, in a course belonging to "pre-calculus", in which the polynomials have domains, and in which $Q[z] = 0$ would be solved for roots in the domain, in order to exclude those values for the ratio. (See for example Juha Pohjanpelto. [3] Apparently this is also used in Wikipedia – a portal, no source. [4]) This version of RF causes the notion of "removable singularity" - which notion is only relevant if there is a domain. [5]
(2) RF-GT: The RF-FL may also be used in *group theory*, but there is a particular version falling under *group theory*, and let us call this RF-GT, in which the polynomials have no domain, and in which $Q[z] = 0$ would be the zero-polynomial (with the value 0 for all z). Also, z is called an *indeterminate* rather than *variable,* since variables have domains,

[1] https://www.encyclopediaofmath.org//index.php?title=Rational_function&oldid=17805
[2] https://www.encyclopediaofmath.org//index.php?title=Rational_function&oldid=17805
[3] http://oregonstate.edu/instruct/mth251/cq/FieldGuide/rational/lesson.html
[4] https://en.wikipedia.org/wiki/Rational_function
[5] https://www.encyclopediaofmath.org/index.php/Removable_singular_point

which are not assumed here. Let us use *nondetermined* rather than *indeterminate.* [6] RF-GT finds that the rational functions form a field. (If these theorists agree that *y* / *x* normally applies to domains, then they would have to agree that they need a new symbol for the RF[*y, x*] rather than abusing the same symbol for other purposes. Paul Garrett suggests to replace "ratio" by "pair". [7])

The fundamental level version of RF requires that the user keeps track of the singularities. This might be seen as needlessly burdensome for some results, whence we understand the attractiveness of the approach in group theory to abstract from the domain. However, when results of RF-GT are applied, then there is a domain again, and RF-FL comes into focus again. (Actually, I find it awkward to presume that there is no domain, since an expression $x - a = 0$ would solve as $x = a$, and then this would be "just an expression" and not be an element in some domain ? Indeed, we find that RF-GT has only limited relevance.)

The theory of polynomials [8] is embedded within RF. Each polynomial *P*[*x*] can be written as *P*[*x*] / 1. However, authors can specialise, and potentially there might be some miscommunications, in particular for transfers from *quotients of factors* in RF to *products of factors* in polynomials.

A key issue in this paper is that "removing a singularity" is actually an adjustment of the domain. Mathematics texts on this issue should be much more explicit. The notion of "dynamic quotient" (see below) fits this explicity.

There are two ways to remove singularities for polynomials:

(a) Use the *coefficients* (and not the variables) to prove a theorem on *factoring*, and rely on *P*[*z*] = *Q*[*z*] *R*[*z*]. This eventually causes Ruffini's Rule, [9] sometimes a.k.a. Horner's Scheme. [10] (PM. The proof on factoring might also be done in RF-GT but this is actually an irrelevant and rather confusing approach, since the idea is to remove singularities, and those only are relevant when there is a domain which isn't assumed for RF-GT.) (PM. Multiplicative factoring does not quite generate the slope of a curve, as only $\Delta f / \Delta x$ contains the notion of division, as in the trigonometric notion of tangent.)
(b) Use limits. This approach isn't considered "algebra" but regarded as "analysis". This method would also work for non-polynomials.

There are some key issues in definition. Is *f*[*x*] = *p*[*x*] / *q*[x] still a RF for points *q*[*x*] = 0 ? For such points *f*[*x*] is not defined and it cannot be written as a fraction. Thus above definition of RF implies that we don't have a RF at roots of *q*. Wikipedia – a portal and no source – has the confusing statement: "The rational function *f*(*x*) = *x* / *x* is equal to 1 for all *x* except 0, where there is a removable singularity." [11] This suggests that *f*[*x*] = *x* / *x* is a rational function. But it is not a rational function for *x* = 0. The proper phrasing would be that *x* / *x* is a rational *expression,* and for $x \neq 0$, *f*[*x*] = *x* / *x* is a rational function. [12] We get a function when we combine an expression with domain and range. Subsequently, if *p*[*x*] can be factored as *p*[*x*] = *q*[x] *r*[*x*] then we can "simplify" and get *f*[*x*] = *r*[*x*] which is a RF (*r*[*x*] / 1). However, *f* still has the undefined spots in the domain. But *r*[*x*] might not have such undefined spots. In this example *r*[*x*] = 1 also for *x* = 0. However, how can we write *f*[*x*] = *r*[*x*] when domains differ ? It may be that users of RF are so used to such domain issues, that they don't properly record them, for doing so might be needless ballast. For students learning about these issues these conventions will however be confusing (but they hate losing points for not writing $x \neq 0$ too).

[6] In Wolfram's *Mathematica* the term *Indeterminate* stands for undefined, comparable to Infinity. Above group theory is better served with "nondetermined".
[7] http://www-users.math.umn.edu/~garrett/m/algebra/notes/06.pdf
[8] https://www.encyclopediaofmath.org/index.php/Polynomial
[9] http://mathworld.wolfram.com/RuffinisRule.html
[10] https://www.encyclopediaofmath.org/index.php/Horner_scheme
[11] https://en.wikipedia.org/wiki/Rational_function
[12] Compare the shifting meanings of "student": (i) "Student John can apply for this scholarship except when he is not at a university", (ii) If you are at a university, then you qualify as a student and can apply for this scholarship."

1.3. Conquest of the Plane (COTP)

Colignatus (2011ab) "*Conquest of the Plane*" (COTP) is a *primer* on both analytic geometry and calculus. Thus it targets (i) teachers of mathematics and (ii) students training for this, and (iii) researchers in the didactics of mathematics. It may also be useful for (iv) matricola of non-math majors and (v) advanced placement students at highschool. COTP is a *proof of concept* for the algebraic approach to calculus. The original approach was given in a few pages by Colignatus (2007) "*A Logic of Exceptions*" (ALOE). COTP shows how this can be developed into a primer with the various intermediate steps and subsequent deductions.

ALOE and COTP present the "dynamic quotient". Let us briefly restate it, as it will be discussed at more length below. Let y / x be as it is used currently in textbooks, and let $y // x$ be the following process or program, called *dynamic division* or *dynamic quotient*, with numerator y and denominator x:

> $y // x \equiv \{ y / x$, unless x is a variable and then: assume $x \neq 0$, simplify the expression y / x, declare the result valid also for the domain extension $x = 0$ }

The algebraic definition of the derivative then follows directly:

$$f'[x] = \{\Delta f // \Delta x, \text{ then set } \Delta x = 0\}$$

Key properties are:

(1) The dynamic quotient refers to expressions that can be simplified, and variables that have domains that can be manipulated. These need not be polynomials.
(2) The derivative (using the dynamic quotient) concerns functions that rely on (1).

The definition of the dynamic quotient can be found in Colignatus (2007:241) or (2011a:57). It appears that mere reference is not sufficient and that some readers require that a discussion is self-contained. This condition is awkward since COTP really deserves a study because of its approach to didactics and essential refoundation of calculus. Yet, a section below thus repeats the definitions of the dynamic quotient and the derivative that uses this.

1.4. Potential relation

A potentional relation between COTP and RF is given by these observations:

(1) With Δx an obvious polynomial, $\Delta f // \Delta x$ would relate to the format of rational function, if $\Delta f[x]$ would be polynomial too.
    (a) Might the domain manipulation be equivalent to Ruffini's bypass (RF-FL) ?
    (b) Might the domain manipulation be equivalent to abstracting from it (RF-GT) ?
    (c) If there is this equivalence:
        (i) Does RF in this manner provide a corroboration for the notion of the dynamic quotient ?
        (ii) If the group theory version of RF corroborates the notion of dynamic quotient, doesn't it then also show that the dynamic quotient is superfluous (for polynomials), since it would imply that there is an algebraic representation of $\Delta f / \Delta x$, with normal division ? (But RF-GT is not taught in highschool, so it couldn't be used to teach an algebraic approach to the derivative there.)
        (iii) Might the notion of domain manipulation make RF *itself* superfluous, so that also the distinction between group theory and fundamental level version becomes rather irrelevant ?
(2) Could these results be translated to cases with f non-polynomial ?
    (a) The algebraic approach to the derivative also works for the exponential function and trigonometry. Looking at calculus in algebraic manner again would be a fundamental redesign, after the Cauchy and Weierstrasz turn to numbers.
    (b) Colignatus (2014) is already an invitation to research mathematicians to see if one can develop a theory of "expressions" that can be "simplified". This present paper extends on that invitation.

(3) For group theory, the step from a ring to a field is given by the inclusion of division for the latter. How would group theory adapt to the notion of the dynamic quotient ? RF would be a key case to start with. (The motivation to focus on division is obviously the problem of 1 / 0 and 0 / 0. The angle on the derivative is a special case for this.)

For all clarity, our issue doesn't concern the derivative of $f[x]$ / $g[x]$ for polynomials f and g.

1.5. Analytic geometry and calculus

Mathematics makes a distinction between *analysis* and *algebra*. Yet now there is the suggestion that there is a connection where that distinction gets blurred.

(i) Ancient Greek geometry is called *synthetic* since one generates proofs by "putting together" the various givens (definitions, axioms and earlier theorems). In *analytic geometry* one generates proofs by decomposing (analysing) issues in terms of algebra (even though this actually means a shift to other kinds of givens).

(ii) Historically, a function was a *proscription* of how to turn an input into an output. Leonhard Euler (1707-1783) worked in this manner. See for example Cha (1999). This generated some study of notations and algorithms, as we see nowaday in *computer algebra,* with the algebra of *variables* and *expressions.* See for example Wolfram (1996) on pattern recognition.

(iii) A subsequent historical development was that even analytic geometry and its system of co-ordinates was seen as not exact enough, whence one looked for foundations in number and arithmetic. This became the field of *analysis.* Corner stones of the latter are notions of numerical continuity and limits. The current perception in mathematics is that calculus can only be done in *analysis*.

A role is played by the distinction between finite and infinite sums. The RF adopts finite sums only. The exponential function and the (basic) functions in trigonometry have a Taylor expansion, which is a polynomial form, but with infinite sum. The expression $(e^x - 1) / x$ generates a problem for $x = 0$, and cannot be treated in RF because of its Taylor infinite sum. We have $\mathrm{Sin}[\varphi] \leq \varphi \leq \mathrm{Sin}[\varphi] / \mathrm{Cos}[\varphi]$ around $\varphi = 0$ and for $\varphi = 0$, but only $1 < \varphi / \mathrm{Sin}[\varphi] < 1 / \mathrm{Cos}[\varphi]$ and no proper outcome for $\varphi = 0$ because of the Taylor infinite sum. With the standard (static) notion of division, the infinite sum requires limits, whence this is *analysis.* (See however COTP (2011ab) and Colignatus (2017b) for algebraic solutions to these cases.)

(iv) Set theory considers functions as pairs $f = \{\{x,y\} \mid x$ and y in their sets$\}$. The x and $y = f[x]$ are *elements* and not quite *variables* (symbols that can be assigned different values). For *analysis*, the sets are (real) numbers. This necessitates notions of *numerical* continuity and limits. (There may be the problem here that continuity is defined by using limits, and limits are defined by using continuity, see Colignatus (2016bc).)

(v) Colignatus (2007) (2011a) rekindles the approach to look at expressions. Information about the function is contained in its expression. There is a notion of "continuity in form" (COTP 224-225). This information can be used when particular methods of arithmetic generate problems, notably with arithmetic division at zero. We can define a notion of "dynamic quotient" that manipulates the domain. This dynamic quotient allows an algebraic definition of the derivative.

1.6. Origin and purpose of this paper

The algebraic approach to calculus originated from didactics and it would be up to mathematicians to see how far the redesign can be developed further. The algebra of expressions is less developed indeed, but this may be just a historical drift due to the focus on numerics. Highschool students should not become a victim of this historical development within mathematics. For didactics the use of the dynamic quotient is well-defined, and, the derivative is a mere consequence. Obviously, students majoring in mathematics would have to know both methods.

This paper (thus) must be seen as an invitation from the realm of didactics of mathematics to research mathematics to see whether issues can be developed further. Are we allowed to refer to insights from analytic geometry (or "geometric intuition") as part of proofs ? Mathematicians might grant that this might be done in didactics. Didacticians might grant that mathematicians have the job to question details, which they do in research mathematics. Views on this might clash when there is the emphemeral notion as if we would withold students essential information by referring to analytic geometry (and some "geometric intuition") instead of requiring that they should be trained to become research mathematicians themselves too.

My approach to this is that a sound training in empirical science (and in what is called "applied mathematics") would be the best base to judge about balance between what is both didactically effective and mathematically required. Research mathematics can speak their mind but without a sound background in the empirics of didactics they should not decide upon math education in highschool and matricola for non-math-majors. Obviously, it are the students who must tell what didactics works for them. Thus I am looking forward to classroom experiments, i.e. not just usage of the methods but randomised controlled trials. This present discussion should also be helpful to determine what to check.

This present discussion has been inspired by reading work by Michael Range, who referred to RF. See **Appendices A and B** and Colignatus (2017e). **Appendix B** has been taken from (2017e) Appendix H.

For the Dutch setting, it appeared that a mathematics training for aspiring teachers of mathematics was organised by research mathematicians rather than teachers. Their training programme puts more emphasis on group theory and analysis (less relevant for highschool education) and doesn't develop the algebraic approach to calculus, see Colignatus (2017f).

1.7. Insulation as the risk of specialisation

Research mathematics (RM) might not understand the need to look at division again. They regard the issue as essentially solved by Analysis with its notions of continuity and limits. In other subjects of study, division by zero is avoided by excluding those points from the domains of functions and from general consideration anyhow, for the reason that the issue is no longer interesting (or would turn the topic into Analysis).

Observe in particular that RF and the theory of polynomials might be separate specialisations, even though RF looks at ratios of polynomials.

- In the subject of "rational function" - group theory version (RF-GT), the roots of the denominator are excluded *by definition*. (This is not standard for polynomial theory in general. Roots often must be found and cannot be specified a priori anyway.)
- In the subject of polynomials, the roots are "factored" such that it is claimed that division is not needed. (Though there still is the need to *isolate* or *identify* factors.)
- A research mathematician can switch between the different subjects of "rational function" (ratio of factors) and "polynomial" (product of factors) without drawing attention to the implication that there is a switch in perspectives and *that the domain of the function is being manipulated*.

This switch in perspectives need not be transparant, and then there will be confusing for students.

The key issue is that the domain is being manipulated. It is transparant to actually *say so*.

It may be so obvious to RM that they are manipulating the domain, that they no longer say so. It may also be that some RM have forgotten what the purpose is, and that they are just working on their specialisation. It may well be that research mathematicians (RM) have insulated themselves against the problems that students in highschool have with these issues. The answer by RM might be that such students should become more like RM, but the real solution is to remove this insulation.

PM. Roots of polynomials can be handled by looking at the *coefficients* in the polynomial. This is the only way known to avoid (long) division. See the discussion below on Range (2016c:16), Figure 5. However, this approach (or Ruffini's Rule) isn't always mentioned, and when it isn't mentioned then this contributes to confusion.

1.8. Overview of this discussion

This discussion will look at the angles mentioned in the table of contents.

Figure 1 gives an overview of those angles. Group theory doesn't have the concept of domain manipulation, and group theory is so dominant in the perception of theorists, that mathematicians currently are forced to manipulate domains without being able to say so. Figure 2 shows that the invoked steps however can be actually incorrect.

It will be useful to start with a review of the algebraic approach to the derivative with the use of the dynamic quotient.

Figure 1. Overview of the relations in this discussion

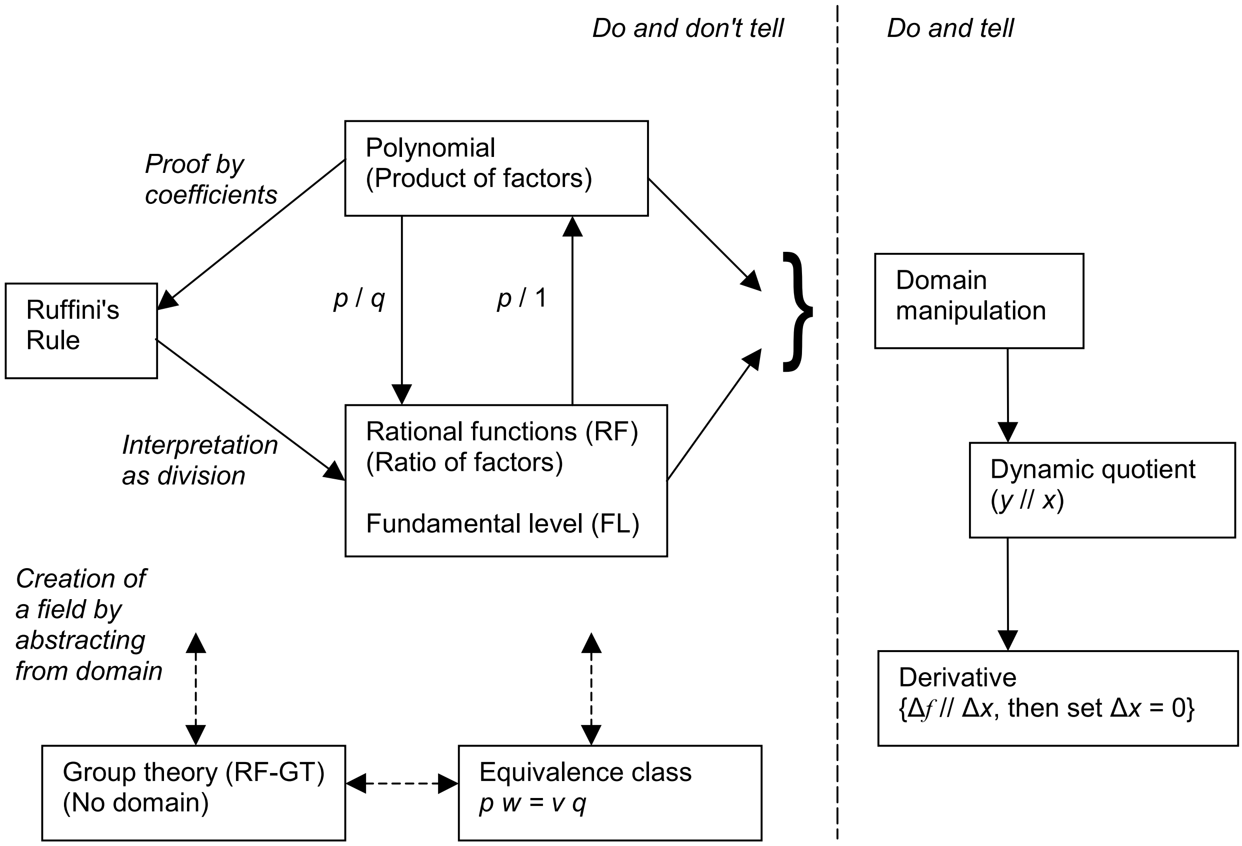
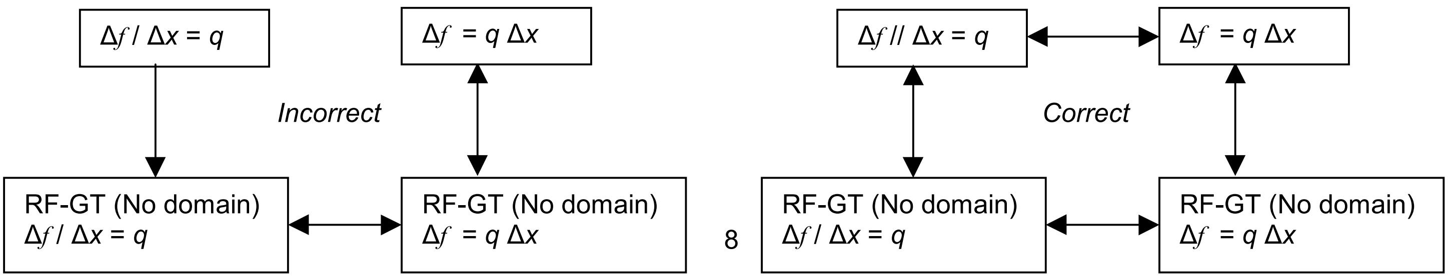


Figure 2. Steps without a memory where they originated (First line with real domain)



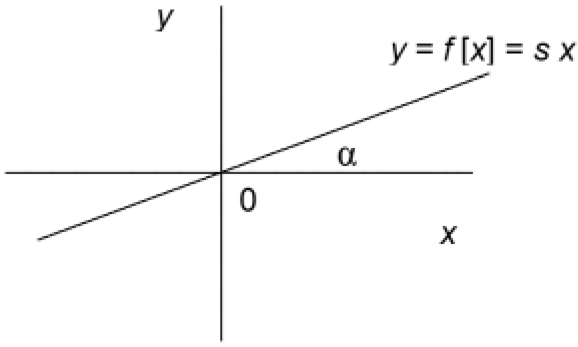
2. Short restatement of dynamic quotient and derivative

The following basically repeats sections from Colignatus (2016ad), but includes some edits and new comments. See COTP for the theoretical development and the approach to calculus in general (integral and derivative).

2.1. Ray through the origin and definition of dynamic quotient

Let us consider a ray – rays are always through the origin – with horizontal axis x and vertical axis y. The ray makes an angle α with the horizontal axis. The ray can be represented by a function as $y = f[x] = s\, x$, with the slope $s = \tan[\alpha]$. Observe that there is no constant term ($c = 0$). See Figure 3.

Figure 3. A ray with angle α and slope *s*



The quotient y / x is defined everywhere, with the outcome s, except at the point $x = 0$, where we get an expression 0 / 0. This is quite curious. We tend to regard y / x as the slope (there is no constant term), and at $x = 0$ the line has that slope too, but we seem unable to say so.

There are at least five responses:

(i) The argument can be that y has been defined as $y = s\, x$, so that we can always refer to this definition if we want to know the slope of the ray. This approach relies on a notion of a "memory of definitions", to be used when algebra lacks richness in expressiveness.

(ii) Standard mathematics can take off with limits and continuity.

(iii) A quick fix might be to redefine the function with a branching point:

$$\text{SlopeOfRay}[y, x] = \begin{cases} y / x = s & \text{if } x \neq 0 \\ s & \text{if } x = 0 \end{cases}$$

We can wonder whether this is all nice and proper, since we can only state the value s at 0 when we have solved the value elsewhere (or rely on the definition as in (i) again). If we substitute y when it isn't a ray, or example $y = x^2$, then we get a curious construction, and thus the definition isn't quite complete, since there ought to be a test on being a ray. Anyway, defining lines in this manner isn't a neat manner. It is really so, that we cannot define a line as $y = s\, x + c$ and that we must specify the branching when $x = 0$?

(iv) The slope y / x is regarded as a special case of "rational functions". See the section above and the discussion below of Range (2016c:16), Figure 5. If we work on coefficients only, then we get Ruffini's Rule (a case of "synthetic division"), see Colignatus (2016ef) also referring to MathWorld. [13] The first problem is that in this approach the issues of "identifying the factors"

[13] http://mathworld.wolfram.com/RuffinisRule.html

and "adjusting the domain" are only indicated and not made explicit via separate notations. The term "synthetic division" indicates that it might not be "proper division". To what extent is there proper division, so that "eliminating" the factor x generates a result that can be understood as the slope of the line at that point (i.e. fitting to the tangent in trigonometry) ? The second problem is that this remains within the realm of polynomials.

(v) The algebraic approach uses the following definition of the dynamic quotient. Let y / x be as it is used currently in textbooks, and let y // x be the following process or program, called *dynamic division* or *dynamic quotient*, with numerator y and denominator x:

> y // x ≡ { y / x, unless x is a variable and then: assume $x \neq 0$, simplify the expression y / x, declare the result valid also for the domain extension $x = 0$ }

Thus in this case we can use y // x = s x // x = s, and this slope also holds for the value $x = 0$, since this has now been included in the domain too.

We thus *extend the vocabulary of algebra*, so that *multiplication with variables* gets an inverse with *dynamic division by variables*. Since this is a new suggestion we must obviously be careful in its use, but the application to the derivative is a case that appears to work.

The case of the line may be seen as a special case of a polynomial. However, the general notion is "simplify", and there might be other ways than just eliminating factors.

2.2. Dynamic quotient has the denominator as a variable

Simplification only applies when the denominator is a variable but not for numbers. Thus x // x = 1 but 4 // 0 generates 4 / 0 which is undefined. Also x / x is standardly undefined for $x = 0$.

This definition assumes a different handling of different parts of the domain. The test on the denominator is a syntactic test. When the denominator is an expression like (p + 2) then the syntactic test shows that the denominator is a variable, $x = p + 2$. One does not substitute "(p + 2) is a variable" for substitution doesn't look at syntax but uses the value of the variable.

It has been an option in the {...} definition above to write "(a) variable" instead of "a variable", which allows a shift from the syntactic test towards the semantic test of variability, and which also allows *substitution* into the definition, like "(p + 2) is (a) variable". After ample consideration, already in 2007 and later explicitly in Colignatus (2014), I think that we are better served with the syntactic test on the denominator, since this directly leads to the question: what is the domain of the denominator ?

The use of the curly brackets {...} also borrows from *Mathematica.* The brackets signify a list, that can be a set, but when the elements are expressions then the sequential evaluation of those turns into a programme.

2.3. From eliminating factors in polynomials to general "simplification"

In multiplication, (x – 1) (x + 1) = (x^2 – 1) holds for all real x. For division we lack an efficient vocabulary to express (x – 1) = (x^2 – 1) / (x + 1), since this is undefined for x = -1. We can introduce branching, but still would have to use a limit to recover the value at x = -1. When we want to identify or *isolate* the factors however then this "isolate" would commonly be tantamount to requiring division.

An alternative way to identify factors (and find the derivative) for polynomials is the use of coefficients and Ruffini's Rule. If multiplication for polynomials is equivalent to manipulating coefficients, then the latter can also be used for the reverse process of division. See Colignatus (2016ef), that was inspired by (with thanks to) Harremoës (2016) also linking to Bennedsen (2004). It works for polynomials but is it general enough, for non-polynomials ?

There remains the notion of a slope however too. There is no clear link between coefficients (Ruffini's Rule) and the slope. We find the proper *values*, which suggests that there is such a link, yet this link must be shown. The method may be an efficient calculation method, but it doesn't explain that when we find outcome *s,* then we may also declare that it is valid for $x = 0$ (for we cannot do 0 / 0). Ruffini's Rule suggests that the user sets up a division, y / x, but when we look at the proof why it works, [14] then we see addition and multiplication, and thus division (or repeated subtraction) is only an *interpretation*. The method works on the coefficients, and it isn't for nought that the term "synthetic division" is used.

The slope of a curve $\Delta f / \Delta x$ contains the notion of division (or ratio). See also the definition of *tangent* in trigonometry for a right-angled triangle. This notion of a slope generates the link between derivative and integral, as the integral uses $(\Delta f / \Delta x) * \Delta x$ to find Δf. The fundamental theorem of calculus is: A function gives the area under its derivative. [15]

A crucial insight:

> When we want to find the root factor $x + 1$ in (x^2 – 1), then we don't have to assume $x \neq -1$, but we can assume the unrelated $x \neq 1$, and then isolate the root factor as (x + 1) = (x^2 – 1) / (x – 1).

One might deem this acceptable. It might be a rationale for the theory of "rational function" – group theory version (RF-GT) to define such singularities away. This theory doesn't seem to care that we must also say something about factor $x + 1$ at $x = 1$, but it would be straightforward to plug those holes in multiplicative form. The key question then is:

> If we are willing to assume $x \neq 1$ and adjust the domain afterwards (in multiplicative form) to again include it, then why would we not do so for $x \neq -1$ directly ?

Reasoning like this generates the notion of the dynamic quotient as a useful extension of our vocabulary.

Students must simplify algebraic expression like (x^2 – 1) / (x – 1) anyhow. Since the dynamic quotient allows them to do so consistently with (x^2 – 1) // (x – 1), there is no reason not to allow them to do so for the derivative too.

Eliminating factors is one way of simplification. There might be more ways. Thus the dynamic quotient uses the general notion of "simplification".

2.4. Perspective on division

The core of the new algebraic approach to the derivative lies in a new look at division. While division is normally defined for numbers, we now use the extension with variables and expressions with variables. Variables have their domains. By default the domain is the real numbers. (There might be symbols with unspecified (only potential) domains though: the "nondetermineds" of RF-GT.)

Let us distinguish the passive division result (noun) from the active division process (verb). For didactics it is important to write y for the numerator and x for the denominator, and not the other way around. In the active mode of dividing y by x we may first simplify algebraically under the assumption that $x \neq 0$, or that 0 is not in the domain of the denominator. Subsequently the result can also be declared valid for $x = 0$. This means *extending the domain*, i.e. not setting $x = 0$ but merely including that element in the domain.

Active division is not an entirely new concept since we find the main element of simplification well-defined in the function Simplify in *Mathematica,* see Wolfram (1996). For us there is the particular application of Simplify[y / x]. This doesn't claim that this well-definedness satisfies conditions for RM. For empirical research, it removes ambiguity, where students will have

[14] https://en.wikipedia.org/wiki/Horner's_method#Description_of_the_algorithm

[15] http://mathworld.wolfram.com/FundamentalTheoremsofCalculus.html

various levels of skills on simplification, and we can refer to the computer output as an empirical standard. The active notion of division still requires a separate notation for our purposes. Denote it as $y // x$ or $(y\ x^D)$ where the brackets in the latter notation are required to keep y and x together, and where the D stands for *dynamic division*. In the same line of thinking it will be useful to choose static $H = -1$, and have $x \,.\, x^H = 1$ for $x \neq 0$. H gives a half turn as imaginary number i gives a quarter turn.

There is already an active notion (verb) in taking a ratio $y : x$. But a ratio is not defined for $x = 0$. Normally we tend to regard division y / x as already defined for the passive result without simplification – i.e. defined except for $x = 0$. Non-mathematicians will tend to take y / x as an active process already (so they might denote the passive result as $y // x$ instead). For some it might not matter much, since we might continue to write y / x and allow both interpretations depending upon context. This is what Gray & Tall (1994) call the "procept", i.e. the use of both concept and process: "The ambiguity of notation allows the successful thinker the flexibility in thought (...)". In that way the paradoxes of division by zero are actually explained, i.e. by confusion of perspectives. It seems better to distinguish y / x and $y // x$.

2.5. Already used in mathematics education

Clearly, mathematics education already takes account of these aspects in some fashion. In early excercises pupils are allowed to divide $2\,a / a = 2$ without always having to specify that a must be nonzero. At a certain stage though the conditions are enforced more strictly. A suggestion that follows from the present discussion is that this process towards more strictness can be smoother by the distinction between / and //.

An expression like $(1 – x^2) / (1 - x)$ is undefined at $x = 1$ but the natural tendency is to simplify to $1 + x$ and not to include a note that there is branching at $x \neq 1$, since there is nothing in the context that suggests that we would need to be so pedantic, see **Table 1**, left column. This natural use is supported by the right column. The current practice in teaching and math exams is to use the division y / x as a hidden code that must be cracked to find where $x = 0$, but it should rather be the reverse, i.e. that such undefined points must be explicitly provided if those values are germane to the discussion. Standard graphical routines also tend to skip the undefined point, requiring us to give the special point if we really want a discontinuity.

Table 1. Simplification and continuity

Traditional definition overload	*With the dynamic quotient*
$f(x) = (1 – x^2) / (1 - x) = 1 + x \quad (x \neq 1)$ $f(1) = 2$	$(1 – x^2) // (1 - x) = 1 + x$

In common life there is no need to be very strict about always writing “//”. Once the idea is clear, we might simply keep on writing “/” as a procept indeed. It remains to be tested in education however whether students can grow sensitive to the context or whether it is necessary to *always* impose strictness. For the mathematically inclined pupils or students graduating at highschool one would obviously require that they are aware that y / x is undefined for $x = 0$ and that they can find such points.

2.6. Subtleties

The classic example of the inappropriateness of division by zero is the equation

$$(x - x)\ (x + x) = x^2 – x^2 = (x - x)\ x,$$

where unguarded "division" by $(x - x)$ would cause $x + x = x$ or $2 = 1$.

This is also a good example for the clarification that the rule, that we should never divide by zero, actually means that we must distinguish between:

- *creation* of a quotient by the choice of the *infix* between (x - x) (x + x) and (x - x)
- *handling* of a quotient such as (x - x) (x + x) *infix* (x - x) once it has been created.

The first can be the great sin that creates such nonsense as 2 = 1, the second is only the application of the rules of algebra. In this case, x - x is a constant (0) and not a variable, so that simplification generates a value Indeterminate, for both infices / and //. (One may notice that $x - x = 0$ is the zero polynomial $Q[z] = 0$ in the reference to RF-GT above.)

Also (a (x + x) / a) would generate $2x$ for $a \neq 0$ and be undefined for $a = 0$. However, the expression (a (x + x) // a) gives $2x$, and this result would also hold for $a = 0$, even while it then is possible to choose $a = x - x = 0$ afterwards: since then it is an instant (and not presented as a variable).

2.7. The derivative

The algebraic definition of the derivative then follows directly:

$$f'[x] = \{\Delta f \,//\, \Delta x, \text{ then set } \Delta x = 0\}$$

This means first algebraically simplifying the difference quotient, expanding the domain of Δx with 0, and then setting Δx to zero.

The Weierstraß $\varepsilon > 0$ and $\delta > 0$ and its Cauchy shorthand $\lim(\Delta x \to 0)\ \Delta f / \Delta x$ are paradoxical since those exclude the zero values that are precisely the values of interest at the point where the limit is taken. Instead, using Δf // Δx on the formula and then extending the domain with $\Delta x = 0$, and subsequently setting $\Delta x = 0$ is not paradoxical at all. Students only need an explanation why one would take those steps.

Much of calculus might well do without the limit idea and it could be advantageous to see calculus as part of algebra rather than a separate subject. This is not just a didactic observation but an essential refoundation of calculus. E.g. the derivative of $|x|$ traditionally is undefined at $x = 0$ but would algebraically be sign[x], see Colignatus (2011b). The derivative gives the change in the area under the curve, and this might not be the same as the slope of the incline (tangent line).

2.8. Differentials

There is the following progress from 2011 to 2016:

- COTP (2011ab) uses "***df*** / ***dx*** " as a icon only, or "***d*** / ***dx***" as an operator, to link up with history only, so that everyone who still uses this notation for the derivative can see that this has the same outcome,
- Colignatus (2016d) proposes to use dx and dy as variables, and to define $dy = f'[x]\, dx$ so that dy // $dx = f'[x]\, dx$ // $dx = f'[x]$. This is actually the situation with the ray that this section started out with. Thus the derivative $f'[x]$ is found by other means, and then is used to set up the ray with dx and dy. The dynamic quotient dy // dx should not be confused with finding the derivative (since dy can only be defined if one already has the derivative).

For users new to the notions of the dynamic quotient and the algebraic approach to the derivative, the relation dy // $dx = f'[x]$ might be confusing since they might think that the dynamic quotient suffices to find the derivative. (An answer to this is: There are various roads to Rome but only few ways to build it.)

2.9. Derivative at a point $x = a$

In the standard notation for the derivative, x is fixed and the new variable is Δx.

There is also a notation when x is retained as a variable, and the fixed value is $x = a$. If we want to find the derivative at a point $x = a$ then we would use above method to find $f'[x]$ and then substitute the value to find $f'[a]$. This suffices.

If one wishes to specify a in the deduction, then use:

$$\{(f[x] - f[a]) \,//\, (x - a), \text{ then set } x - a = 0\} = f'[a]$$

The following notation would be advised against, since it mixes changes of perspectives:

$$\{\Delta f \,//\, \Delta x, \text{ then set } \Delta x = x - a = 0\} = f'[a]$$

2.10. Definition of the incline

I will use the word "incline" instead of "tangent (line)" since the incline may also cut the function, see Colignatus (2016e). Let us use "tangent" in trigonometry only.

The core notion is that the slope s must be taken as the *slope of the curve* at the point of consideration. We don't have just the line. First we determine the slope of the curve, and then create the incline with it.

The point-slope form with Δx is: $y - f[x] = s\,\Delta x$ at the point of inclination $\{x, f[x]\}$.

The point-slope form with a is: $y = \text{incline}[x] = s\,(x - a) + f[a]$ at the point of inclination $\{a, f[a]\}$.

The standard form is $y = c + s\,x$, with slope is s and constant $c = f[a] - s\,a$.

3. The use of an equivalence class in RF

This paper assumes that most readers will have a background in RF.

Readers unfamiliar with RF would be advised to skip this section. The subsequent sections discuss RF-FL and RF-GT, and one can return to this present section later.

Let us consider the idea that RF already resolved the issue of the dynamic quotient, namely via the notion of an equivalence class.

Danilov & Shokurov (1998:212) define a rational function on variables T, and then create an algebraic variety [16] on some X with an equivalence class. [17]

[16] https://en.wikipedia.org/wiki/Algebraic_variety
[17] https://en.wikipedia.org/wiki/Equivalence_class

1.3. Rational Functions. A rational function in the variables $T_1, \ldots, T_n$ is defined as the ratio f/g of two polynomials, f and g, in $T_1, \ldots, T_n$, with $g \neq 0$. Note that it is not a function on the whole of $\mathbf{A}^n$, but only on the open subset $\mathcal{D}(g) \subset \mathbf{A}^n$ where g is different from zero. It is thereby uniquely determined by its restriction to any nonempty open subset $U \subset \mathcal{D}(g)$. Conversely, any regular function on an open set $U \subset \mathbf{A}^n$ can be represented by a rational function.

This suggests the following generalization to any algebraic variety X. A *rational function* on X is an equivalence class of regular mappings $f: U \to K$, where U is an open dense subset of X. Two such maps, say, $f: U \to K$ and $f': U' \to K$, are regarded as equivalent if they agree on $U \cap U'$. This really is an equivalence relation, because $U \cap U'$ is also dense in X. (The naïve definition of a rational function as the ratio of two regular functions is of little interest, since on $\mathbb{P}^n$ there are few regular functions.)

Rational functions can be added and multiplied together, so that the set $K(X)$ of all rational functions on the variety X is a ring. It is clear that $K(X)$ is the direct limit $\varinjlim \mathcal{O}_X(U)$ of the rings $\mathcal{O}_X(U)$, as U runs through the open dense subsets of X. If X is irreducible, $K(X)$ is even a field. Indeed, if $f: U \to K$ is a nonzero function, it is invertible on the nonempty (and therefore dense) open subset $U - f^{-1}(0)$. Further, for X irreducible, the field $K(X)$ coincides with the quotient field of the integral domain $K[U]$, where U is any affine chart of X. For arbitrary X, the ring $K(X)$ is the direct sum of the fields $K(X_i)$, where the X_i denote the irreducible components of X.

Thus:

- There is a manipulation of the domain, with *U* and *U'*. (It is recognised that these are different, and then the equivalence class is invoked.)
- Division by zero is removed by referring to density, which relies on limits.

Thus, this is not "algebraic" but rather "analysis". (Unless one regards algebra as how can say the same thing without using epsilon and delta.) (PM. I used Google Books to find this quote, and though I might grasp largely what these authors do, check also Colignatus (2016bc), I am at a loss on what this *X* is, while Google Books doesn't make it easy to get more on this. I leave it here.)

Wikipedia – a portal, no source – is being filled by MIT students copying there textbooks, and this "encyclopedia" becomes increasingly less transparant. The following quote however is transparant, and let us hope (in vain) that we can rely on this quote (Feb 3 2017): [18]

[18] https://en.wikipedia.org/wiki/Rational_function

A function $f(x)$ is called a rational function if and only if it can be written in the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials in x and Q is not the zero polynomial. The domain of f is the set of all points x for which the denominator $Q(x)$ is not zero.

However, if P and Q have a non constant polynomial greatest common divisor R, then setting $P = P_1 R$ and $Q = Q_1 R$ produces a rational function

$$f_1(x) = \frac{P_1(x)}{Q_1(x)},$$

which may have a larger domain than $f(x)$, and is equal to $f(x)$ on the domain of $f(x)$. It is a common usage to identify $f(x)$ and $f_1(x)$, that is to extend "by continuity" the domain of $f(x)$ to that of $f_1(x)$. Indeed, one can define a rational fraction as an equivalence class of fractions of polynomials, where two fractions $A(x)/B(x)$ and $C(x)/D(x)$ are considered equivalent if $A(x)D(x) = B(x)C(x)$. In this case $\frac{P(x)}{Q(x)}$ is equivalent to $\frac{P_1(x)}{Q_1(x)}$.

Thus, there is a manipulation of the domain too. But this relies on (numerical) continuity (again), and not on the algebra of expressions.

If $p[x] = q[x] (x – a)$, then $p[x] / (x – a)$ would be *equivalent* to $q[x]$, while the ratio in the LHS is not defined at $x = a$, and while $q[x]$ in the RHS could be defined at $x = a$. I am afraid that this overstretches the notion of "equivalence". Still, when something is *equivalent,* then why *identify* ?

In comparison, the statement $p[x] // (x – a) = q[x]$ makes perfect sense, since the domain on the LHS is adjusted so that it is the same domain as on the RHS. We namely have:

$$p[x] // (x – a) = q[x] (x – a) // (x – a) = q[x]$$

In the parlance of "equivalence class" for numbers there is identity like 2 / 4 = 1 / 2, even though the expressions are different. But it is awkard to write $p[x] / (x – a) = q[x]$, if we know that the domains differ. (Unless the mind quickly switches to RF-GT without saying so.)

This statement (today) from Wikipedia – a portal, no source – shows the same tricky use of words (and see also the *Introduction* for issues of vocabulary):

> "The rational function $f(x) = x / x$ is equal to 1 for all x except 0, where there is a removable singularity. The sum, product, or quotient (excepting division by the zero polynomial) of two rational functions is itself a rational function. However, the process of reduction to standard form may inadvertently result in the removal of such singularities unless care is taken. Using the definition of rational functions as equivalence classes gets around this, since x / x is equivalent to 1 / 1."

I really don't understand how one "gets around this". The latter equivalence class merely means that $x = x$. Obviously, for a rational function x / x the value $x = 0$ is excluded and in $x = x$ the value $x = 0$ is included. These are different situations. If this is called an "equivalence class" then this defines the problem away. It is a great leap of magic to make 0 / 0 also *seem* equivalent to 1 / 1, but you can perform this magic by abusing the word "equivalence".

What I understand, I have put into the definition of the dynamic quotient: the reliance on expressions and the manipulation of the domain.

4. Linking up to school mathematics

This section assumes readers that might have skipped the discussion on equivalence class.

4.1. The theory of the "rational function" cannot be used in highschool (yet)

Consider the subject matter of a "rational function" – see the *Introduction*. This is defined as $R[z] = P[z] / Q[z]$ for real or complex z such that $Q[z] \neq 0$. [19] Thus the denominator cannot become zero. This latter option has been removed surgically by definition, since it is now seen as no relevant topic of discussion . (This holds for both RF-GT and RF-FL.) It is easy to claim now that there are no singularities here (other than the poles in the extended complex plane). If one would study ratios such that $Q[z]$ might be zero, then this would not be catalogued under the notion of "rational function". In the same manner, the "polynomial remainder theorem" excludes such zero denominators, whence its proof can avoid limits.

One might argue that COTP also avoids that $Q[z]$ is zero, so that this falls under "rational functions" (if there would be polynomials). Yet there remains a subtlety w.r.t. the handling of the domain for *Q.*

A mathematician wrote me, and I paraphrase and adapt to current notation, and observe that this mathematician refers to a domain whence he uses RF-FL:

> "$P[x] / Q[x] = R[x]$ (no dynamic quotient here) makes perfectly good sense: On the left we have a rational function, which, algebraically, turns out to be equal to the polynomial on the right. No need to worry that the left side may not be defined for some values ($x = a$ in this case). It is always understood that the domain of a rational function does *not* include points where the denominator vanishes. I just don't understand the point of // in $P[x]$ // $Q[x] = R[x]$, i.e., viewing this particular statement as a dynamic quotient."

The point that this mathematician doesn't understand here is that students have no background in research mathematics. We have been teaching them since kindergarten that they cannot divide by zero. We need to keep up that discipline. The systematic neglect as research mathematicians are doing is risky, since students might not have the same discipline, and still make such a division where it isn't allowed. They would become quite confused by the trick in research mathematics to simply define the problem away. (Or, for RF-GT: use the same symbol y / x for RF[y, x].)

4.2. A picture tells more than a thousand words

A picture tells more than a thousand words. Let us see how Range (2016c:16) in the first proof of "Proposition 5.1" switches between "rational function" (ratio of factors) and the theory of polynomials (product of factors), and manipulates the domain during the switch without actually saying so.

[19] https://www.encyclopediaofmath.org//index.php?title=Rational_function&oldid=17805

Figure 4. Range (2016:16c) Proposition 5.1

Proposition 5.1. *If the polynomial P of degree $n \geq 1$ has a zero at the point $x = a$, then $(x - a)$ is a factor of P, i.e., there exists a unique polynomial q of degree $n - 1$ such that*

$$P(x) = q(x)(x - a).$$

Proof 1. This is a well known simple consequence of the division algorithm for polynomials, as follows. By that algorithm, $P(x)/(x - a) = q(x) + R(x)/(x - a)$ for some polynomial q, where the remainder R is a polynomial of degree *less* than the degree of $x - a$, which is one. So R has degree 0 and hence must be a constant R_0. Thus $P(x) = q(x)(x - a) + R_0$, and evaluation at a shows that $0 = q(a) \cdot 0 + R_0$, so that $R_0 = 0$. This completes the proof of the proposition. ■

Apparently $R = R[x] / (x – a)$. Observe that Range in the above:

- writes "$P[x] / (x – a)$" but then he works within the confines of "rational functions" such that the root $x = a$ is disregarded,
- while the domain is later extended with $x = a$ (*without making an issue of it*) in "$P[x] = q[x] (x – a) + R_0$".

For highschool this is *inconsistent.* This might be acceptable for the target group of students of Range (2016c), who must learn about (the professional schizophrenia of) "rational functions" (ratio of factors) and "polynomial theory" (product of factors). COTP considers it necessary to introduce the notion of "dynamic division" for the general case (for all students).

PM. See also the second proof of "Proposition 5.1" below, Figure 5

4.3. State clearly that we are manipulating the domain

Thus we must now find a way for students to eliminate a denominator that might cause a singularity.

There is the difference in approach by Parmenides ("existence is timeless") [20] and Heraclitus ("panta rei"). [21] Parmenides contributed to the notion that the laws of nature are given. It would be awkward to see each event as a law of nature though. Functions are defined for domain and range. If these would change, then there would be another function. We better stick to this. Thus if we allow for the flexibility of manipulating the domain, then this best is put into an operator. Thus use $y // x$ instead of y / x (when relevant).

My suggestion is also that such manipulation of the domain is already being done actually within mathematics. Texts switch between (a) $P[x] / Q[x] = R[x]$ in which $Q[x] \neq 0$ within the subject of "rational functions" and (b) forms $P[x] = R[x]\ Q[x]$ in which $Q[x]$ can be 0 within the subject of polynomials. The issues thus are compartmentalised, and this seems neatly reasonable, but the whole generates confusion when the switches between the compartments are not handled clearly. (For example, wikipedia – a portal, no source – has mere HTML links, without a pop-up window that explains the key switch in perspective and that the domain is being manipulated.) Historically we can understand the rise of these different "subjects", but if we allow the manipulation of the domain then these subjects can join up.

[20] https://en.wikipedia.org/wiki/Parmenides
[21] https://en.wikipedia.org/wiki/Heraclitus

5. Range's second proof

For the factoring of polynomials, we already saw part of this above in Figure 4 but let us look at it now more fully, see Figure 5.

Figure 5. Range (2016c:16-fuller)

Before continuing with the tangent problem, let us review an important fundamental fact about zeroes of polynomials. Recall that a polynomial P is a function whose value at the real number x is given by a formula $P(x) = c_n x^n + c_{n-1}x^{n-1} + ... + c_1 x + c_0$, where the coefficients $c_0, ..., c_n$ are certain fixed numbers. If $c_n \neq 0$, the polynomial P is said to have degree n.

Proposition 5.1. *If the polynomial P of degree $n \geq 1$ has a zero at the point $x = a$, then $(x - a)$ is a factor of P, i.e., there exists a unique polynomial q of degree $n - 1$ such that*

$$P(x) = q(x)(x - a).$$

Proof 1. This is a well known simple consequence of the division algorithm for polynomials, as follows. By that algorithm, $P(x)/(x - a) = q(x) + R(x)/(x - a)$ for some polynomial q, where the remainder R is a polynomial of degree *less* than the degree of $x - a$, which is one. So R has degree 0 and hence must be a constant R_0. Thus $P(x) = q(x)(x - a) + R_0$, and evaluation at a shows that $0 = q(a) \cdot 0 + R_0$, so that $R_0 = 0$. This completes the proof of the proposition. ■

Because this result is so important for our discussion, we shall also verify it by a different argument that does not rely on the division algorithm. The reader eager to proceed may surely skip this alternate verification.

Proof 2. We rewrite x as $(x - a) + a$ and note that

(For readers who jump to this section without having seen the above: Observe that Range in the above writes "*P*[*x*] / (*x* – *a*)" but then he works within the confines of "rational functions" such that the root *x* = *a* is disregarded, while it is later restored (*without making an issue of it*) in "*P*[*x*] = *q*[*x*] (*x* – *a*) + R_0". For highschool this is *inconsistent.* This might be acceptable for the target group of students of Range (2016), who must learn about the "compartments" of "rational functions" (ratio of factors) and "polynomial theory" (product of factors). COTP considers it necessary to introduce the notion of "dynamic division" for the general case (for all students).)

The second proof rewrites the coefficients. Thus it is important to see:

- The *existence* of this factorisation can be shown *without division*.
- The actual *isolation* of that factor *q*[*x*] is a separate issue. (How can we identify it ?)
- Proofs can use this factorisation too if they do not rely on isolation of the factor.
- If we work on the coefficients only, then we get Ruffini's Rule. Thus there is a numerical algorithm to work on the coefficients only, that allows isolation of such a factor. (That is, if we have the factor, then we can calculate the remainder.)

- If a discussion uses the identification and isolation of factors without using coefficients (Ruffini's Rule), then a trick is used, in which the domain is being manipulated.
- The dynamic quotient is a *definition.* If $P[x]$ // $(x – a) = q[x]$ follows from its application and then meets with an independent derivation via the method of coefficients, then this consistency is wellcome. (So use it.) A theorist coming from the realm of polynomial theory and RF who hasn't heard about the dynamic quotient before might argue that the approach via coefficients "proves" that the dynamic quotient can be used. There is no reason however to claim that polynomial theory would be more fundamental than algebra.

6. Rational Functions – Fundamental Level (RF-FL)

When we have Δf / Δx where Δf is polynomial (or algebraic) then we might link up to the theory of "rational functions".

> "A rational function is a function $w = R(z)$, where $R(z)$ is a rational expression in z, (...). A rational function can be written (non-uniquely) in the form $R(z) = P(z) / Q(z)$ where P, Q are polynomials, $Q(z) \neq 0$." [22]

We now look at the fundamental level (not group theory). The domain of Q excludes the z such that $Q(z) = 0$.

6.1. RF - Fundamental Level: using a domain and not using group theory

An example of the fundamental level is given by Juha Pohjanpelto at Oregon State, for his (2002) course on differential calculus. This allows for functions that would have domains:

> "Rational function" is the name given to a function which can be represented as the quotient of polynomials, just as a rational number is a number which can be expressed as a quotient of whole numbers. Rational functions supply important examples and occur naturally in many contexts. All polynomials are rational functions." [23]

This remains closer to the perceptions of students (non-math-majors).

> "The domain of the rational function $p(x)/q(x)$ consists of all points x where $q(x)$ is non-zero. This domain really depends on the way in which $p(x)$ and $q(x)$ are chosen. For example, the function $g(x)$ above can be written as
>
> $$g(x) = \frac{(x - 2)(x + 2)}{(x - 3)(x - 2)}$$
>
> which (...) simplifies to $(x+2)/(x-3)$, but the $x = 2$ is NOT in the domain of $g(x)$ whereas it IS in the domain of $(x+2)/(x-3)$."

Unfortunately, Pohjanpelto doesn't explain the relation between $g[x]$ and the simplified expression. In the group theory version (see below) they would claim identity, but there they neglect the domain.

Also:

> "Functions which are quotients of functions other than polynomials are not called rational functions, but the same considerations apply: The domain of a quotient includes only points where
>
> - both the numerator and the denominator are defined, and

[22] https://www.encyclopediaofmath.org//index.php?title=Rational_function&oldid=17805
[23] http://oregonstate.edu/instruct/mth251/cq/FieldGuide/rational/lesson.html

- the denominator is not 0. ”

As cautious as Pohjanpelto has been up to now, in the following quote he takes this caution for granted, but effectively forgets it, since the equation $f(x) = p(x)/q(x) = b$ is *not* the same as $p(x) = b\ q(x)$ when the domains differ.

> "If $f(x) = p(x)/q(x)$ is a rational function, the equation $f(x) = b$ is the same as the equation $p(x) = b\ q(x)$ or
>
> $$p(x) - b\ q(x) = 0.$$
>
> The function $p(x) - b\ q(x)$ is a polynomial, so solving an equation involving a rational function reduces to finding roots of a polynomial."

One reason why mathematicians have switched to group theory with the "rational function" without domains, must be to avoid this kind of omission. If the possibility of error has been swept under the rug, then you cannot make it anymore.

6.2. Removable singularities

It is useful to refer to the analysis of removable singularities, as this refers to domains (and is categorised rather as analysis than as algebra).

This still is RF-FL, which differs from the approach of rational functions in group theory, or RF-GT.

Observe that mathematicians thinking in terms of RF-GT may still use the term "removable singularity" but then actually in the meaning of "removed singularity". (One should not expect group theorists to start removing what is yet only removable. They already removed it, by neglecting it, and not by using limits to show that it actually is removable.)

6.3. Ruffini's Rule

However, the proof on factoring by means of coefficients, that leads to Ruffini's Rule, would also bypass the singularities (or provide an alternative way to remove those).

Bennedsen (2004) and Harremoës (2016) are examples of this application in RF-FL.

The major conceptual issue w.r.t. factoring is whether the multiplicative form $df = f\,'[x]\ dx$ can still be recognised as the slope $df\ /\ dx$ (since a slope is given by the tangent in trigonometry). Ruffini's Rule factors and solves $df\ /\ dx$ by "synthetic division", but to what extent is "synthetic" also proper division, so that "eliminating" the factor dx generates a result that can be understood as the slope of the function at that point ?

Potentially, the group theory version was created by those who were not charmed by the available proof via coefficients.

7. Rational Functions – Group Theory (RF-GT)

When we have $\Delta f\ /\ \Delta x$ where Δf is polynomial (or algebraic) then we might link up to the theory of "rational functions".

> "A rational function is a function $w = R(z)$, where $R(z)$ is a rational expression in z, (...). A rational function can be written (non-uniquely) in the form $R(z) = P(z)\ /\ Q(z)$ where P, Q are polynomials, $Q(z) \neq 0$." [24]

[24] https://www.encyclopediaofmath.org//index.php?title=Rational_function&oldid=17805

We now look at the approach in group theory that excludes domains. (There may be group theory that doesn't exclude domains, but this would be a mere extension of RF-FL.)

The $Q[z] \neq 0$ means that $Q[z]$ is not the zero element in the group. This zero element is $Q[z] = 0$ for all *z.*

7.1. Group theory: "function" without a domain

Ahmet Feyzioglu (1990) chapter 3 paragraph 36 usefully clarifies (p430): [25]

- "Thus a rational function over *D* is a fraction *f* / *g* of two polynomials over *D* with $g \neq 0$."
- "This terminology is unfortunate and misleading, because a rational function is not a function (...). A rational function is *not* a function of the 'rational' kind, whatever that might mean."
- "The technical term we defined is *rational function,* a term consisting of two words "rational" and "function". The meaning of the words "rational" and "function" do not play any role in [its] definition (...)"

PM. There is no definition what a "fraction" is. Note also that Feyzioglu uses "an indeterminate" rather than "variable", presumably because *variables* imply *domains*.

Then on p431:

rational function is a fraction of polynomials over D. The reader should exercise caution about this point. One should not conclude that

$$\frac{x^2 - 1}{x - 1} \quad \text{and} \quad \frac{x + 1}{1} \quad \text{in } \mathbb{C}(x)$$

are different rational functions, on grounds that that their domains are different, since the domain of the first one does not contain 1, whereas 1 is in the domain of the second one. Neither of them has a domain, for neither of them is a function. And these rational functions are equal because the polynomials $(x^2 - 1)1$ and $(x - 1)(x + 1)$ in $\mathbb{C}[x]$ are equal.

This refers to a definition on p340: ($f / g = h / j$) if and only if ($f j = h g$).

Really, if the RHS is the relevant criterion, then what is the *use* of writing the LHS ?

7.2. Not writing *f* / *g* but referring to a pair {*f, g*}

Paul Garrett at UMN has this explanation: [26] The "fraction" is only a pair {*f, g*}, but then still see his footnote 19.

He happens to write "if" instead of "iff".

Still, what is the *use* of writing the LHS, even as a pair ?

7.3. Group theory finds that rational functions form a field

Aha. In both cases, a nonzero element *f* / *g* would have a multiplicative inverse *g* / *f,* such that $f / g * g / f = 1$. And multiplicative inverse means that we have a field.

[25] http://feyzioglu.boun.edu.tr/book/chapter3/ch3(36).pdf
[26] http://www-users.math.umn.edu/~garrett/m/algebra/notes/06.pdf

In other words, y / x represents division of variables that have a domain, and, abusing this notation for *nondetermineds* that have no domain, we can apply the group theory notion of a multiplicative inverse.

If we avoid that notation with /: in the equality ($f\,j = h\,g$) the pair $\{f, g\}$ is called a "rational function" and the set of those is a field, *because* there is a $\{h, j\} = \{g, f\}$ such that $f\,g = g\,f$.

My problem then is that obvious phenomena are recorded in needlessly complicated fashion, while the true problem, namely the domain and division by zero, is swept under the carpet.

PM 1.It makes one also wonder whether the plain property of commutativity $f\,g = g\,f$ is not abused for the notion of multiplicative inverse. [27] (Potential theorem: For all commutative groups with elements a and b and $a\,b = b\,a$, it is possible to create sets of pairs $\{a, b\}$ such that there is a field with inverse $\{b, a\}$, namely $\{a, b\} * \{b, a\} = \{a\,b, b\,a\} = \{1, 1\}$. And the trick is to eliminate the zero elements too, of course.)

PM 2. The above also needs:

- $\{f, g\} * \{h, j\} = \{f\,h, g\,j\}$
- $\{a, b\} = \{1, 1\}$ iff $a = b$.

PM 3. It might be an exercise for students to test the properties of a field, *as if* this would be a relevant exercise, but I doubt that it is. [28]

7.4. Summary finding

A group theory version RF-GT with "rational functions" thus "works" because the notion of a domain is eliminated. There is only one zero element and not a bunch of zero points and singularities.

Statements like QUOTE $f[x] = q[x]\,(x - a)$ is equivalent with $f[x] / (x - a) = q[x]$ UNQUOTE are possible, because RF-GT assumes that there are no domains.

When "variables" are associated with domains, apparently "nondetermineds" are not. It is just a "placeholder". An expression like "$x - a$" or even "x" itself would only be a term (a simple polynomial) and no number itself.

The approach works for group theory itself. It allows group theorists to do what they do, like showing that "rational functions" with this definition form a field.

But it is not clear what this further amounts to. When this field is applied to functions with a domain, as in RF-FL, then all properties must be checked and properly translated, and apparently we still need an approach for singularities: namely, when the carpet is rolled up, and these show from under the carpet. Thus, there doesn't seem to be much use for this exercise in group theory.

Thus:

- Results from RF-GT cannot by implemented just like that, without translation. They need to be translated into RF-FL with proper handling of singularities.

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[27] https://www.encyclopediaofmath.org/index.php/Commutativity

[28] http://mathonline.wikidot.com/the-field-of-rational-functions and http://www.math.cornell.edu/~protsak/hiahw.html the question for Feb 19: "A rational function is a ratio of two polynomials with real coefficients, R(x)=P(x)/Q(x), Q≠0. Equality between rational functions and the operations of addition and multiplication are defined similarly to the case of the usual fractions (i.e. the rational numbers). Prove that the set of rational functions with these operations is a field. You may assume standard facts about fractions and polynomials."

- This translation works because there is a proof on factoring that uses coefficients, which results into Ruffini's Rule. (This latter proof is not a result of RF-GT.)
- RF-GT might look impressive, but it sweeps the real problem under the carpet, and its main result, that the rational functions (without domain) form a field, is only interesting for group theory itself without relevance. (What is relevant namely is only relevant because of Ruffini's Rule.)

7.5. Perhaps some potential uses

Potentially, the theory of "rational functions" RF-GT might be turned into part of a *theory of expressions* (namely polynomial expressions). The "Encyclopedia of Mathematics" article already referred to "expression". See Colignatus (2014).

Potentially, we might include domains, so that statements like QUOTE $f[x] = q[x] (x - a)$ is equivalent with $f[x] // (x - a) = q[x]$ UNQUOTE are possible (to start with polynomials).

8. Disclaimer. What is the use of group theory ?

A disclaimer is: I am new to this theory of "rational functions". It appears that much of the theory of rational functions is formulated in terms of group theory. I am only aware of some basics of group theory too.

Colignatus (2016h) (2017ae) give my misgivings about group theory.

8.1. Numbers and their notation are didactically much more important

For education it is very important that students develop a good understanding of numbers and how to denote numbers. E.g. keep 2 + ½ as it is, and don't write 2½ since the latter might read as 2 * ½ like 2*a* or 2 meters. E.g. define real numbers as (potentially infinite) strings of digits, see Timothy Gowers (undated). I regard "group theory" as partly an excuse for theoretical mathematicians to avoid such issues of definition and notation. They rather look at "existence" of numbers rather than at how to actually use them.

8.2. Confusion about division

Statements in group theory tend to suggest that groups are defined by operators (addition, multiplication) but in reality the elements are important.

- A ring supposedly has no division, but there is repeated subtraction like 12 – 4 – 4 – 4 = 0, and the main issue is that there is e.g. no element x that $1 - x - x = 0$.
- A field has division except for zero. (Each $x \neq 0$ has a x^H such that $x\, x^H = x^H\, x = 1$.) However, the main point is that there is such element x with $1 - x - x = 0$ now. This element is defined as Arg[$1 - x - x = 0$], namely the x that solves this condition.

The crux of the matter is the existence of elements, not the notion of division (repeated subtraction).

8.3. Other confusing statements

Also other statements in group theory can be quite confusing. For example there is a notion that one group is contained in the other. For example, it is claimed here that the "integral domain" (taken from "integer" and not from "integration") contains fields. [29]

While I am aware of the distinction between an inverse with e.g. the construct $f[f^{-1}[x]]$ and a multiplicative inverse for group theory $f * f^{-1} = 1$, I get nervous again when a discussion shifts

[29] https://en.wikipedia.org/wiki/Integral_domain

to "cancellation" [30] as if this would be relevant for the topic (without an explanation why it would be relevant).

But what does "contain" mean ? The crux of the matter is the existence of elements, not the notion of division. I understand what elements are, and that the integers form a subset of the reals. But "contain" for group theory might mean the inverse order of subsets of elements ?

If *f* is a field, are all its properties also covered by the integral domain ? I would not think so because a field has division and an integral domain doesn't have divison but only cancellation. Does "contain" mean that if *id* is an integral domain, then all its properties are also covered by a field ? I would not think so, for the *property of not having division* is not covered by the field, that has division. Thus, I am quite confused about what this "class containment" might mean.

9. Conclusions

The conclusions are also put into the Summary:

The derivative $df / dx = f\,'[x]$ concerns the slope of the function. If df is polynomial then we have a rational function $r = df / dx$. The theory of rational functions at the fundamental level (RF-FL) recognises domains and singularities. Conventionally singularities must be resolved by using limits. For polynomials there is the possibility of factoring, $df = f\,'[x]\, dx$. Multiplicative factoring can be proven by use of coefficients only, which leads to Ruffini's Rule.

The major conceptual issue w.r.t. factoring is whether the multiplicative form $df = f\,'[x]\, dx$ can still be recognised as the slope df / dx (since a slope is given by the tangent in trigonometry). Ruffini's Rule factors and solves df / dx by "synthetic division", but to what extent is "synthetic" also proper division, so that "eliminating" the factor dx generates a result that can be understood as the slope of the function at that point ?

This conceptual problem is resolved as follows. We better state explicitly that the domain must be manipulated. Let $y // x$ be the following process or program, called *dynamic division* or *dynamic quotient*, with numerator y and denominator x:

> $y // x \equiv$ { y / x, unless x is a variable and then: assume $x \neq 0$, simplify the expression y / x, declare the result valid also for the domain extension $x = 0$ }

The algebraic definition of the derivative then follows directly:

$$f'[x] = \{\Delta f // \Delta x, \text{ then set } \Delta x = 0\}$$

This implies that the expression "df / dx" only has proper meaning as an operator "d / dx" applied to *f,* without proper division. This also means that we finally have a sound interpretation for differentials. These would not be infinitesimals. The differentials df and dx are better seen as variables, so that, when $f\,'[x]$ has been found by other methods of algebraic manipulation, we can define $df = f\,'[x]\, dx$ for the incline (tangent) to *f.*

This gives:

$$df // dx = f\,'[x]\, dx // dx = f\,'[x]$$

The group theory approach to rational functions (RF-GT) (the version that we looked at) appears to have limited value, because of the assumption that these "functions" don't have domains. If its results are to be useful, they must be translated, and domains and singularities come into consideration anyhow.

The notion of an equivalence class relies on limits and continuity, and the manipulation of the domain is not explicit enough.

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[30] https://en.wikipedia.org/wiki/Cancellation_property

An algebraic approach to calculus is possible that relies on algebra and expressions only, and that manipulates the domain to find the slope of the function. The formal continuity given by the expression is sufficient, and there is no need for numerical continuity and limits.

Appendix A. Michael Range (2016abc) WIC Prelude

This paper was inspired by reading Range (2011) (2014) (2016b) "*Front Matter*" – including "*Preface*" and "*Notes for instructors*" – and (2016c) "*Prelude*". For a review, see Colignatus (2017e).

Notions of continuity and limit make analysis and calculus complicated, both on content and in didactics, see Figure 6 for Range (2016b).

Figure 6. Range (2016b:xvi) WIC "Preface"

> Unfortunately, the transition from high school mathematics to calculus is not easy. Students are usually exposed to deep new concepts right at the beginning. In particular, important central applications such as variable velocity, slopes of tangents, and more general rates of change and derivatives are introduced by an approximation process that involves "limits" of certain expressions that formally approach the meaningless quotient 0/0. Therefore it becomes necessary to investigate and understand such "limits" in order to proceed. Algebraic examples involving polynomials, rational functions, roots, and so on, often tend to confuse matters: The limit as the input x approaches the value a, where x must be assumed $\neq a$, is ultimately found—after algebraic manipulations to remove the troublesome zero from the denominator—by what is de facto *evaluation* of an algebraic expression by setting $x = a$. Thus limits tend to get mixed up with evaluation, often leaving one wondering about what seem unnecessary complications. The confusing relationship between limits and evaluation had surfaced already at the origins of calculus in the 17th century, but that did not stop the pioneers from moving forward. The difficulties were only resolved in the 19th century, when mathematicians introduced precise—and necessarily complicated—technical descriptions of *limits*. Since then, these new abstract concepts—in varying degrees of technical detail—have become a major component of any introduction to calculus. Even when discussed in intuitive non-technical language, they present quite a challenge right at the beginning for anyone who wants to learn and understand calculus.

My comments on this:

(1) In Holland a bit less that 11% of highschool graduates has the Math B (beta) profile with quite a bit of calculus preparing for university. Forty years ago demands were tougher and nowadays the limit is mentioned by handwaiving, and the focus is on mastering rules and applications. Bressoud (2004) gives some information about the USA and the link from highschool to tertiary education. Beta students like in Holland or in the USA with Advanced Placement will not have quite such difficulties as Range refers to.

(2) The real problem is indeed, what he refers to, the confusion about the *need* for limits. The confusion lies not with the students but with the mathematicians. There is no need for a limit if it suffices to find the derivative by an evaluation (of a function with manipulated domain).

(3) The notion of a limit is not so hard to grasp or explain, see the asymptotes of 1 / *x*. It are rather the current definitions in mathematics that make the limit more complicated than needed. A first step towards deconstruction is in Colignatus (2016b).

(4) An approach of "only teach rules and applications" (even embellished by handwaiving on limits) sacrifices both rigour and understanding what derivative and integral actual are. Thus the stage is set for a major redesign.

Appendix B. An introductory view on group theory from programming

We might look at group theory from the viewpoint of programming. The following programming example uses *input* → *output*. This is asymmetric, since the output isn't necessarily the same as the input. However, let us use = instead of →. This helps to clarify a potential source of confusion.

Steps in programming

Let us suppose that the computer programme knows what *plus* is, and what a *variable* (storage location) is. Consider a repeated addition of a variable and storing the outcome into a new variable.

Computer screen	*Programmer's view*
$a + a + a = c$	This is just addition.

An inverse operation:

$c - a - a - a = 0$	Subtraction, repeatedly, given the above.

We may extend the programme with another feature, called "multiplication", with "factors":

$a + a + a = 3\ a = c$	Introduction of a recorder for the number of repeats in the addition.

The operation calls also for an inverse operation, called "division", and "factor" iff "divisor":

$c / 3 = a$	Introduction of a recorder of the number of repeats in the subtraction. Or a feature to eliminate the number of repeats in addition.

The number of repeats in the addition can be replaced by a variable too:

$a\ b = c$	The factors *a* and *b* give *c.*

Key step: (*) seems like an equivalent statement

$a\ b = c$	(*) *a* and *b* are factors or divisors of *c.*

With two substatements (**):

$c / a = b$	(**) *a* is a factor or divisor of *c.*
$c / b = a$	(**) *b* is a factor or divisor of *c.*

Discussion of these steps

(*) seems innocent when the process is read from input to output. If a and b generate c then one might say that a and b are factors of c. Even when $a = 0$ or $b = 0$ then the outcome $c = 0$ doesn't make it wrong to say that a times b generated c.

The problem emerges in (**), when the order of the process is reversed. The input now consists of both c (numerator) and a particular factor (denominator). The output should be the other factor. If the denominator is zero, then any value in the outcome might be possible, and c should be zero too. (There is no way how c can be nonzero and still be produced by a zero factor and a nonzero factor.) The programmer hasn't introduced this condition yet.

Thus the statement "The factors a and b give c" is *not* equivalent to "a and b are factors or divisors of c".

It is proper to say that *divisors are nonzero, a divisor is a factor, and only a nonzero factor is a divisor.*

It is elementary school stuff, yet, when a group theory version of the rational functions (GT-RF) eliminates the domain and neglects the removable singularities, then there might still arise the confusion that all factors are also divisors. (This confusion may happen when this is translated to proper functions with domains.)

PM. The above gives the rule: If $c \neq 0$ and a and b are factors with $a\ b = c$, then $a \neq 0$, and we can write $c / a = b$ without problem. However, in the main body of this paper, we are looking at $c = f[x] - f[a] = 0$, whence this rule doesn't help us, even though we know that $x - a$ is a factor. We only know that if $f[x] - f[a] \neq 0$, then $x - a \neq 0$, and then we can determine $q[x] = (f[x] - f[a]) / (x - a)$. Our problem however is what happens when $x = a$. The mathematical theory of Analysis states that we need limits for this. Colignatus (2011ab) shows that we only need a theory of algebraic expressions and the ability to manipulate the domain.

Hidden asymmetry (or dynamics) in =

In conventional mathematics the equality sign is symmetric. Thus $a = b$ iff $b = a$.

There is symmetry in arithmetic for $3\ 0 = 0$. For algebra, the above shows that there is a hidden asymmetry, for then we have variables with domains.

When $3\ 0 = 0$ is read left to right (LTR) it is true that the input $3\ 0$ generates output 0. Read right to left (RTL) the input 0 actually generates: For all x, $0 = x\ 0$. Perhaps we can write this as: $0 = \{\text{any } x\}\ 0$. It is partially true that $x = 3$ is one of the possible solutions, yet it isn't the only one, and thus in algebra $3\ 0 = 0$ doesn't give the full truth and symmetry. In algebra $0 = 3\ 0$ becomes an incomplete statement of all possible solutions for solving $0 = x\ 0$. [31]

In RF-GT – see Figure 1 – there is apparently the intention (if I understand this well) to interprete $a\ b = c$ as "a and b are divisors of c", which then includes the possibility that $c = 0$. This happens where RF-GT considers polynomials only (thus polynomials $p / 1$ as rational functions that have nonzero denominator 1). For this apparent intention, e.g. compare Range (2014:389) where the left column has the difference quotient and the right column has the multiplicative format (with "removed singularity"). In that case, the multiplication for polynomials is *interpreted* as division. For $c = 0$ the implied suggestion is that there would be unique combinations of factors a and b, restricted only by an equivalence class, just like when $c \neq 0$. However, when $c = 0$ and $a = 0$ then any b might be possible. This is still a *function* when you interprete this as Div[c, b] $\rightarrow a$ for nonzero b. But this kind of interpretation also

[31] This is not the question "Give the factors of zero". For example, "Give the factors of 12" would generate $12 = 2\ 6 = 4\ 3$, thus multiple outcomes too. The question is "Solve $0 = x\ 0$" and in this context even better: "Find the factor x such that $0 = x\ 0$".

generates *not* a function but a *correspondence* Div[c, a] → b. Thus the (presumed) interpretation of multiplication as division fails.

Range (2016c:14) actually recognises this for an equation 0 = k 0. See Colignatus (2017e) **Appendix I** for a discussion that his argumentation there isn't convincing. It is better to grow aware of the distinction between polynomial and "rational function" and the need to adjust the domain.

(For the ratio format, the denominator must be nonzero, but for the interpretation of multiplication of factors as divisors it is not quite clear whether the theory of rational functions imposes such a restriction. If the restriction is used, then there would be no difference with division, so why do so difficult by *interpreting multiplication as divison* (if it is division by nonzero elements anyway) ?)

I am not at home in RF, either RF-FL or RF-GT. My suggestion is that group theory first resolves this hidden asymmetry in its use of the equality sign. Let one adopt a notation such that equality is symmetric, and confusion is avoided. It seems okay that we use this hidden asymmetry in elementary school, since we teach pupils not to divide by zero. But when group theory creates the impression that it interpretes a b = c such that factors are seen as divisors, then RF-GT doesn't adopt that rule anymore, and then it better be put into the notation.

The notational problem might be resolved by using a b = c only for nonzero factors and use a b → 0 if there is a zero factor. I am blank about the option whether this would actually also be better for elementary school. (Would kids understand 3 0 ≠ 0 but 3 0 → 0 ? Three times zero reduces to zero, and isn't quite equal to zero. For, 0 → x 0 for any number x.)

There is a key difference between numbers and variables that have domains. My suggestion is that the dynamic quotient likely provides the required notation also for group theory. The current RF-GT would survive as a theory of expressions, but not for functions with domains. Yet I am only a teacher of mathematics and no research mathematician, and I am not qualified to judge on this, and thus this remains a suggestion only.

PM. We might use the property that polynomials put restrictions on the solution space. For (x^2 – 1) = (x – 1) (x + 1) there are these discussion steps:

- The interpretation (x^2 – 1) / (x + 1) = (x – 1) seems to work when x ≠ -1 even when x = 1, and we have the form 0 / {a particular nonzero, now 2} = 0.
- The interpretation (x^2 – 1) / (x – 1) = (x + 1) seems to work for x ≠ 1 even when x = -1, and we have the form 0 / {a particular nonzero, now –2} = 0.
- These restrictions are actually no different from the earlier rule: *divisors are nonzero, a divisor is a factor, and only a nonzero factor is a divisor*.
- Observe that 0 / {a particular nonzero} = 0 is only partially true if read RTL. Symmetry would require: 0 / {any denominator ≠ 0} = 0. Thus 0 / 2 = 0 and 0 / -2 = 0 use a hidden asymmetry.
- However, the polynomial has restricted the solution space to roots {-1, 1} where factors would be zero. When a particular root x is used is used to create the 0 in both numerator and result, then the denominator *must* use an element in the *remaining* solution set, and thus be nonzero.
- In this case the polynomial has two factors and thus {any denominator ≠ 0} reduces to {a particular nonzero}. For more factors, the solution set however would be larger. For polynomials we should write {any denominator ≠ 0 in the remaining solution set}.
- However, we still can use: c / {n – 1 factors} = {1 remaining factor}. In that case, it is fair to use {a particular nonzero} as denominator.
- Thus, since polynomials restrict the solution space, we can allow the structure:
  0 / {a particular nonzero} = 0 as also a symmetrical expression, however *conditional* on the assumption that the 0 has been created from only 1 remaining factor.

Thus, the correspondence now is replaced by a conditionality. This still deviates from the notion of a function.

It remains that $(x^2 - 1) = (x - 1)(x + 1)$ allows $x = -1$ and $(x^2 - 1) / (x + 1) = (x - 1)$ requires $x \neq -1$.

These conceptual problems are resolved by $(x^2 - 1) // (x + 1) = (x - 1)$ that has a symmetrical equality sign (without a hidden asymmetry).

Equivalence class

The above on a nonsingle solution set should not be confused with the notion of an equivalence class.

Speaking about equivalence classes, it may be noted that the theory of rationals (Q) declares 1 / 2 and 2 / 4 equivalent, like the theory of rational functions does for $x / (2x)$ and $(2x) / (4x)$. Part of the problem here is that the sign "/" is used both as an operator and for denoting a number. It might again be that group theory facilitates confusions in elementary school. If we denote 1 / 2 as 0.5, then we can reduce 1 / 2 and 2 / 4 to operations and phases in a computation that aren't a final result yet. This doesn't entirely resolve the matter since we must establish that 0.5 and 0.49999... would be equivalent too. Yet, this comment may be an eye-opener that group theory focuses on *existence of numbers* while the crucial question for students and didactics is on *notation* without confusion, see Colignatus (2016h) (2017a).

(Group theorists will object that there are a / b for huge numbers a and b so that it would be humanly impossible to determine the decimal expansion, so that there is value in the notion of an equivalence class. Yet the principle $a / b = c$ is already given by the very operation of division. It aren't actually the operations that are relevant but rather the elements in the set.)

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Colignatus is the name in science of Thomas Cool, econometrician and teacher of mathematics, in Scheveningen, Holland.

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