# A CHILD WANTS NICE AND NO MEAN NUMBERS

# A child wants nice and no mean numbers

Mathematics in primary education

Major parts were written at the occasion of M’s sixth birthday in 2012

Thomas Colignatus

Samuel van Houten Genootschap

Colignatus is the preferred name of Thomas Cool in science. He is an econometrician (Groningen 1982) and teacher of mathematics (Leiden 2008). http://thomascool.eu, cool at dataweb.nl



August 20 2015 z



Supported by

Samuel van Houten Genootschap
Wetenschappelijk bureau van het Sociaal Liberaal Forum
Scientific bureau of the Social Liberal Forum
The Hague

ISBN 978-946318970-5
Published by MijnBestseller.nl

American Mathematical Society: MSC2010
00A35 Methodology and didactics
(97-XX Mathematics education)
97F02 Arithmetic, number theory ; Research exposition
97G70 Analytic geometry. Vector algebra
97H20 Elementary algebra

Journal of Economic Literature: JEL
P16 Political economy
I20 General education
A20 Economics education

Nederlandstalige Uniforme Rubrieksindeling: NUR
841 Onderwijskunde
918 Wiskunde algemeen
780 Economie en bedrijf algemeen (Onderwijs van wiskunde en beheer ervan, Political Economy)

# Prologue

Mathematics education is a mess. Earlier books *Elegance with Substance* (EWS) (2009, 2015) and *Conquest of the Plane* (COTP) (2011) [1] present a diagnosis:

*Mathematicians are trained for abstraction while education is an empirical issue.*

Stepping into a classroom the abstract thinking mathematicians meet with real life pupils. They resolve their cognitive dissonance by relying on tradition. This ought to work, aren't they themselves not evidence that the system can work ? However, mathematical formats have grown historically and aren't necessarily designed for didactics. EWS and COTP re-engineer mathematics education for didactic purpose. The advice is that each nation has a parliamentary enquiry into the education in mathematics, in order to identify the proper policy for improvement and make funds available for change.

This book looks at mathematics in primary education. It can be included in the list of examples where tradition is not as friendly to pupils as it can be re-engineered.

I am professionally involved in mathematics education at the level of highschool and the first year of higher education, and thus these thoughts on elementary school are prospective only. Perhaps the proper word is *amateurish.* My very plea is for professional standards, and thus I am sorry to say that I cannot provide this myself for elementary school. For example, Domahs et al. (eds) (2012) discuss finger counting and numerical cognition, with theory and empirical research: which I haven't read or studied, and thus it is quite silly of me to discuss the topic. This qualifier holds for this whole book.

My only defence for this book – or the articles that it collects and re-edits – is that I want to organise my thoughts on this. If parliaments will already need to investigate the issue, with much more funds than I can muster, then it seems acceptable that I organise my marginal comments on primary education too. There is also a good reason why I must collect my thoughts on this. Thinking about education in highschool and the first year of higher education caused questions about more elementary mathematics. It seems rather natural to wonder whether such issues cannot be dealt with in elementary school.

To be sure: it is not at all clear whether the world is served by this book. However, I am still under the impression that these articles support the general diagnosis in EWS and COTP. It may also be that my intuition is wrong and that the questions posed here have good answers, which I only missed because I did not study the issues fully. The book however achieves its goal when it provides some new ideas and perspectives for the *true* researchers of elementary education, and when it indeed provides some additional support for the general diagnosis of EWS and COTP that parliaments must take steps.

This book has a Dutch counterpart in Colignatus (2012a) that was written at the occasion of my son M.'s sixth birthday. These books only partly overlap. Various Dutch texts on local conditions are not interesting for an English translation. The present book includes some new articles since 2012. I thank Yvonne Killian for her permission to use some of her ideas on presenting the Pythagorean Theorem in elementary school.

A shocking discovery in 2014 w.r.t. Holland was that abstract thinking Hans Freudenthal (1905-1990) sabotaged the empirical theory by Pierre van Hiele (1909-2010); [2] see also the discussion in Colignatus (2014). Readers interested in primary education will not quickly read §15.2 of COTP on the right approach by Van Hiele and the erroneous approach by Freudenthal. For that reason page 101+ below copies that text.

[1] Reviews by Gamboa (2011) and Gill (2012).

[2] https://boycottholland.wordpress.com/2014/07/06/hans-freudenthal-s-fraud/

## Contents in brief

# Contents

# Introduction

The West reads and writes **text** from the left to the right, while Indian-Arabic **numbers** are from the right to the left. English pronounces 14 as *fourteen* instead of *ten·four,* and switches order from 21, to *twenty·one.* This order is already better, yet there still is an issue, for structurally the latter is *two·ten·one*. Pronunciation like *ten·four* and *two·ten·one* gives so much more clarity that pupils could learn arithmetic much faster. [3]

Seen from the perspective of the pupil, the traditional pronunciation can be called *mean* and the mathematically proper way is *nice*.

Teaching arithmetic does not only deal with number but also language. Education errs in regarding English as perfect. English as a language appears to be a crummy dialect of mathematics. There is no learning goal yet of recognising the dialect *for what it is*.

This book develops the proposal (i) to teach in a nice language (ii) to clarify the translation to the dialect of English so that pupils grow aware of its pitfalls. The translation of mathematical pronunciation to standard English would be like handling any dialect. Given that children learn other languages with ease, while this concerns only a small set of words and concepts, this translation cannot be much of a burden. Perhaps the English and American reluctance to learn other languages and accept dialects is a larger bottleneck than possible doubts about the didactic advantages of using mathematics. The key notion thus is to regard traditional English as a dialect indeed, and extend lessons on arithmetic with clarification of the dialect.

The chapter *Marcus learns counting and arithmetic with ten* contains a stylized lesson for six-year olds. This is not intended for actual use in class but provides an example to start thinking about this for research and development. Six-year olds can still be orienting on left and right, for example (perhaps also because of this), and it could take more refined material to handle the issues (like particular feedback on progress and error).

A related issue is to call for a greater awareness of the role of the positional system in arithmetic. In *ten* + *ten* = *two·ten,* the result is immediately available in the positional system itself. Thus it would also be advantageous for pupils to grow aware, as a learning goal by itself, of the positional system and its relation to language.

Sadly, though, all fingers are already in use for the numbers 1 – 10, and there are no fingers available to practice on the decimal system itself. Perhaps lower arms help out.

The second type of issues below are more directly on arithmetic (algebra) and (analytic) geometry. The texts relate to the ideas of Pierre van Hiele and Dina Van Hiele - Geldof about levels of insight (understanding, abstraction). These didactic ideas directly transfer from my experience with highschool. Education in highschool requires algebraic insight, and this is based upon arithmetic mastered in elementary school. Van Hiele thought that algebra could be started in elementary school already, and would even be the best subject to start in elementary school with formal deduction and proof.

Pierre van Hiele also proposed to have vectors in elementary school. He was hesitant about formal proofs with geometry there. Killian (2006) (2012) designed a proof of the Pythagorean Theorem that however feels very natural for this environment, and I have seen it work wonderfully for pupils in an enrichment course in elementary school.

Our order of discussion thus is: numbers, arithmetic, geometry. We close with didactics.

[3] The middle dots are better than hyphens in numbers, to prevent possible confusion with the negative sign.

# Medical School as a model for education

2014-07-18 [4]

In Medical School, doctors are trained while doing both research and treating patients. Theory and practice go hand in hand. We should have the same for education. Teachers should get their training while doing theory and learning to teach, without having to leave the building. When graduated, teachers might teach at plain schools, but keep in contact with their alma mater, and return on occasion for refresher updates.

Some speak about a new education crisis (e.g. in the USA). The above seems the best solution approach. It is also a model to reach all existing teachers who need retraining. Let us now look at the example of mathematics education.

Professor Hung-Hsi Wu [5] of UC at Berkeley is involved in improving K12 math education since the early 1990's. He explains how hard this is, see two enlightening short articles, one in the *AMS Notices* 2011 [6] and one interview in the *Mathematical Medley* 2012. [7] These articles are in fact remarkably short for what he has to tell. Wu started out rather naively, he confesses, but his education on education makes for a good read. It is amazing that one can be so busy for 30 years with so little success while around you Apple and Google develop into multi-billion dollar companies.

Always follow the money, in math education too. A key lesson is that much is determined by textbook publishers. Math teachers are held on a leash by the answers books that the publishers provide, as an episode of *The Simpsons* shows when Bart hijacks his teacher's answers book. [8] As a math teacher myself I tend to team up with my colleagues since some questions are such that you need the answers book to fathom what the question actually might be (and then rephrase it properly).

At one point, the publishers apparently even ask Wu whether he has an example textbook that they might use as a reference or standard that he wants to support. The situation in US math education appears to have become so bad that Wu discovers that he cannot point to any such book. Apparently he doesn't think about looking for a UK book or translating some from Germany or France or even Holland or Russia. In the interview, Wu explains that he only writes a teacher's education book now, and leaves it to the publishers to develop the derived books for students, with the different grade levels, teacher guides and answers books. One can imagine that this is a wise choice for what a single person can manage. It doesn't look like an encouraging situation for a nation of 317 million people. One can only hope that the publishers would indeed use quality judgement and would not be tempted to dumb things down to become acceptable to both teachers and students. In a world of free competition perhaps an English publisher would be willing to replace "rigour" by "rigor" and impose the A-levels also in the US of A.

In my book *Elegance with Substance* (2009, 2015) I advise the parliaments of democratic nations to investigate their national systems of education in mathematics. Reading the experience by Wu suggests that this still is a good advice, certainly for the US.

---

[4] https://boycottholland.wordpress.com/2014/07/18/the-medical-school-as-a-model-for-education/
[5] http://math.berkeley.edu/~wu/
[6] http://www.ams.org/notices/201103/rtx110300372p.pdf
[7] http://math.berkeley.edu/~wu/Interview-MM.pdf
[8] http://www.wired.com/2013/11/simpsons-math/

About the subject of logic, professor Wu in the interview p14 suggests that training math teachers in mathematical logic would not be so useful. He thinks that they better experience logic in a hands-on manner, doing actual proofs. I disagree. My book *A Logic of Exceptions* (1981 unpublished, 2007, 2011) would be quite accessible for math teachers, shows how important a grasp of formal logic is, and supports the teaching of math in fundamental manner. The distinction between necessary and sufficient conditions, for example, can be understood from doing proofs in geometry or algebra, but is grasped even better when the formal reasons for that distinction are seen. I can imagine that you want to skip some parts of ALOE but it depends upon the reader what parts those are. Some might be less interested in history and philosophy and others might be less interested in proof theory. Overall I feel that I can defend ALOE as a good composition, with some new critical results too.

Thus, apart from what parliaments do, I move that the world can use more logic, even in elementary school.

Update 2015: Editing the 2nd edition of *Elegance with Substance* (2015), now available, I was struck again by the empirical observation on the diversity of students and pupils. Evidence based education may never attain the sample sizes that are required for statistical testing of theories that allow for such diversity. This fits the Medical School model: there is an important role for individual observation and personal hands-on experience to deal with empirical variety. Methodology and statistics remain important, of course, but in balanced application.

# English as a dialect for a didactic number system

## The problem

The issue came to my attention by Gladwell (2008:228):

> "Ask an English-speaking seven-year-old to add *thirty·seven* plus *twenty·two* in her head, and she has to convert the words to numbers (37 + 22). Only then can she do the math: 2 plus 7 is 9 and 30 plus 20 is 50, which makes 59. Ask an Asian child to add *three·tens·seven* and *two·tens·two,* and then the necessary equation is right there, embedded in the sentence. No number translation is necessary: It's *five·tens·nine*." (Hyphens in numbers replaced by middle dots.)

My alternative suggestion is to use *five·ten·nine,* thus (i) no 'tens' and (ii) the use of a middle dot. The hyphen is unattractive since it is too similar to subtraction. The dot is not pronounced, like the hyphen or comma. Thus there is not only the notation of 59 and the pronunciation, but also the notation of the pronunciation. [9]

Gladwell (2008:228) also emphasizes the importance of mental working space:

> "(…) we store digits in a memory loop that runs for about two seconds."

English numbers are cumbersome to store. He quotes Stanislas Dehaene:

> "(…) the prize for efficacy goes to the Cantonese dialect of Chinese, whose brevity grants residents of Hong Kong a rocketing memory span of about 10 digits."

The problem has an internationally quick fix: Use the Cantonese system and sounds for numbers. It would be good evidence based education (EBE) to determine whether this would be feasible for an English speaking environment (e.g. start in Hong Kong).

## Decimal system

There is more to it. The decimal number system has, for digits *a, b, c, d, ….*:

$$...dbca = a \times 10^0 + b \times 10^1 + c \times 10^2 + d \times 10^3 + \dots$$

The West reads and writes text from the left to the right while Indian-Arabic numbers are from the right to the left. Thus 19 is *nineteen* instead of *ten·nine.* Human psychology apparently focuses on the lowest digits that have been learned first. The order switches in English at *twenty-one*, when attention shifts to the most important weight. While English switches order at 21, Dutch continues in the wrong order till 99 (*negen-en-negentig*). Thus instead of saying *...dcba* (most important weight first) we have reversed pronunciation ...*dcab* for the numbers below 20 (English) or 100 (Dutch). See Ejersbo & Misfeldt (2011) for the Danish convolutions.

Can we do something about these linguistic pecularities ? A key observation is that for higher numbers like 125 the Indian-Arabic writing order happily co-incides with our attention for the most important weights of the digits. Let this order be the guide. Let us agree that 21 is *two·ten·one.* The article *Marcus learns counting and arithmetic with ten* (page 19, pronounce *ten·nine*) explains how this works. The idea is that the most important weight is pronounced first, and that *ten* is the weight (and not *tens*).

Conclusion: We can apparently handle the pecularities of the natural languages. But also at an appalling cost of teaching in primary education. Instead, there is a number system

[9] 2015: A relevant reference to Barrow (1993) is discussed on page 114 below.

with didactic clarity so that pupils could learn arithmetic more easily: the decimal positional system. The translation to English would be a mere matter of learning another dialect, which cannot be a burden given the ease by which children learn other languages, and also given the small set of words and concepts. Perhaps the English and American reluctance to learn other languages and accept dialects is a larger bottleneck than possible doubts about the didactic advantages. The key notion is to regard English as a dialect indeed, and extend lessons on arithmetic with clarification of the dialect.

## Language pecularities

For English it may be easier to switch from *nineteen* to *ten·nine* and from *twenty* to *two·ten.* Other languages may have to make a greater adjustment. Consider Dutch as an example for handling such pecularities.

English distinguishes *ten* and *teen* in *nineteen* while Dutch uses *tien* everywhere, such as *negentien* for 19. A possible switch in Dutch to *tien·negen* runs against the problem that the new pronunciation of 90 would be *negen·tien* (English *nine·ten*). It would wreak havoc that the new pronunciation of 90 would be the old 19.

An option in Dutch is to use a new plural: *tienen* (rather than *tientallen* for the numbers of ten). However, the plural *tens* is not needed, and may cause later problems for higher powers such as *ten·ten·ten* for *thousand.* Thus *tens* and *tienen* can be used in discussion but not official pronunciation.

The solution in Dutch is to introduce a new label *tig* which can be done since 20 = *twintig* and 30 = *dertig* and so on. This is presented in Colignatus (2012a).

The equivalent for English would be to use *ty* so that we would get *two·ty* and *three·ty*. The latter is not necessary since we can already use *ten.* Perhaps *two·ty* is better than *two·ten* but *ten* does fine. Better to have *hundred* = *ten·ten* than *ty·ty*.

English tends to use a hyphen: *twenty-two*. Dutch tends to concatenate words and has *tweeëntwintig* with the sudden umlaut to prevent "confusion" over vowels. (Thus an original confusion is solved by introducing another one.) For pupils learning the structure of the number system it is useful to avoid complexity. The middle dot then is better than a hyphen since the subject area is arithmetic and there might be a confusion with the minus sign. Thus Dutch *twee·tig·twee* is fine. Or switch to English or Cantonese.

## Positional system and multiplication

It is a question at what age pupils can understand and actually learn multiplication. It is an option to see whether they already can multiply for the numbers up to 5 before progressing with the numbers above 20. When multiplication is known then it is easier to highlight the numerical structure. We can write (using 'times' and 'to time' rather than 'multi-plus' and 'multiplication'):

$$\ldots c \times \text{hundred} + b \times \text{ten} + a = \ldots cba$$

and then explain to the pupils that the number on the right is pronounced like on the left but without pronouncing the operators. [10] This is how the positional system supports understanding of arithmetic. At some points this may conflict with the assumed abstraction level of the pupils and perhaps the need to first learn to pronounce numbers before understanding the structure in the pronunciation. But when pupils are learning arithmetic, then we should also discuss how the positional system supports this.

[10] An idea to use * and & is too distractive. (Thus '… *c* * hundred & *b* * ten & *a*' where the star indicates *times* and the ampersand *plus,* written but not pronounced, with possibly a colour code.)

## Notes on *Marcus learns counting and arithmetic with ten*

This discussion quickly becomes more complicated than needed. It is better to proceed with *Marcus learns counting and arithmetic with ten,* since this clarifies what the ideas entail. This is not spelling reform but targeted bilingualism. Please keep in mind:

(1) This text contains a stylized presentation for six-year olds. This is not intended for actual use in class but contains the framework for starting to think about that.
(2) The idea is to write *five·ten·nine,* where the dot is not pronounced and the order helps to decode the position.
(3) Much of arithmetic can be already done by proper pronunciation. Having this creates room for the operators plus and minus.
(4) Numbers are called *low* and *high* instead of *small* and *large,* since the latter would refer to absolute sizes, and a wrong convention might become a block in the later introduction of negative numbers.
(5) Putting the tables of addition together in a big table gives the opportunity to discuss patterns.
(6) Addition of many numbers uses the separation of numbers of ten (or higher) as an intermediate step. After some experience the pupils will use the direct method.

## Multiplication is a long word

Before we can proceed with *Marcus* there is the issue that the word *multiplication* itself is long and rather awkward. In Dutch it is *vermenigvuldigen.* Apparently multiplication does not belong to the Indo-European core words like *mom* or *water.* Pupils in elementary school seem to lack easy words to express what they are doing.

Surprisingly, David Tall mentions that *of* is used for multiplication, see page 74 below. Thus *five of two makes ten* would be unambiguous.

I would explain that as *grouping five groups of two makes ten,* and then erase the *group* words. [11] We could call × the *of-sign*, and say *John ofs five and two to get ten,* rather than *John multiplies ....* The surface of a rectangle as *five by two makes ten* might perhaps also be used: *Mary bies two and five to get ten*. But verbs *to of* and *to by* are absent from online dictionaries and even Mathworld. [12]

My guess is that historically the development of *five of two is ten* into a verb *to of* was blocked by prim mathematicians who stuck to Latin *multiplex*. The Italian *volta* generates the English *times* with a reference to Father Time [13] – like in French *fois* and Dutch *keer, maal.* [14] Even when emphasis is put upon the notion of repetition, it is actually distractive. Multiplication is not only repetition of same sizes, but rather the *grouping* of those: creating a set of sets.

*Times* actually is a prefix *((five times) two) gives ten.* Five times two hamburgers need not be the same event as two times five hamburgers, if we allow for different days. The *times* prefix forces a demonstration that *((two times) five) gives ten too.* Once symmetry has been established we can create a *new* infix *five times two gives ten.* This is needlessly complex, and only gives an infix, i.e. without a rich and easy vocabulary with verb,

[11] Set theory has *joining five sets of two gives a set of ten.*
[12] http://mathworld.wolfram.com/Multiplication.html
[13] Amusing is http://math.stackexchange.com/questions/1150438/the-word-times-for-multiplication. But informative is Mauro Allegranza: "This latin plicō, like the ancient Greek : πλεκτός - "plaited, twisted", comes from Indo-European pleḱ- : "to plait, to weave"." Apparently related to *fold.* A weaving loom indeed reminds of a rectangle for multiplication. Folding a piece of paper however is an example for exponential growth.
[14] https://translate.google.com/#nl/en/keer

adjective and so on. It makes more sense to directly use an infix that actually has a rich and easy vocabulary that expresses symmetry directly.

The question becomes what synonyms for *times* there are. The Webster thesaurus on *times* is absent, with *time* only as a noun, and it is disappointing on *multiply.* [15] The idea of *double*, *triple*, *quadruple*, ... invites to think about a *multiple*, or *multi-plus,* indeed. But *run* is not *multi-walk.* When you are running then you don't want to be reminded continuously that you are actually walking but only faster.

It appears to be a relevant research objective to establish easy words for arithmetic so that pupils can discuss what they are doing without stumbling over the syllables and losing places in working memory. It is fine that mathematicians have developed the words *multiply* and *multiplication* so that adults may know what they are speaking about, but these words are overly complex for First or even Second Grade.

It is not clear how the verb *to group* is used for other applications, but if it is not used much then *group five of two makes ten* would be clearer than *times* on what multiplication is. My proposal in 2012 for Dutch was to use the verb *malen* (English *to mill*), given the already conventional Dutch *vijf maal twee geeft tien.* This was my first reaction to get rid of *vermenigvuldigen.* But *groepeer vijf van twee geeft tien* looks better.

For now, the paper *Marcus learns counting and arithmetic with ten* uses the verb *to time*. Hopefully there is scope for *to group*, or *to of* or *to by* eventually, with *tables of to of.*

# Appendices

Some issues have been put in appendices.

## Appendix: A novel way to look at numbers

An option is to mirror the numerals. Thus 19 becomes 91. It does not take much time to get used to, and when working from left to right then the handling of the overflow in addition feels rather natural, see Table 1.

**Table 1. Addition, also in the mirror**

| | |
|---|---|
| 4321 | 1234 |
| 765 | 567 |
| 98 | 89 |
| 0981 | 1890 |

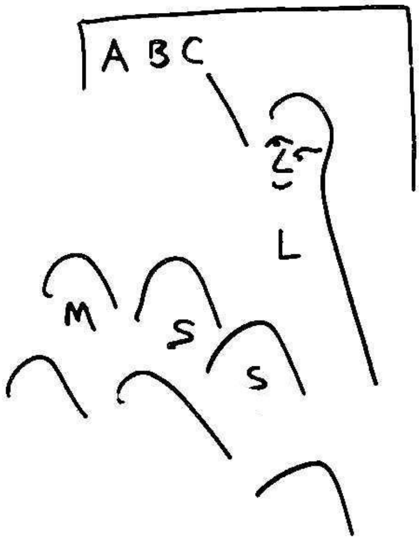
However, the number system is well established, and given the psychological preference to know the size (the digit with the most weight) the present graphical order might be alright. A discussion on the four combinations of Indian / mirror and writing / pronouncing is put in an Appendix: *Number sense and sensical numbers.*

## Appendix: Fingers and hand

Embodiment or gestures are important for the development of number sense. The decimal positional system can be supported by using fingers and ells (lower arms), see the subsequent chapter. There is an appendix on using only fingers and hands. The particular system that is presented there will not be quickly used in first grade itself. It may be of use for students who are training for teachers in elementary school, and who want to re-experience what it is to learn a positional system.

[15] http://www.merriam-webster.com/thesaurus/multiply

# Marcus learns counting and arithmetic with ten


A B C
L
M
S
S

## 1. Marcus and his friends at school

Marcus is now at school.

His friends Sam and Susan are in his class too.

They have reading, writing and arithmetic.

The teacher is called Linda.

Miss Linda shows how to do it.

## 2. Marcus knows ten digits

Marcus knows the letters of the alphabet.

He uses the letters to make words.

Marcus also knows the ten digits.

We use these to make the first numbers.

| | |
|---|---|
| zero | 0 |
| one | 1 |
| two | 2 |
| three | 3 |
| four | 4 |
| five | 5 |
| six | 6 |
| seven | 7 |
| eight | 8 |
| nine | 9 |
| ten | 10 |

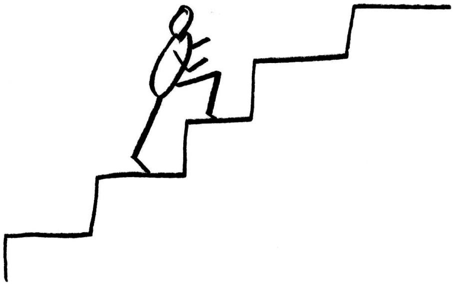
Do you see the difference between a digit and a number ?
A number is recorded with the digits.

A hand has 5 fingers.

Two hands have 10 fingers.

When you calculate with zero you better use candy. (It must be able to disappear.)

It is Marcus's birthday and he brought cookies !



# 3. Count and add

Numbers can be used for counting.

You count when you say: 0, 1, 2, 3, 4, 5, … and so on.

Numbers can be used for addition.

You add when you say plus and then what it adds up to.

Or when you write numbers with + and then =.

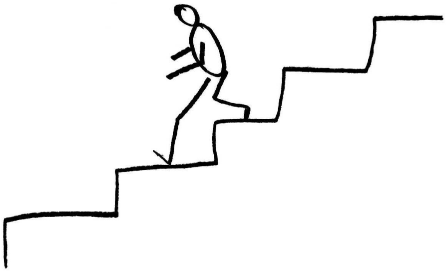
This adds 1.

| | |
|---|---|
| zero plus one is one | 0 + 1 = 1 |
| one plus one is two | 1 + 1 = 2 |
| two plus one is three | 2 + 1 = 3 |
| three plus one is four | 3 + 1 = 4 |
| four plus one is five | 4 + 1 = 5 |
| five plus one is six | 5 + 1 = 6 |
| six plus one is seven | 6 + 1 = 7 |
| seven plus one is eight | 7 + 1 = 8 |
| eight plus one is nine | 8 + 1 = 9 |
| nine plus one is ten | 9 + 1 = 10 |

You can add also in a column.

| number | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| plus | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| is | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

You may switch the first and second row, with the same outcome.



## 4. Count down and subtract

Numbers can be used to count down.

This is when you say: 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0

Numbers can be used for subtraction.

You subtract when you say minus and then what is the difference.

Or when you write numbers with – and then =.

This subtracts 1.

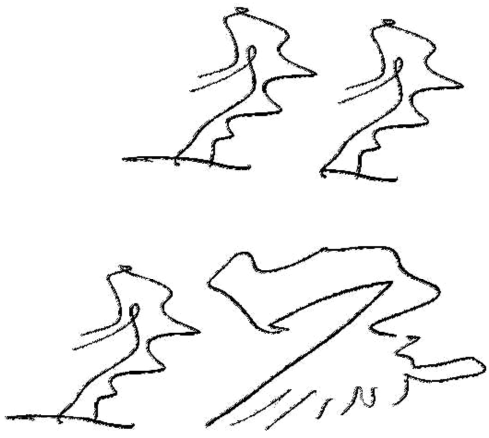
| | |
|---|---|
| one minus one is zero | 1 – 1 = 0 |
| two minus one is one | 2 – 1 = 1 |
| three minus one is two | 3 – 1 = 2 |
| four minus one is three | 4 – 1 = 3 |
| five minus one is four | 5 – 1 = 4 |
| six minus one is five | 6 – 1 = 5 |
| seven minus one is six | 7 – 1 = 6 |
| eight minus one is seven | 8 – 1 = 7 |
| nine minus one is eight | 9 – 1 = 8 |
| ten minus one is nine | 10 – 1 = 9 |

Check: 9 – 2 = 7 because 7 + 2 = 9.

You can subtract also in a column.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| number | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| minus | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| is | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

You may not switch the first and second rows, because the outcome is different. (You will learn this later on.)



## 5. From ten to two·ten

Sam says: ten is the highest number.

Not true, Marcus says, eleven is higher.

Eleven is a weird number, Susan says.

It is the same as ten·one but people also say eleven.

Yes, Marcus says, for ten·two they say twelve.

That is easy for telling the hour.

| number | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| plus | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| is | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |

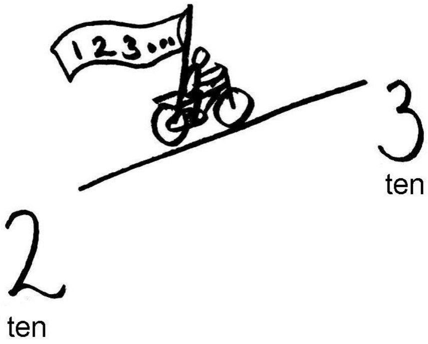
Ten plus ten is two·ten. You write a dot but don't say it.

Miss Linda explains that people say the numbers in different orders. Then the names sound differently. It is useful to know this. But words like eleven and twelve will not be used in calculation.

Marcus, Sam and Susan learn the numbers to two·ten.

They also learn that they can say twenty. But not in calculation.

| | | *Reverse order but not in calculation* |
|---|---|---|
| ten | 10 | ten |
| ten·one | 11 | eleven |
| ten·two | 12 | twelve |
| ten·three | 13 | thirteen |
| ten·four | 14 | fourteen |
| ten·five | 15 | fifteen |
| ten·six | 16 | sixteen |
| ten·seven | 17 | seventeen |
| ten·eight | 18 | eighteen |
| ten·nine | 19 | nineteen |
| two·ten | 20 | twenty |


123...
3
ten
2
ten

## 6. From two·ten to three·ten

Sam says: two·ten is the highest number.

Not true, Marcus says.

Two·ten plus one gives two·ten·one.

This is higher.

And so on, Marcus says.

Miss Linda explains that people say also *twenty·one*.
But not in calculation.

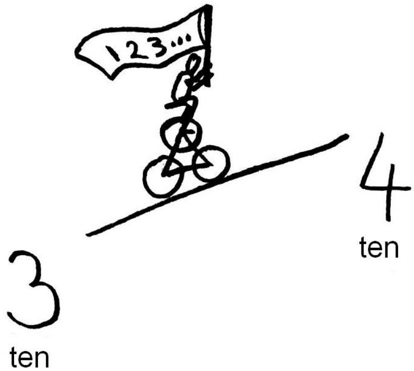
| number | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|
| plus | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| is | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |

Two·ten plus ten gives three·ten.

They learn that people also can say thirty. But not in calculation.

Marcus, Sam and Susan now learn the numbers to three·ten.

| | | *Also used but not in calculation* |
|---|---|---|
| two·ten | 20 | twenty |
| two·ten·one | 21 | twenty·one |
| two·ten·two | 22 | twenty·two |
| two·ten·three | 23 | twenty·three |
| two·ten·four | 24 | twenty·four |
| two·ten·five | 25 | twenty·five |
| two·ten·six | 26 | twenty·six |
| two·ten·seven | 27 | twenty·seven |
| two·ten·eight | 28 | twenty·eight |
| two·ten·nine | 29 | twenty·nine |
| three·ten | 30 | thirty |


123...
3
ten
4
ten

## 7. From three·ten to four·ten

Sam says: three·ten is the highest number.

Not true, Marcus says.

Three·ten plus one gives three·ten·one.

This is higher.

And so on, Marcus says.

Sam and Susan don't believe it.

Marcus says: if you don't believe it, then calculate it yourselves.

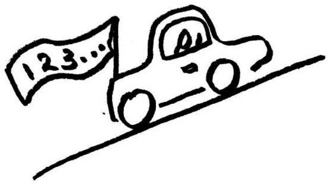
| number | 30 | 30 | 30 | 30 | 30 | 30 | 30 | 30 | 30 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|
| plus | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| is | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |

Three·ten plus ten is four·ten.

They learn that they also can say forty. But not in calculation.

Marcus, Sam and Susan now learn the numbers to four·ten.

| | | *Also used but not in calculation* |
|---|---|---|
| three·ten | 30 | thirty |
| three·ten·one | 31 | thirty·one |
| three·ten·two | 32 | thirty·two |
| three·ten·three | 33 | thirty·three |
| three·ten·four | 34 | thirty·four |
| three·ten·five | 35 | thirty·five |
| three·ten·six | 36 | thirty·six |
| three·ten·seven | 37 | thirty·seven |
| three·ten·eight | 38 | thirty·eight |
| three·ten·nine | 39 | thirty·nine |
| four·ten | 40 | forty |




5
ten

4
ten

## 8. From four·ten to five·ten

Sam says: four·ten is the highest number.

Not true, Marcus says.

Four·ten plus one gives four·ten·one.

And so on, Marcus says.

Sam and Susan now agree with him.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| number | 40 | 40 | 40 | 40 | 40 | 40 | 40 | 40 | 40 | 40 |
| plus | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| is | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |

Four·ten plus ten gives five·ten.

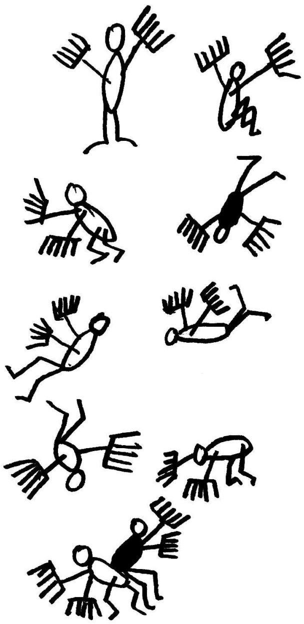
Five children with each ten fingers have five·ten fingers in total.

They learn to count to five·ten.

| | | *Also used but not in calculation* |
|---|---|---|
| four·ten | 40 | forty |
| four·ten·one | 41 | forty·one |
| four·ten·two | 42 | forty·two |
| four·ten·three | 43 | forty·three |
| four·ten·four | 44 | forty·four |
| four·ten·five | 45 | forty·five |
| four·ten·six | 46 | forty·six |
| four·ten·seven | 47 | forty·seven |
| four·ten·eight | 48 | forty·eight |
| four·ten·nine | 49 | forty·nine |
| five·ten | 50 | fifty |

Miss Linda applauds.

They are such smart kids !



## 9. Ten·ten is hundred

Miss Linda says:

Shall I show you the numbers to a hundred ?

Hundred, Susan asks, what is that ?

Hundred, Miss Linda explains, that is ten·ten.

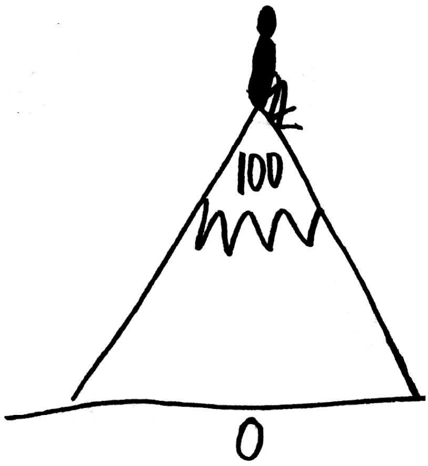
Ten children with ten fingers have ten·ten fingers jointly.

Hundred is a word that we use in calculation too.

And so on, Marcus says, raising his hand with one finger.

Miss Linda laughs.

Yes, she says, that is a hundred and one.


100
0

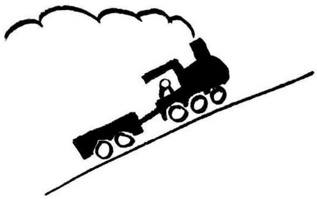
## 10. Hundred and one numbers

Miss Linda shows the numbers to hundred.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 |
| 1 | 11 | 21 | 31 | 41 | 51 | 61 | 71 | 81 | 91 |
| 2 | 12 | 22 | 32 | 42 | 52 | 62 | 72 | 82 | 92 |
| 3 | 13 | 23 | 33 | 43 | 53 | 63 | 73 | 83 | 93 |
| 4 | 14 | 24 | 34 | 44 | 54 | 64 | 74 | 84 | 94 |
| 5 | 15 | 25 | 35 | 45 | 55 | 65 | 75 | 85 | 95 |
| 6 | 16 | 26 | 36 | 46 | 56 | 66 | 76 | 86 | 96 |
| 7 | 17 | 27 | 37 | 47 | 57 | 67 | 77 | 87 | 97 |
| 8 | 18 | 28 | 38 | 48 | 58 | 68 | 78 | 88 | 98 |
| 9 | 19 | 29 | 39 | 49 | 59 | 69 | 79 | 89 | 99 |
| 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |

These are the numbers of ten.

| | | *Also used but not in calculation* |
|---|---|---|
| ten | 10 | ten |
| two·ten | 20 | twenty |
| three·ten | 30 | thirty |
| four·ten | 40 | forty |
| five·ten | 50 | fifty |
| six·ten | 60 | sixty |
| seven·ten | 70 | seventy |
| eight·ten | 80 | eighty |
| nine·ten | 90 | ninety |
| ten·ten | 100 | hundred |



11

ten

10

ten

## 11. Above hundred

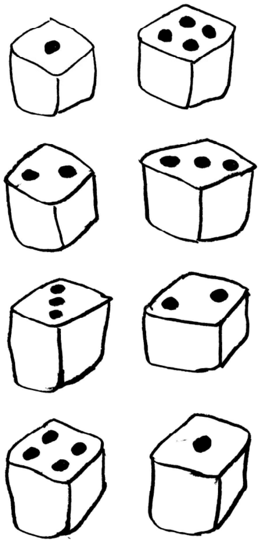
Sam says: hundred is the highest.
Not true, Marcus says.
Hundred plus one is hundred·one.
And so on, Marcus says.

Didn't you pay attention, Sam ?
Miss Linda already said this, didn't she ?
Sam and Susan now agree with him.

Miss Linda nods. Hundred·one is 101.

| number | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|
| plus | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| is | 101 | 102 | 103 | 104 | 105 | 106 | 107 | 108 | 109 | 110 |

Miss Linda says: let us look at the numbers less than hundred.



## 12. The tables of addition to ten

Miss Linda says: let us look at the table of addition.

When we add 1, 2 and 3 with themselves and each other, then we get this table.

| + | 1 | 2 | 3 |
|---|---|---|---|
| **1** | 1 + 1 = 2 | 1 + 2 = 3 | 1 + 3 = 4 |
| **2** | 2 + 1 = 3 | 2 + 2 = 4 | 2 + 3 = 5 |
| **3** | 3 + 1 = 4 | 3 + 2 = 5 | 3 + 3 = 6 |

And so on, Marcus says.

Miss Linda nods.

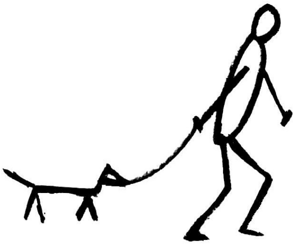
When we add the numbers to ten then we get this table.

| + | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| **1** | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| **2** | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| **3** | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| **4** | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| **5** | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| **6** | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| **7** | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
| **8** | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| **9** | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| **10** | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |

Do you see that five fingers plus five fingers is ten fingers ?

And four fingers plus six fingers is ten fingers too.

Do you see that ten plus ten is two·ten ?



## 13. Mental addition in steps

Susan may pick a number. She says 4.

Sam may pick a number. He says 8.

Miss Linda ask Marcus to add these up. What is 4 + 8 ?

Marcus counts down from 4 to 3.

For the second number he counts up from 8 to 9.

| | | |
|---|---|---|
| number | 4 | 3 |
| plus | 8 | 9 |
| is | | |

Marcus counts down from 3 to 2, and up from 9 to 10.

| | | | |
|---|---|---|---|
| number | 4 | 3 | 2 |
| plus | 8 | 9 | 10 |
| is | | | 12 |

Marcus looks in the table. Yes, 4 + 8 = 12.

Miss Linda explains what is easy to do:

- If the first number is less than 5 you count down, and for the second number you count up.
- If the first number is 5 or more you count up, and for the second number you count down.

## 14. Mental addition with jumps

When you learn the table of addition by heart then it goes faster.

Then you don't make steps but jumps.

How do you do these sums ?

Does everyone in class have the same outcome ?

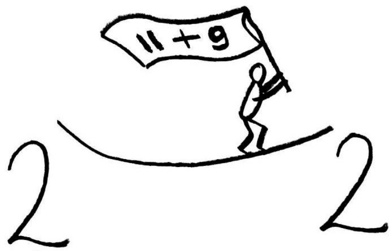
5 + 6 =

7 + 8 =

9 + 3 =

2 + 6 =

4 + 7 =




ten ten

15. The table of addition of two·ten

Miss Linda says: When I use small writing then I can make the table of addition for 1 to 20.

Two·ten plus two·ten is four·ten.

| **+** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** | **12** | **13** | **14** | **15** | **16** | **17** | **18** | **19** | **20** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **1** | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
| **2** | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
| **3** | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
| **4** | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
| **5** | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| **6** | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 |
| **7** | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 |
| **8** | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 |
| **9** | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| **10** | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| **11** | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 |
| **12** | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 |
| **13** | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 |
| **14** | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 |
| **15** | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 |
| **16** | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 |
| **17** | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 |
| **18** | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 |
| **19** | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 |
| **20** | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |

Susan may pick a number. She says 9.

Sam may pick a number. He says 14.

Miss Linda asks Marcus to add these. What is 9 + 14 ?

Marcus counts from 9 to 10, and down from 14 to 13.

| | | |
|---|---|---|
| number | 9 | 10 |
| plus | 14 | 13 |
| is | | 23 |

Marcus checks the table. Yes, 9 + 14 = 23.

## 16. Adding more numbers

Sam may pick a number. He says 7.

Susan may pick a number. She says 11.

Marcus may pick a number. He says 6. It is his sixth birthday.

What is 7 + 11 + 6 ?

The friends start adding the three numbers.

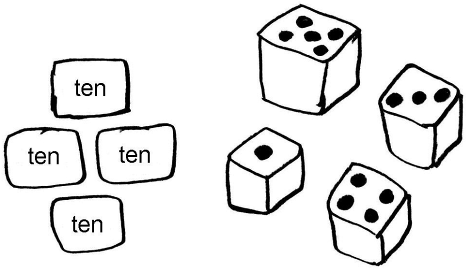
When they find 0 or 10 then they stop changing them.

| | | | | |
|---|---|---|---|---|
| number | 7 | 8 | 9 | 10 |
| plus | 11 | 10 | 10 | 10 |
| plus | 6 | 6 | 5 | 4 |
| is | | | | 24 |

You can also add numbers one by one:

7 + 11 + 6 =

18 + 6 = 24


ten
ten
ten
ten

## 17. Adding many numbers

They may pick one or two numbers each.

Sam says 5 and 11. Susan says 20 and 3. Marcus says 14.

What is 5 + 11 + 20 + 3 + 14 ?

The class wants to find out what these numbers add up to.

Miss Linda shows a fast way.

She takes the numbers of ten apart.

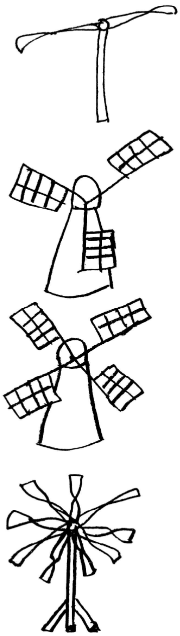
| | | | | |
|---|---|---|---|---|
| number | 5 | | 5 | 5 |
| plus | 11 | 10 | 1 | 11 |
| plus | 20 | 20 | 0 | 20 |
| plus | 3 | | 3 | 3 |
| plus | 14 | 10 | 4 | 14 |
| is | | 40 | 13 | 53 |

Five·ten·three. That is a high number !

Marcus shows another way to do it.

5 + 11 + 20 + 3 + 14 =
16 + 20 + 3 + 14 =
36 + 3 + 14 =
39 + 14 =
40 + 13 =
50 + 3 = 53

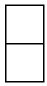
He thinks that the way by Miss Linda is faster.



# 18. Group, of, by, times

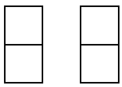
The class counts how many tiles a stoop has.
How many groups are there ? How many are there in a group ?

Here is a group of two tiles. How many tiles are there ?
One group of two = one of two = one by two = one times two = ?



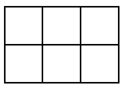
one times two tiles (one by two)
1 × 2 = 2 tiles together (one of two)

Two groups of two tiles. How many tiles are there ?



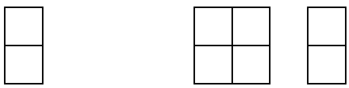
two times two tiles (two by two)
2 × 2 = 4 tiles together (two of two)

Three groups of two tiles. How many tiles are there ?



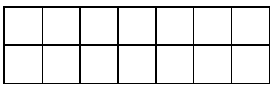
three times two tiles (three by two)
3 × 2 = 6 tiles together (three of two)

Four groups of two tiles. How many tiles are there ?



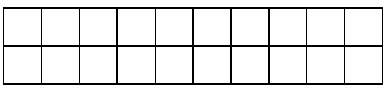
four times two tiles (four by two)
4 × 2 = 8 tiles together (four of two)

Seven groups of two tiles. How many tiles are there ?

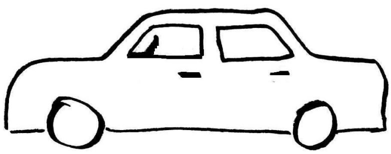
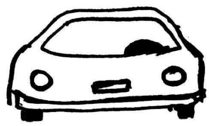


seven times two tiles (seven by two)
7 × 2 = 14 tiles together (seven of two)

Ten groups of two tiles. How many tiles are there ?



ten times two tiles (ten by two)
10 × 2 = 20 tiles together (ten of two)

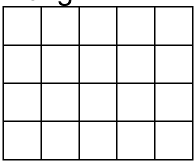
## 19. Length by width

A stoop has length and width.
We take length horizontal (*laying*) and width vertical (*standing*).

This stoop is 5 tiles long en 4 tiles wide.
How many tiles are there ?

Long

Wide

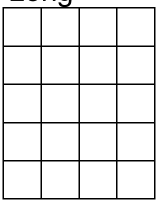


length times width is all
5 times 4 tiles (5 by 4) (5 of 4)
5 × 4 = 20 tiles all together
5 groups of 4 gives 20

This stoop is 4 tiles long (horizontally) en 5 tiles wide (vertically).
How many tiles are there ?

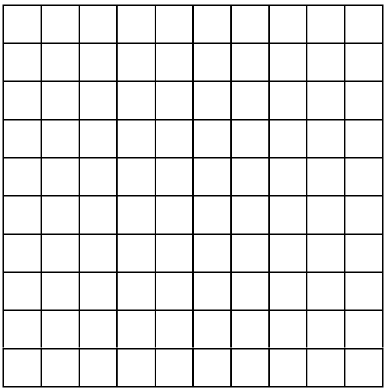
Long

Wide



length times width is all
4 times 5 tiles (4 by 5) (4 of 5)
4 × 5 = 20 tiles all together
4 groups of 5 gives 20

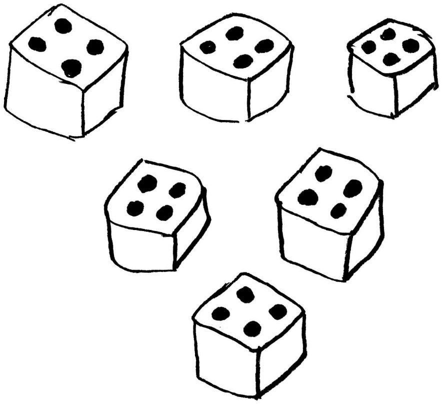
This stoop is 10 tiles long and 10 tiles wide.
How many tiles are there ?



ten times ten tiles
10 × 10 = 100 tiles all together

PM. What is the difference with
§10. Hundred and one numbers (p39) ?

Give an example when you cannot do times ?



## 20. The table of group, of, by, times

Miss Linda says: now we look at the table of *group, of, by, times.*

When we time 1, 2 and 3 with themselves and each other, then we get the following table.

| **×** | **1** | **2** | **3** |
|---|---|---|---|
| **1** | 1 × 1 = 1 | 1 × 2 = 2 | 1 × 3 = 3 |
| **2** | 2 × 1 = 2 | 2 × 2 = 4 | 2 × 3 = 6 |
| **3** | 3 × 1 = 3 | 3 × 2 = 6 | 3 × 3 = 9 |

And so on, Marcus says.

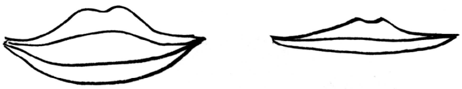
Miss Linda nods.

When we time 1 to 10 with themselves and each other, then we get the following table.

| **×** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** |
|---|---|---|---|---|---|---|---|---|---|---|
| **1** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| **2** | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| **3** | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 |
| **4** | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 | 40 |
| **5** | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 |
| **6** | 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 | 60 |
| **7** | 7 | 14 | 21 | 28 | 35 | 42 | 49 | 56 | 63 | 70 |
| **8** | 8 | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 | 80 |
| **9** | 9 | 18 | 27 | 36 | 45 | 54 | 63 | 72 | 81 | 90 |
| **10** | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |

5 Children with each 10 fingers is 5 × 10 = 50 fingers.

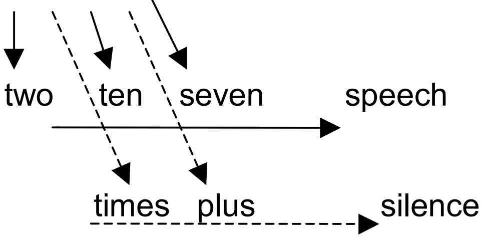
And 4 children and each 6 marbles is 4 × 6 = 24 marbles.



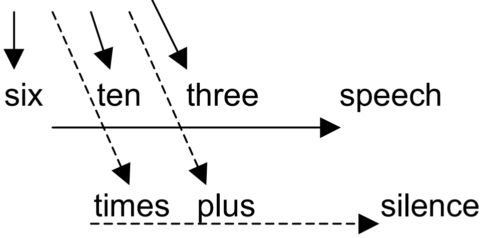
## 21. Speech is silver, silence is golden

Miss Linda shows these sums:

2 × 10 + 7 = 20 + 7 = 27 = two·ten·seven




6 × 10 + 3 = 60 + 3 = 63 = six·ten·three




The name of a number is how it is calculated with ten.

You can understand how numbers are spoken now that you have learned what *group, of, by, times* is.

Idea: write × with red, and + with green, when you don't pronounce them.

## 22. A present for Marcus

Miss Linda says:

Marcus has his birthday and I have a present for him.

Marcus, here are the very high numbers.

| | | | *Short - also in calculation* |
|---|---|---|---|
| 10^1 | ten | 10 | ten |
| 10^2 | ten·ten | 100 | hundred |
| 10^3 | ten·ten·ten | 1,000 | thousand |
| 10^4 | ten·ten·ten·ten | 10,000 | ten·thousand |
| 10^5 | ten·ten·ten·ten·ten | 100,000 | hundred·thousand |
| 10^6 | ten·ten·ten·ten·ten·ten | 1,000,000 | million |

In this way you make a high number:

| | | |
|---|---|---|
| number | 5000 | five·thousand |
| plus | 300 | three·hundred |
| plus | 80 | eight·ten |
| plus | 7 | seven |
| is | 5387 | five·thousand·three·hundred·eight·ten·seven |

Miss Linda explains:

There are three·hundred·million people in the USA.

Sam says: that is the highest number that I know.

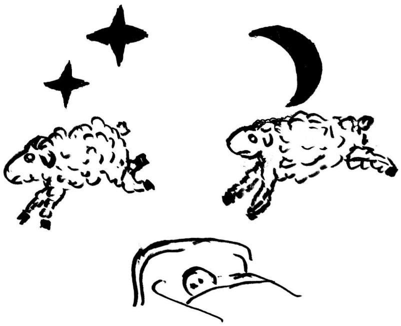
Not true, Marcus says.

300·million plus one gives 300·million·one.

And so on, Marcus says.

Miss Linda laughs.

She says: Today your name is *Marcus And so on*.



## 23. Marcus counts sheep

It is evening and Marcus is in bed.

His head is full of numbers.

He cannot sleep.

He counts sheep.

One, two, three, four, five, …

Thousand, thousand and one, thousand and two, …

Million, million and one, million and two, million and three, ….

Million·million, million·million and one, ….

Marcus says: and so on.

He falls asleep happily.

# Decimal positions using fingers and ells

## Introduction

The idea is to allow pupils to grow aware of the positional system much earlier. This will allow them to achieve faster insight in the structure of numbers and arithmetic, including multiplication. Whether this is so, of course needs to be tested in research.

I did not quickly see a developed system on the internet that satisfies some basic conditions: (i) use of fingers and (ii) use of the positional system for those finger signs. My intention here is to clarify what those conditions are.

For example, the American Sign Language drops out, since it doesn't use the positional system. [16] There is some conflicting information on the German DGS (base 5 ?). [17]

The following develops a method to use the lower arms (ells) to signify the numbers of ten: 10, 20, ..., 100. This leaves the hands free to fill in the intermediate digits. The method is a proposal for further research, not a proposal for implementation.

## Number sense, process and result

A *sense of number* is natural to many mammals and at least humans, see Piazza & Dehaene (2004). We teach children to use their fingers to count to ten. Milikowski (2010):

> "Kaufmann concludes: a brain doing arithmetic needs the fingers for a long while for support. They apparently help to build a bridge from the concrete to the abstract. In other words: the use of the fingers helps the brain to learn the meaning of the digits."

There is a problem for the numbers higher than 10, since there are no more fingers. Pupils find it difficult to master the positional shift.

In Holland, First Grade is limited to addition and subtraction with the numbers to 20 – a bit comparable to the US *Common Core*. This will be related to the positional shift, the illogical pronunciation of the numbers (*nineteen* instead of *ten·nine* and *twenty* instead of *two·ten*), and the fact that multiplication may quickly give such awkward numbers. When we take a fresh look at the issue then we may agree that learning the numbers to 20 does not have a priority in itself. Unless research would show that First Grade can only grasp number size but not multiplication.

A calculation like 2 × 10 + 4 is not much of a calculation since it is precisely the definition for the number 24 within the positional system (two·ten·(and)·four). There is a distinction between calculation as the process and number as the result. EWS:29 has:

> Gray & Tall developed this distinction into the idea of a 'procept'. Tall (2002) seems to embed the 'procept' into the 2nd Van Hiele level:
>
> > "The Symbolic-Proceptual World of symbols in arithmetic, algebra and calculus that act both as PROcesses to do (eg 4+3 as a process of addition) and conCEPTs to think about (eg 4+3 as the concept of sum.)"

---

16 https://www.youtube.com/watch?v=teK9oqqOo6g

17 http://journal.frontiersin.org/article/10.3389/fpsyg.2011.00256/full
and http://www.sign-lang.uni-hamburg.de/alex/lemmata/oberbegr/zahlen.htm

I have a small problem with this use of vocabulary, in that a concept is not necessarily static and may well be a process too. It is not necessary to limit the distinction between verb and noun to symbols only. It is not entirely clear whether it is really useful to use a new word “procept” to indicate that verbs and nouns are connected, and that processes hopefully give a result and that results tend to be created by processes. That said, the Gray & Tall idea remains important. It points to the phenomemon that mathematics can use *deliberate vagueness* in order to make efficient use of the same symbols. See Colignatus (2014) for Van Hiele and Tall.

For process, result and their connection, it seems advantageous for pupils to be aware how the positional system works. With that awareness pupils not only count but know that the number system supports both counting and arithmetic. This awareness may already start when they begin using fingers to learn to count.

## Caveat

The Prologue of this book has indicated my lack of knowledge about this topic. Libraries have been written about the education on numbers, counting and arithmetic. Domahs, Kaufmann and Fischer (eds) (2012) show a mature field of research which I have not looked into. I have not read the latter reference and lack time to do so. The following is only prospective. These comments have been triggered by the apparant lack of a system that satisfies the mentioned two conditions, but perhaps I did not look well enough.

## Base 10 versus base 6

Originally in 2012, I wrote *Numbers in base six in First Grade ?,* here put in the Appendix. This article wonders about a training on the positional system itself, by using fingers and hand in base 6, before using base 10. The fingers on the right hand count the single digits, and the fingers on the left hand count the number of (completed) right hands. The idea is rather radical and will not be quickly adopted. Few parents will offer their children to experiment with. (The pupils might become confused between the senary and decimal systems, for example.) [18]

The only reason to include that article here in the Appendix is that it was a useful stepping stone to think more generally about gestures to indicate number position:

(a) Pupils use the fingers because of their great educational value.
(b) Cognition about the positional system better is an explicit learning goal so that pupils can achieve insight in the structure of numbers and arithmetic, including multiplication.
(c) A question for empirical research is: can pupils in First Grade already multiply ?

The objective becomes: can we think of a *positional sign system* ? The following develops a suggestion how the lower arms (ells) might be used to identify the numbers of ten (10, 20, ..., 100). The fingers are used for the intermediate numbers.

---

[18] In 2012 I wrote seductively: "We might agree on this: Counting the fingers on the back of the hand (with the thumbs in the middle) we use the decimal system, and, counting the fingers on the palm of the hand (with the thumbs sticking out) we use base six, i.e. the senary system. In a senary system with two hands, the right hand for the units 0 to 5 and the left hand for the number of right hands, in the order of the Indian-Arabian positional system. When we have this foundation in First Grade and below then the later change to the decimal system seems a repetition of moves, relatively simple and enlightening." Of course I advised to get evidence, but now in 2015 it seems better to develop a system of gestures for the decimal system anyway, to use from the beginning.
PM. This discussion and the Appendix cover the same subject except for base 10 or 6. It is useful that both discussions can stand by themselves. Some texts thus are copies.

## Design principles

A system of signs is in Table 2. Design principles have been:

(1) For arithmetic it is easier to look at your palm and check how the thumb holds down other fingers.

(2) Zero is given by the neutral position of two fists, palms up.

(3) Only the ell (lower arm) is used (since stretching the full arm causes problems in class).

(4) The numbers are assigned in clockwise rotation.

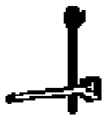
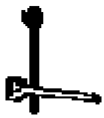
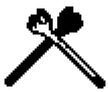
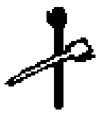
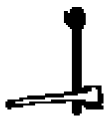
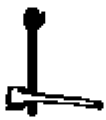
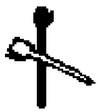
(5) To support the positional shift: *All ten fingers out* is equivalent to the next position with *fists.* For example, ten can also be presented by two fists, crossed at the wrists, left over right, see Table 2. This allows a stepwise transition from fingers to fists. Eventually one of these phases may be skipped. (Thinking continually in terms of equality and replacement will slow down the process of counting.)

(6) The numbers up to and including 50 use the distinction between a fist at the wrist versus a fist at the inside of the elbow. This suits younger pupils. For 60-100 we must use the middle positions of the ells too.

(7) At 50 the hands turn over (from palms up to palms down). At 50 the right ell over left ell (for 40) also switches to left ell over right ell, to allow a new clockwise round.

(8) The table needs only mention the tens (10, 20, ..., 100). Numbers in-between have some fingers out. There is no need for a scheme on fingers.

## Table 2. Number, gesture (sign), description

| *Number* | *Gesture* | *Description (ell = lower arm)* |
|---|---|---|
| 0 |  | Two fists parallel, palms up |
| 10 |  | Two hands parallel, palms up, all fingers stretched |
| 10 |  | Two fists crossing at the wrists, palms up (thumbs together), left ell over right ell |
| 20 |  | Two fists, palms up (thumbs facing each other), left wrist over right elbow (inside of the elbow) |
| 30 |  | Two fists, palms up (thumbs facing each other), right wrist over left elbow (inside of the elbow) |
| 40 |  | Two fists crossing at the wrists, palms up (thumbs together), right ell over left ell |
| 50 |  | Two fists crossing at the wrists, palms down (thumbs not facing each other), left ell over right ell |
| 60 |  | Two fists, palms down (thumbs not facing each other), left wrist over middle of right ell |
| 70 |  | Two fists, palms down (thumbs not facing each other), left wrist over right elbow (inside of the elbow) |
| 80 |  | Two fists, palms down (thumbs not facing each other), right wrist over left elbow (inside of the elbow) |
| 90 |  | Two fists, palms down (thumbs not facing each other), right wrist over middle of left ell |
| 100 |  | Two fists crossing at the wrists, palms down (thumbs not facing each other), right ell over left ell |

# Related research questions

The advantage of having above signs is that we can consider the introduction of multiplication. For these pupils it seems better to speak about ‘times’ and ‘to time’ (or to repeat) rather than the long terms ‘multiplication’ and ‘multiply’ (multi-plus).

The curious point is:

> When pupils in First Grade can master above sign system (say to 50) then this itself shows that they can master elementary multiplication. Counting groups of ten namely is multiplication by ten. Can they multiply different numbers ?

The discussion of a rectangle and its surface shows that *times* is commutative. Thus, the order of *times* does not matter. When there are five cats with each two eyes then there are 5 × 2 = 10 eyes in total. With five cats you have five left eyes and five right eyes, thus 5 + 5 = 2 × 5 = 10.

A calculation like 2 × 12 = 24 contains operations that seem doable at this level, using the property that 2 × 10 + 4 is the formula for the number 24. Table 3 uses those higher numbers to make the issue nontrivial. How high can the numbers be for First Grade ?

**Table 3. Example multiplications**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 12 | | 10 + 2 | | 19 | | 10 + 9 | |
| 2 | × | 2 | × | 4 | × | 4 | × |
| 4 | | | | 36 | | | |
| 20 | + | | | 40 | + | | |
| 24 | | | | 76 | | | |

Many pupils of age six could learn this. Would there be a sufficient number of them to introduce the approach in the general curriculum ?

Counting groups of ten is a higher level of abstraction (the levels identified by Pierre van Hiele). Counting is the ticking-off of the elements of a set. It is a higher level of abstraction to see a set as a new unit of account, and then tick off the sets.

The following is an important insight with respect to *times*:

> A result like 5 × 2 = 10 is trivial for us but only since we learned this by heart.

Some authors argue that pupils need not learn the *table of times* by heart but must first feel their way. This runs against logic. If you don’t learn the *table of times* by heart then you remain caught in the world of addition. This is very slow and does not contribute to understanding. Remember what *times* is:

(1) Taking a set of sets
(2) To know how you can count single elements but that it is faster to only count the border totals
(3) To know which table to use to look it up (namely × instead of +)
(4) And get your result faster because you know the table by heart
(5) To know all of this.

This discussion shows the advantage of knowing what *times* means. Who knows what it is can understand how the numbers are contructed, and can also understands what arithmetic is (the collection of the weights of the powers of the base number). For this reason it is didactically advantageous to have *times* available as quickly as possible.

It becomes a serious research question whether more should be done with set theory in primary education. Apparently pupils are willing and able to memorize long lists of data but it might be enlightening for them to discuss what a set is, and that multiplication concerns the determination of the cardinal number of a set of sets.

## Conclusions

Current problems in teaching arithmetic may have to do less with the number system itself, see for comparison the 1950s. In Holland since then there has been a curious move towards *not* learning the tables by heart, see Milikowski (2004). We may already see a big improvement when misunderstandings like these are resolved. That said, it still is a separate issue to think about the number system and its relation to arithmetic.

Libraries have been filled on number and arithmetic but the present discussion seems to includes these useful points:

(1) The sign system with ells seems doable.
(2) This book gives another perspective as well, with the proposal to revise the names of the decimal numbers (with 11 = ten·one and so on).
(3) Research in both didactics and brains could look with priority whether First Grade can multiply. When pupils can learn about the positional system then this already shows their elementary grasp of *times*. Can pupils also multiply with other numbers ? Five cats with two eyes each gives ten eyes. Seems doable as well. When a range of numbers can be found then this can be exploited to develop arithmetic.
(4) Above discussion may also help to beter target learning aims for Second Grade. Problems like 2 × 10 + 4 = 24 highlight the structure of number as well.

# Re-engineering arithmetic

## Confusing math in elementary school

2014-08-25 [19]

The problems in Russia-Ukraine, Irak-Syria and Israel-Gaza are so large since the combatants are hardly aware of the concept of fair division and sharing. Something must have gone wrong in elementary school with division and fractions. Let us see whether we can improve education, not only for future dictators but for kids in general.

### *English as a dialect*

In 2012 I suggested that English can best be seen as a dialect of mathematics. [20] The case back then was the pronunciation of the integers, e.g. 14 as *fourteen* (English) instead of *ten·four* (math & Chinese). The decimal positional system isn't merely a system of recording but it contains switches in the *unit of account*. In this system the step from 9 to 10 means that *ten* becomes a new unit of account, and the step from 99 to 100 means that *hundred* (ten·ten) becomes a new unit of account. This relies on the ability to grasp a whole and the notion of cardinality. Having a new unit of account means that it is valid to introduce the new words *ten* and *hundred*, so that 1456 as a number differs from a pin-code with merely mentioning of the digits. When the numbers are pronounced properly then pupils will show greater awareness of these elements and become better in arithmetic – and arithmetic is crucial for division and fractions.

When education is seen as trying to plug mathematics into the mold of English as a natural language, then this is an invitation to trouble. It is better to free mathematics from this mold and teach it in its own structural language. It is a task for the teaching of English to show that it is a somewhat curious dialect.

### *Rank numbers*

After the recent discussion of ordinal or cardinal 0, [21] it can be mentioned that the ordinals are curiously abused in the naming of fractions. Check the pronunciation of 1/2, 1/3, 1/4, 1/5, … With number 4 = four and the rank 4th = fourth, the fraction 3/4 is pronounced as *three·fourths*. What is rank *fourth* doing in the pronunciation of 3/4 ? School kids are excused to grow confused.

Supposedly, when cutting up a cake in four parts, one can rank the pieces into the first, second, third and fourth piece. Assuming equal pieces, or fair division, then one might borrow the name of the last rank number *fourth* to say that all pieces are *a fourth*. This is inverse cardinality. Presumably, this is how natural language developed in tandem with budding mathematics. Such borrowing of terms is conceivable but not so smart to do. It is confusing.

The creation of *a fourth*, as a separate concept in the mind, also takes up attention and energy, but it doesn't produce anything particularly useful. Malcolm Gladwell alerted us to that the Chinese language pronounces 3/4 as "*out of four parts, take three*". [22] Shorter

---

[19] https://boycottholland.wordpress.com/2014/08/25/confusing-math-in-elementary-school/
[20] https://boycottholland.wordpress.com/2012/04/01/english-as-a-dialect-of-mathematics/
[21] https://boycottholland.wordpress.com/2014/08/01/is-zero-an-ordinal-or-cardinal-number-q/
[22] http://gladwell.com/outliers/rice-paddies-and-math-tests/

would be “3 out of 4″. This directly mentions the parts, and there is no distracting step in-between.

For a reason discussed below we better avoid the “of” in “out of”. Thus it might be even shorter to use “3 from 4″, but a critical reader alerted me to that his might be seen as subtraction. Thus “3 out 4″ seems shortest. However, there is also the issue of ratio versus rate. In a ratio the numerator and the denominator have the same dimension (say apples) while in a rate they are different (say meter per second).

Thus the overall best shortest pronunciation would be “3 **per** 4″, which is neutral on dimensions, and actually can be used in most European languages that are used to “percent”.

This pronunciation facilitates direct calculation, like “*one per four* plus *three per four* gives *four per four,* which gives *one”.*

### *Dividing and sharing*

The Dutch word for “divide” ("delen") also means “share”. Sharing a cake tends to generate a new unit of account, namely the *part.* In fair division each participant gets a part of the same size, which becomes: the same part. This process focuses on the denominator and generates a larger number and not a smaller number. [23] It actually relies on multiplication: the denominator times the new unit of account (the part) gives the original cake again. The process of sharing is rather opposite to the notion of division that gives a fraction, that maintains the old unit of account and generates a smaller number on the number line.

A fraction 3/4 or *three per four*, when three cakes must be shared by four future dictators, requires the pupil to establish the proportional ratio with *three cakes per four cakes* (virtually giving each a cake even though there are no four cakes but only three), and then rescale from the four hypothetical cakes down to one cake.

PM. The pupil must have a good control of *active* versus *passive* voice. [24] The relation is that “4 kids share 3 cakes” (active) and “3 cakes are being shared by 4 kids” (passive). Thus *3 per 4* or *3 out of 4* is shorthand for “3 units taken out of 4 units” (or “4 (kids) take out of 3 (cakes)”) but not for “3 (kids) take out of 4 (cakes)”. The latter would give 1 + 1/3 per kid, and would require a discussion of mixed numbers).

Hence it is unfortunate that the Dutch language uses the same word for both *sharing* and *dividing*. Fraction 3/4 reads in Dutch as “3 shared by 4 gives three·fourths” (“3 gedeeld door 4 geeft drie·vierde”), which thus combines the two major stumbling blocks: (a) the sharing/dividing switch in the unit of account, (b) the curious use of rank words. When 3/4 = *three per four* would be used, then the stumbling blocks disappear, and teaching could focus on the difference between the *process* of dividing and the *result* of the fractional number on the number line.

David Tall (2013) points to a related issue in the language on sharing and dividing:

> “The notion of a fraction is often introduced as an *object,* say ‘half an apple’. This works well with addition. (…) What does ‘half an apple *multiplied by* half an apple’ mean? (…) However, if a fraction is seen flexibly as a *process,* then we can speak of the process ‘*half* [halve] an apple’ and then take ‘a *third* of half an apple’ (…) the idea is often simply introduced as a rule, ‘of means multiply’, which can be totally opaque to a learner meeting the idea for the first time.” (p97)

[23] Indeed in absolute sizes: not only greater but also larger, not only lesser but also smaller.
[24] http://en.wikipedia.org/wiki/English_passive_voice

Note that Tall's book is rather confused [25] so that you better wait for a revised edition. He indeed does not mention above issues (a) and (b). But this latter observation on the *process* and *result* of division is correct.

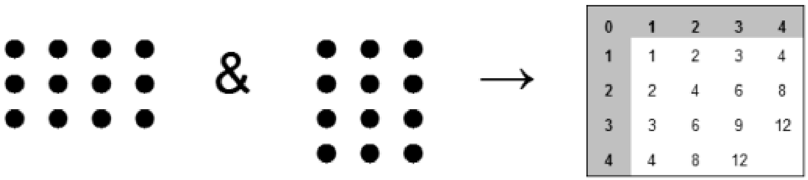
The rank words thus are abused not only as nouns but also as verbs ("take a *third* of half of an apple"). We better translate into "(*one per three*) of (*one per two*)", which gives "(one times one) per (three times two)". The mathematical procedure quickly generates the result. The didactic challenge becomes to help kids understand what is involved rather than to master confused language.

## *Multiplication*

Speaking about Tall and multiplication: Apparently the English pronunciation of the tables of multiplication can be wrong too. E.g. *'two fours are eight'* refers to two groups of four, and thus implies an order, while merely *'two times four is eight'* gives the symmetric relation in arithmetic. Tall's book p94 contains a table with 3 rows and 4 columns – see Figure 1 – and Tall argues:

> "the idea of three cats with four legs is clearly different from that of four cats with three legs. The consequence is that some educators make a distinction between 4 x 3 and 3 x 4. (…) I question whether it is a good policy to *teach* the difference. (…) [ reference to Piaget ] (…) So a child who has the concept of number should be able to see that 3 x 4 is the same as 4 x 3."

**Figure 1. An exercise in marbles**




| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 |
| 2 | 2 | 4 | 6 | 8 |
| 3 | 3 | 6 | 9 | 12 |
| 4 | 4 | 8 | 12 | |

Tall doesn't provide this explanation: Pierre van Hiele focuses on the distinction *'concrete versus abstract',* and would focus on the table, so that children would master the insight that the order does not matter for arithmetic. Once they have mastered arithmetic, they might consider *'reality versus model'* cases like on the cats and their legs without becoming confused by arithmetical issues hidden in those cases. Instead, Hans Freudenthal with his 'realistic education of mathematics' (RME) would present kids with the 'reality versus model' cases (e.g. also five cups with saucers and five cups without saucers, a 3D table), and argue that this would inspire kids to re-invent arithmetic, though with some guidance ("guided re-invention"). Earlier, I wondered why Freudenthal blocked empirical research in what method works best (and my bet is on Van Hiele). [26] See p 101.

## *Conclusion*

Overall, the scope for improvement is huge. It is advisable that the Parliaments of the world investigate failing math education and its research. When kids have improved skills in arithmetic and language, they would have more time and interest to participate in and understand issues of fair division. Hurray for World Peace !

---

[25] http://arxiv.org/abs/1408.1930

[26] https://boycottholland.wordpress.com/2014/07/06/hans-freudenthal-s-fraud/

PM 1. *Conquest of the Plane* pages 77-79 & 207-210 discuss proportions and fractions.

PM 2. See also COTP for the distinction between standard static division *y* / *x* and dynamic division *y* // *x.*

PM 3. Some say "3 over 4″ for 3/4, hinting at the notation with a horizontal bar. I wonder about that. The "3 per 4″ is actually shorter for "3 taken out of 4″, and this puts emphasis on what is happening rather than on the shape of the notation. An alternative is "3 out of 4″ but my inclination was to avoid the "of" as this is already used for multiplication. Also, my original training has been to reserve "*n* over *k*" for the binomial coefficient [27] (that can be taught in elementary school too). However, a reader alerted me to Knuth's suggestion [28] to use "*n* choose *k*" for the binomial coefficient, and that is better indeed. In any case I would still tend to avoid the "over".

It was also commented that "3 from 4″ sounded like subtraction: but my proposal is to adhere to "3 minus 4″ for "3 – 4″ as opposed to "3 plus 4″ for addition. It is just a matter to introduce plus and minus into general usage, so that it is always clear what they are. Note that we are speaking about mathematics as a language and not about English as a natural language. Also -1 would be "negative-1″ or "min-1″, with the sign "min" differing from the operator "minus".

[27] http://en.wikipedia.org/wiki/Binomial_coefficient
[28] http://arxiv.org/abs/math/9205211

# Taking a loss

2014-08-30 [29]

Sharp readers will have observed that Vladimir Putin of Russia closely follows the suggestions in this weblog. After the last discussion of "*To invade or not to invade ?"* [30] we now see the *"Alea iacta est"* with Russian tanks crossing the Ukrainean border.

Putin's dilemma reminded of Shakespeare and the Danish prince Hamlet: *"To be or not to be ?"* We shouldn't be surprised that we got a response from Peter Harremoës from Denmark as well. [31]

On the issue of taking a loss, be it the Crimea or now larger parts of the Ukraine, or children losing their fingers in Iraq-Syria or Israel-Gaza, but rather mathematically more general in the form of the subtraction of numbers in arithmetic, and thus the creation of negative numbers, Harremoës has developed a creative new approach that might stop the combatants in amazement. His 2000 article [32] might stop you too, since it still is in Danish, and Google Translate still isn't perfect. Harremoës mentions that he considers an extension in English at some time, so let us keep our fingers crossed till then – while we still have those.

In the mean time I would like to take advantage of some minor points on subtraction, partly relying on Peter's article and thanking him for some additional explanation too.

## *Notation of negative numbers*

Namely, in the last weblog discussion on confusing math in elementary school [33] I stated that it is important to distinguish the *operator* **minus** from the *sign* **min**. Peter referred to *a* – (-*b*) and commented that problems of subtraction better be transformed into addition, and that subtraction can be seen on an abstract level as much more complex (or mathematically simple) than commonly thought.

One of his proposals is to create a separate symbol for -1 without the explicit showing of the min-sign. He took an example from history in which *1-with-a-dot-on-top* already stood for -1. I have wondered about this, and would suggest to take a symbol that is available on the keyboard without much ado, where we e.g. already have *i* = Sqrt[-1].

A-*ha* ! Doesn't the reader hear the penny drop ? Let us take *i* = quarter turn*, H* = half turn = $i^2$ = -1, then $i^3$ = *H i* = – *i* = 3 / 4 turn = three per four turn, and *H H* = full turn = 1.

It would appear that *H* best be pronounced as 'eta', [34] both for international exchange, and in sympathy for German teachers who would otherwise have to pronounce *H H* as 'haha', which would form a challenge for the German sense of humour. I considered suggesting small *η* or *h* but the nice thing about *H* is that it has a shade of -1 in it. In elementary school we can use just the Harremoës-operator *H* = -1 without the complex numbers. Later in highschool when complex numbers would arise we can usefully refer to *H* as something that would already be known (or forgotten).

[29] https://boycottholland.wordpress.com/2014/08/30/taking-a-loss/
[30] https://boycottholland.wordpress.com/2014/08/17/putins-ultimatum-to-himself-to-invade-or-not-to-invade/
[31] http://www.harremoes.dk/Peter/
[32] http://www.harremoes.dk/Peter/talnot.pdf
[33] https://boycottholland.wordpress.com/2014/08/25/confusing-math-in-elementary-school/
[34] http://en.wikipedia.org/wiki/H

## *Properties of* H

Kids can understand that a debt is an opposite from a credit, or that losing the Ukraine is opposite to winning it. Thus if $a$ is an asset then $H\ a$ is a liability of the same absolute size.

- Calculation of gains and losses could be done with $a + H\ b$ for counting down, or $H\ b + a$ for counting up.
- If you lose a debt, then you gain. Losing a debt $H\ b$ then would be introduced as $a + H\ (H\ b) = a + b$.

  (2015: This might be misunderstood. Having a debt might be written as $a + H\ b$. Losing that debt then is $a + H\ b + H\ H\ b = a$. The above takes $a' = a + H\ b$.)

Actually, I suppose that it would be even better to start with the absolute difference between two numbers, $\Delta[a, b]$. A sum would be to determine that $\Delta[a, H\ b] = a + b$, presuming that $a$ and $b$ are nonnegative integers.

Thus $H$ would be used in the creation of the negative numbers and the introduction of subtraction, and for later remedial teaching for who didn't get it or lost it. Peter Harremoës seems to be of the opinion that there would be no need, in principle, to introduce minus and min, but agrees that people would currently want to stick to common notions. Once the basics of $H$ are grasped, it is no use to grind them in, since it is better to switch to minus and min that must be ground in because of that commonality.

First the **min** sign and the negative integers are introduced by extending the number line: $-1 = H\ 1$ , $-2 = H\ 2$, … $-100 = H\ 100$ and so on. The teacher can show that applying $H$ means making half turns, or moving from the right to the left, or back.

Subsequently the **minus** operator is introduced as $a – b = a + H\ b$.

Hence there arises the exercise $a – (-b) = a – H\ b = a + H\ (H\ b) = a + 1\ b = a + b$.

Or the relation between minus and min: $–b = 0 – b = 0 + H\ b = H\ b$.

A pupil who has mastered arithmetic will do $a – (-b) = a + b$ directly. Otherwise return to remedial teaching and practice with $H$ again.

## *Positional system*

Arithmetic seems simplest in a positional system. Earlier, we already discussed that English better is regarded as a dialect of mathematics. [35] A number like 15 is better pronounced as *ten·five* than as *fifteen*. A sum 15 + 36 then fluently (yes!) translates into "*ten·five* plus *three·ten·six* equals (*one* plus *three*)·*ten*·(*five* plus *six*), equals *four·ten* plus *ten·one*, equals *five·ten·one*" which is 51. Let me introduce the suggestion that pupils can use balloons in handwriting or brackets in typing to indicate not only the digits but also the values in the positional system. See Figure 2.

**Figure 2. Adding 15 and 36 using the positional system with balloons or brackets**

$$15 + 36 = (1+3)(5+6) = (4)(11) = 40 + 11 = 51$$

15 + 36 = [ 1 + 3 ][ 5 + 6 ] = [ 4 ][ 11 ] = 40 + 11 = 51

In the same manner, the positional system allows us to state –1234 = [-1][-2][-3][-4], where we might rely on $H$ if needed.

[35] https://boycottholland.wordpress.com/2012/04/01/english-as-a-dialect-of-mathematics/

## *Subtraction in the positional system*

For subtraction, the algorithm for *a* – *b* is to keep that order if *a* ≥ *b,* or otherwise reverse and calculate -(*b* – *a*). But, it is useful to show pupils the following method if they forget about reversing the order. For example, 16 – 34 = 16 + [-3][-4] and the rest follows by itself, see Table 4. [36]

**Table 4. Do 16 minus 34 when forgetting to reverse the order**

| Introduction / Remedial | With typewriter using [ ... ] | Mastered |
|---|---|---|
| 16<br>-34<br>---- plus<br>1 - 3 6 - 4<br>-2 2<br>-20 + 2<br>-18 | 16<br>-34<br>---- plus<br>[ 1 - 3 ][ 6 - 4 ]<br>[ -2 ][ 2 ]<br>-20 + 2<br>-18<br>or<br>16-34 = [1-3][6-4] = [-2][2]<br>= -20 + 2 = -18 | 16<br>-34<br>---- plus<br>2<br>-20<br>---- plus<br>-18 |

## *Comparing to subtraction in American or Austrian ways*

One might compare the above with other expositions on subtraction. An obvious portal is the wikipedia article on subtraction, [37] while google gives some pages e.g. from the UK [38] or the USA. [39] Some texts seem somewhat overly complex.

Originally I thought that the subtraction *a* – *b* for *a* ≥ *b* would be harmless, but on close consideration there is a snake in the grass.

A point is that corrections are made *above* the plus-bar, so that the original question is altered.

- In the Wikipedia example of the Austrian method the final sum doesn't add up anymore.
- The Wikipedia example of the American method is okay, provided that indeed 7 is replaced by 6, and 5 is replaced by 15.

But this is not a proper positional notation anymore. The method also assumes that you use pen and paper, which is infeasible in a keyboard world.

## *A new method for subtraction*

In Table 5 below there are two examples on the right that keep the original sum intact, and that only use the working area *below* that original sum.

One approach is to rewrite 753 = [6][15][3] and the other approach is to do the borrowing a bit later, which is faster.

---

[36] 2015: The original weblog article has a minus-bar instead of a plus-bar, but it is better to consistenly use only plus-bars.
[37] http://en.wikipedia.org/wiki/Subtraction
[38] http://www.cimt.plymouth.ac.uk/projects/mepres/book7/bk7i15/bk7_15i1.htm
[39] http://www.themathpage.com/arith/subtract-whole-numbers-subtract-decimals.htm

These methods rely on the trick of using balloons or brackets to put values and sub-calculations into a positional place. If we allow for adaptation above the plus-bar, then the use of *H* = -1 and *T* = 10 would work as well, without the need to dash out digits. The second column combines the American & Austrian methods with the Harremoës operator *H,* indeed treated as a digit, and using [H][T][0] = HT0 = 0. See Table 5. [40]

**Table 5. Do 753 minus 491, in the American manner (source Wikipedia), with comments and an alternative new way on the right**

| Wikipedia (American) | *H* = -1, *T* = 10 | American [ ... ] | Direct |
|---|---|---|---|
|  | HT0<br>753<br>-491<br>------- plus<br>262 | 753<br>-491<br>------- plus<br>REWRITE<br>6[15]3<br>-(4 9 1)<br>----------- plus<br>262 | 753<br>-491<br>------- plus<br>3[-4]2<br>2[10-4]2<br>262 |

## *Evaluation*

Evaluating these methods, my preference is for the last column:

- It follows the work flow, in which the negative value is discovered by doing the steps.
- The method accepts negative numbers instead of creating some fear for them.
- A pupil with experience would not need the 2[10-4]2 line and directly jump to the answer, so that the number of lines is the same as in the first and second column.

The American method (also used in Holland) with HT0 = 0 inserted as a help line creates the suggestion as if borrowing is required before one can do the subtraction, which goes against the earlier training to be able to do a subtraction that results into a negative value. The borrowing is only required to finalize into a final number in standard notation.

Overall, my conclusion is that the emphasis in teaching should be on the positional system. The understanding of this makes arithmetic much easier.

Secondly, the Harremoës operator *H* indeed is useful to first understand the handling of credit and debt, before introducing the number line and the notation *a* – *b*.

Thirdly, in a combination of the two earlier points, this operator also appears useful into decomposing -1234 = [-1][-2][-3][-4].

I want to thank Peter again for starting all this (apart from the more advanced ideas in his article). For completeness, let me refer to the 2012 paper *A child wants nice and not mean numbers*, [41] with a discussion of the pronunciation of the numbers and some more exercise on the positional system.

But these mathematical operations don’t explain that Ukraineans first lose the Ukraine but subsequently gain it once they have turned into Russians.

[40] 2015: The original weblog article uses minus-bars, but this causes a problem in the second column when there are three rows. The consistent use of plus-bars is better.
[41] No longer available: since it has become this present book.

# With your undivided attention

April 9 2014 [42]

Both President Obama of the USA and President Putin of the Russian Federation have somewhat illogical positions. Obama repeats the ritual article 5 *“An attack on one is an attack on all”* but the Ukraine is not a fraction of NATO. So what is the USA going to do about the Ukraine ? Putin holds that Russia defends all Russians everywhere but claims that Russia is not involved with the combatants in the Ukraine. His proposed 7 point plan contains a buffer zone so that he creates fractions in a country on the other side of the border. Overall, we see the fractional division of the Ukraine starting, as already predicted in an earlier entry in this weblog.

What is it with fractions, that Presidents find so hard, and what they apparently didn’t master in elementary school, like so many other pupils ? There are two positions on this. The first position is that mathematics teachers are right and that kids must learn fractions, with candy or torture, whatever works best. The second position is that kids are right and that fractions may as well be abolished as both useless and an infringement of the *Universal Declaration of Human Rights* (article 1). [43] Let us see who is right.

## *An abolition of fractions*

Could we get rid of fractions ? We can replace 1 / *a* or *one·per-a* by using the exponent of -1, giving $a^{\urcorner}$ that can be pronounced as *per-a.* In the earlier weblog entry on subtraction [44] we found the Harremoës operator $H$ = -1. [45] The clearest notation is $a^H = 1 / a$. Before we introduce the negative numbers we might consider to introduce the new notation for fractions. The trick is that we do not say that. We just introduce kids to the operator with the following algebraic properties:

$$0^H = \textit{undefined}$$

$$a\, a^H = a^H a = 1$$

$$( a^H )^H = a$$

Getting rid of fractions in this manner is not my idea, but it was considered by Pierre van Hiele (1909-2010), [46] a teacher of mathematics and a great analyst on didactics, in his book *Begrip en Inzicht* (1973:196-204), thus more than 40 years ago. His discussion may perhaps also be found in English in *Structure and Insight* (1986). Note that $a^H = 1 / a$ already had been considered before, certainly in axiomatics, but the Van Hiele step was to consider it for didactics at elementary school.

From the above we can deduce some other properties.

**Theorem** 1

$$(a\, b)^H = a^H b^H$$

[42] In FMNAI and https://boycottholland.wordpress.com/2014/09/04/with-your-undivided-attention/
[43] http://www.un.org/en/documents/udhr/
[44] https://boycottholland.wordpress.com/2014/08/30/taking-a-loss/
[45] It is one single symbol but still reminds of "-1". Pronounce the operator as "eta".
[46] http://en.wikipedia.org/wiki/Van_Hiele_model

Proof. Take $x = a\ b$. From $x^H\ x = 1$ we get $(a\ b)^H\ (a\ b) = 1$. Multiply both sides with $a^H\ b^H$, giving $(a\ b)^H\ (a\ b)\ a^H\ b^H = a^H\ b^H$, giving the desired. Q.E.D.

**Theorem** 2

$$H^H = H$$

Proof. From addition and subtraction we already know that $H\ H = 1$. Take $a\ a^{\ H} = 1$, substitute $a = H$, get $H\ H^H = 1$, multiply both sides with $H$, get $H\ H\ H^H = H$, and thus $H^H = H$. Q.E.D.

It remains to be tested empirically whether kids can follow such proofs. But they ought to be able to do the following.

**Simplification**

The expression $10 * 5^H$ or *ten per five* can be simplified into $10 * 5^H = 2 * 5 * 5^H = 2$ or *two each*.

**Equivalent fractions**

Observing that 6 / 12 is actually 1 / 2 becomes $6 * 12^H = 6 * (2 * 6)^H = 6 * 2^H * 6^H = 2^H$. Alternatively all integers are factorised into the primes first. Note that equivalent fractions are part of the methods of simplification.

**Multiplication**

$$a\ b^H * c\ d^{\ H} = (a\ c)\ (b\ d\ )^{\ H}$$

**Comparing fractions**

Determining whether $a\ b^H > c\ d^H$ or conversely: this reduces by multiplication by $b\ d$, giving the equivalent question whether $a\ d > c\ b$ or conversely.

**Rebasing**

That $(a / b = c) \Leftrightarrow (a / c = b)$ may be shown in this manner:

$$a\ b^H\ = c$$

$$a\ b^H\ \ (b\ c^H) = c\ \ (b\ c^H)$$

$$a\ c^H\ = b$$

**Addition**

Van Hiele's main worry was that we can calculate 2 / 7 + 3 / 5 = 31 / 35 but without much clarity what we have achieved. Okay, the sum remains smaller than 1, but what else ? Translating to percentages 2 / 7 ≈ 28.5714% and 3 / 5 = 60%, so the sum ≈ 88.5714%, is more informative, certainly for pupils at elementary school. This however requires a new convention that says that 0.6 is an exact number and not an approximate decimal, see *Conquest of the Plane* (2011c). The argument would be that working with decimals causes approximation error, and that first calculating 31 / 35 and then transferring to decimals would give greater accuracy for the end result. On the other hand it is also informative to see the decimal constituants, e.g. observe where the greatest contribution comes from.

Another argument is that 2 / 7 + 3 / 5 = 31 / 35 would provide practice for algebra. But why practice a particular format if it is unhandy ? The weighted sum can also be written in terms of multiplication. Compare these formats, and check what is less cluttered:

$$a / b + c / d = (a / b + c / d)\ (b\ d) / (b\ d) = (a\ d + c\ b) / (b\ d)$$

$$a\ b^H + c\ d^{\ H}\ = (a\ b^H + c\ d^{\ H})\ (b\ d)\ (b\ d)^{\ H} = (a\ d + c\ b)\ (b\ d)^{\ H}$$

### Subtraction

In this case kids would have to see that *H* can occur at two levels, like any other symbol.

$$a\ b^H + H\ c\ d^{\ H} = (a\ b^H + H\ c\ d^{\ H})\ (b\ d)\ (b\ d)^{\ H} = (a\ d + H\ c\ b)\ (b\ d)^{\ H}$$

### Mixed numbers

A number like *two-and-a-half* should not be written as *two-times-a-half* or 2½ . *Elegance with Substance* (2009) already considers to leave it at 2 + ½. Now we get $2 + 2^H$ .

### Division

Part of division we already saw in simplification. The major stumbling block is division by another fraction. Compare:

$$a\ /\ b\ /\ \{c\ /\ d\} = (a\ /\ b)(d\ /\ c)\ /\ \{\ (c\ /\ d)\ (d\ /\ c)\ \} = (a\ /\ b)(d\ /\ c)\ \ /\ \{\ 1\ \} = (a\ d)\ /\ (b\ c)$$

$$a\ b^H * (c\ d^{\ H})^H = a\ b^H\ * c^{\ H}\ d = (a\ d)\ (b\ c\ )^{\ H}$$

Supposedly, kids get to understand this by e.g. dividing 1/2 by 1/10 so that they can observe that there are 5 pieces of $10^H$ = 1/10 that go into $2^H$ = 1/2. Once the inversion has been established as a rule, it becomes a mere algorithm that can also be applied to arbitrary numbers like $34^H\ (127^H)^H$ = 1/34 / (1/127). The statement "divide *per-two* by *per-10*" becomes more general:

(divide by per-*a*) = (multiply by *a*)

### Dynamic division

A crucial contribution of *Elegance with Substance* (2009:27) and *Conquest of the Plane* (2011c:57) is the notion of dynamic division, that allows an algebraic redefinition of calculus. [47]

With $y\ x^H = y\ /\ x$ as normal static division then dynamic division $(y\ x^D) = y\ //\ x$ becomes:

$(y\ x^D) \equiv \{y\ x^H$, unless $x$ is a variable and then: assume $x \neq 0$, simplify the expression $y\ x^H$, declare the result valid also for the domain extension $x = 0$ }.

A trick might be to redefine *y* / *x* as dynamic division. It would be somewhat inconsistent however to train on $x^H$ and then switch back to the *y* / *x* format that has not been trained upon. On the other hand, some training on the division slash and bar is useful since it are formats that occur.

## *Van Hiele 1973*

Van Hiele in 1973 includes a discussion of an axiomatic development of addition and subtraction and an axiomatic development of multiplication and division. This means that kids would be introduced to group theory. This axiomatic development for arithmetic is much easier to do than for geometry. Since mathematics is targeted at 'definition, theorem, proof' it makes sense to have kids grow aware of the logical structure. He suggested this for junior highschool rather than elementary school, however. It is indeed likely that many kids at that age are already open to such an insight in the structure of arithmetic. This does not mean a training in axiomatics but merely a discussion to kindle the awareness, which would already be a great step forwards.

His 1973 conclusions are:

[47] https://boycottholland.wordpress.com/video/

**Advantages**

1. In the abolition of fractions 1/*a* a part of mathematics is abolished that contains a technique that stands on its own.
2. One will express theorems more often in the form of multiplication rather than in the form of division, which will increase exactness. (See the problem of division by zero.)
3. Group theory becomes a more central notion.
4. In determining derivatives and integrals, it no longer becomes necessary to transform fractions by means of powers with negative exponents. (They are already there.)

**Disadvantages**

1. Teachers will have to break with a tradition.
2. It will take a while before people in practice write $3\ 4^H$ instead of 3 / 4.
3. Proponents will have to face up to people who don't like change.
4. We haven't studied yet the consequences for the whole of mathematics (education).

His closing statement: "We do not need to adopt the new notation overnight. It seems to me very useful however to consider the abolition of the algorithms involving fractions."

## *Conclusion*

Given the widespread use of 1 / *a*, we cannot avoid explaining that $a^H = 1 / a$. The fraction bar is obviously a good tool for simplication too, check $6 * (2 * 6)^H$ .

Similarly, issues of continuity and limits $x \to 1$ for expressions like $(1 + x)\ (1 - x^2)^H$ would benefit from a bar format too. This would also hold alternatively for $(1 + x)\ (1 - x^2)^D$.

But, awareness of this, and the ability to transform, is something else than training in the same format. If training is done in algorithms in terms of $a^H$ then this becomes the engine, and the fraction slash and bar merely become input and output formats that are of no significance for the actual algebraic competence.

Hence it indeed seems that fractions as we know them can be abolished without the loss of mathematical insight and competence.

Addendum 2015: Page 63 mentions the powers 10^*n* so that pupils can get used to them at an early stage. Killian observed that they may become confused between 10 *n* and $10^n$. Discussing this with her we agreed that 10^*n* is best till pupils are used to the concept and might become relaxed with $10^n$. There is reason to do this before division. The confusion on powers namely could also hold for 10 *H* and $10^H$. Testing this innovation thus requires attention. The formulas above look less appealing with x^*H* everywhere. In that case we might perhaps as well write 1 / *x* but retain the idea of multiplication, i.e. that the meaning of 1 / *x* is that $x \times (1 / x) = 1$, and maintain this consistently. For example 1 / 4 = (1 / 2) × (1 / 2) because 2 × 2 × (1 / 4) = 1; and don't use 1 / (2 × 2) with its unnecessary concepts. Overall, it seems better to make sure that pupils do not grow confused between 10 *H* and $10^H$.

A comparable issue in the design of the curriculum is that it is better to first introduced the system of co-ordinates before introducing fractions. Because, once co-ordinates are available, then one can introduce Proportion Space (see COTP) too, and explain more about $x^H$.

# Vectors in elementary school 1

## Introduction

Vectors can already be introduced in elementary school since pupils already know about co-ordinates. The *Common Core* has co-ordinates in Grade 5 (ages 10-11). [48] Here are some exercises with co-ordinates. [49] When the Pythagorean Theorem is known then pupils can calculate distances, and thus also lengths of vectors.

Vectors are not difficult at all. Pierre van Hiele who was a celebrated researcher on the didactics of mathematics was a strong proponent that they are taught in elementary school. He did not succeed in convincing the world, however. There may have been some stumbling blocks to the discussion of vectors in elementary school:

- Presentation of a subject must respect the Van Hiele levels of insight. These are: concrete, ordering and analysis. [50] Pupils must first feel the water, then create some structure, and then may be open to see the *reason* for that structure. [51] If this didactic approach is not respected, then teaching may be impossible. That Van Hiele did not succeed in getting his proposal accepted has more to do with the training of elementary school teachers than with the difficulty of the subject.
- There may be a *missing link* in the education on geometry, but that was resolved in 2011 by proposing *named lines,* see page 104.
- When the Pythagorean Theorem is not known then one can do little with vectors.

The subsequent chapters consider these steps: (1) The basic geometry of co-ordinates and vectors, (2) The Pythagorean Theorem, (3) Calculating distances and lengths.

It is a bit silly that we repeat the introduction of co-ordinates, but it is useful to create a sandwich with the Pythagorean Theorem in the middle. This introduction into co-ordinates and vectors is largely taken from *Conquest of the Plane* (COTP) (2011). That book targets a higher level audience than elementary school, but it was felt at that place too that there is value in showing how simple the notion is.

The subsections below give a lesson plan for pupils of ages 10-13, thus Grade 5-8, or the last two years of elementary school or the first two years of middle school.

The exercise assumes:

- Hours 9 – 12 AM, 50 minutes per Van Hiele Level (1, 2, 3) with breaks of 10 minutes.
- The pupils have pencil, grid paper, ruler, set square with protractor, calculator with a √ button. They need not "know what √ means" but must have an operational understanding of "input-button-output" with examples "4-button-2" and "25-button-5".
- The teacher has a blackboard.

---

[48] http://www.cde.ca.gov/be/st/ss/documents/ccssmathstandardaug2013.pdf
[49] http://mathszone.co.uk/shape/coordinates/
[50] https://en.wikipedia.org/wiki/Van_Hiele_model
[51] Hans Freudenthal mistook the Van Hiele ideas and created his own "realistic mathematics education". This RME misinterpretes the process as "applied mathematics". Pupils are presented with a context from "real life" and have to discover how this might be modeled mathematically. The confusion is that the latter already assumes a mathematical competence that must first be developed. See page 101 below and Colignatus (2014).

# Two axes

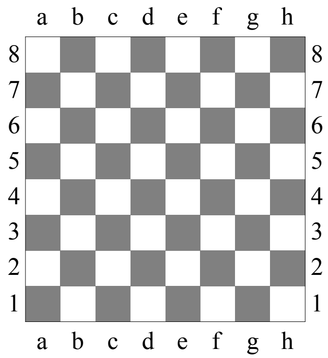
## *What co-ordinates are*

Co-ordinates give information to locate something. For a person it might be a telephone number or an address. When you meet people and want to contact them later then you can ask for their co-ordinates and they will give you their business card.

In the same way for the plane: we use a system of co-ordinates so that every point on the plane can be identified.

A chess board is a familiar system of co-ordinates, see Figure 3. The columns are labelled with the first eight letters of the alphabet (lower case makes for better reading) and the rows are just counted. White starts at the bottom and black at the top. The square at the bottom right hand at h8 will be white. The queen of white will start at d1 and the queen of black will be opposite at d8.

**Figure 3. Chess board**




## ***X* and *Y***

With a ruler on a piece of paper we draw a *horizontal* line and we call it the *x*-axis. Perpendicular to it we draw a *vertical* line and call it the *y*-axis. To identify what axis is what, we label the axes *x* and *y.*

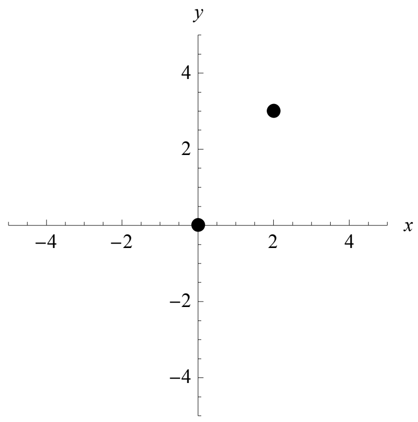
Where the lines cross will be called the *point of origin*. From there we can step right, left, up or down.

We can put numbers on the axes. We copy numbers from the ruler to the axes. The origin will get the number 0. On the horizontal axis we count positive numbers to the right and negative numbers to the left. On the vertical axis we count positive numbers up and negative numbers down. When we go along an axis from 1 to 2, or from 2 to 3, etcetera, then we will call this a full *step*.

We can use curly brackets around two numbers to identify a point on the plane. To start with, {0, 0} will denote the *point of origin*. Then, for example, {2, 3} will mean the point that we can find by moving from the origin, first stepping to number 2 on the horizontal axis and then making 3 steps up.

When you have copied this then you would get a graph like the one below. In this present graph we have put thick dots at {0, 0} and {2, 3}.

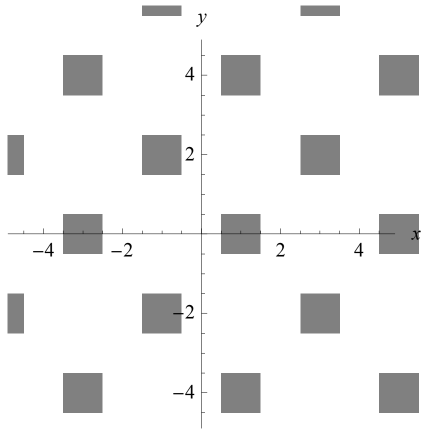
**Figure 4. Co-ordinates *x* and *y***




## *Practice makes perfect*

It can be good practice to step through this maze in Figure 5 using integer points only and without hitting a square. Start at {1, 2} and try to get to {-4, -3}. [52]

**Figure 5. Blocking steps of integer size**




Another exercise is to assign letters to points and translate a word into a list of numbers, so that we get a coded message. Try to code FINE using F = {0, 0}, I = {-3, 4}, N = {4, -2} and E = {-4, -3}.

This [53] is a tool for practice, and here [54] is an example of professional use of a grid system for Planet Earth. Well, the Earth is a globe, and henceforth we will only use the plane.

[52] A path is {1, 2} to {2, 2} to {2, -3} to {-4, -3}.
[53] http://www.taw.org.uk/lic/itp/coords.html
[54] http://geology.isu.edu/geostac/Field_Exercise/topomaps/grid_sys.htm

# Vectors

When we have a point {*a, b*} and a point {*x, y*} then the novel idea is that we add these two and get {*a + x, b + y*}. That is basically it. It is addition of more things *at the same time*.

For example, count the numbers of pens and pencils that kids in class have, but separately.

### ***Arrows have a direction***

Consider a soda can on a deck of a ship. In 10 seconds it rolls 7 meters from port to starboard. In those 10 seconds the ship itself has sailed 67 meters straight forward. People on the ship may see only the movement of the can on the ship. A land-based observer sees a combined movement. The object of discussion is how we could best handle this kind of case.

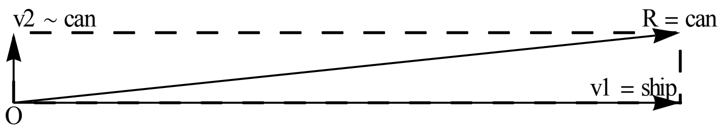
Let us consider two points *P* = {*a, b*} and *Q* = {*x, y*}. We can draw an arrow that starts from *P* and the arrow head ending in *Q.* We shall call that arrow a *vector* and write *v* = {*P, Q*}.

The ship's portside moves along the horizontal axis from {0, 0} to {67, 0}, and this will be vector v1. If the ship would be at rest then the soda can moves along the vertical axis across the deck from {0, 0} to {0, 7}, and this will be vector v2. The resultant movement is *R*. After 10 seconds the can is at position {67, 7}.

Thus the vector from {0, 0} to {67, 7} is *R* = {{0, 0}, {67, 7}}.

The ship moves a distance of v1, the soda can a distance of v2 on the ship, and the soda can has a resulting movement *R* seen by a land-based observer. In those 10 seconds, the soda can moves over a greater distance, and thus it must move faster than the ship.

**Figure 6. A can rolls over the deck of a sailing ship**




While the earlier discussion used points, we now have arrows, as combinations of points. The news is that we now have a model for motion. Co-ordinates are static, vectors are dynamic. What are the properties of such arrows ?

Before we continue this discussion, we must look at the Pythagorean Theorem.

# The Pythagorean Theorem in elementary school

## Introduction

Killian (2006)(2012) gives a great way to present the Pythagorean Theorem in elementary school. Some of the innovations are:

(1) to use rectangles rather than right triangles: since pupils are more familiar with rectangles, and the proof uses rectangles anyway

(2) to link up with the formulas for circumference and surface that pupils are familiar with

(3) to avoid exponentiation ("squares") and write out the multiplications: which is what pupils are familiar with.

I asked whether she planned to give an English translation in the foreseeable future. She didn't plan to, and gave me permission to use the ideas here. [55]

You can compare the result below with another more conventional but less accessible treatment for the *Common Core,* designed by the universities of Nottingham & Berkeley, (a) a "discovery" for grade 8 ($2^{nd}$ class of middle school), that is needlessly complex, [56] and (b) a proof for highschool, that is not as straightforward. [57]

Rather than translating Killian's articles my preference is to link up to the ideas by Pierre van Hiele and Dina van Hiele-Geldof. These ideas concern (i) the levels of insight (or abstraction) [58] and (ii) that pupils in elementary school can already deal with co-ordinates (former chapter) and vectors (next chapter).

The Van Hiele levels are: *concrete, ordering and analysis*. Pupils must first feel the water, then create some structure, and then may be open to see the *reason* for that structure. [59]

The subsections below give a lesson plan for pupils of ages 10-13, thus Grade 5-8, the last two years of elementary school or the first two years of middle school.

The exercise assumes:

- Hours 9 – 12 AM, 50 minutes per Van Hiele Level (1, 2, 3) with breaks of 10 minutes.
- The pupils have pencil, grid paper, ruler, set square with protractor, calculator with a √ button. They need not "know what √ means" but must have an operational understanding of "input-button-output" with examples "4-button-2" and "25-button-5". Potentially the class has practiced the table of squares some lessons ago.
- The teacher has a blackboard, with a section where a large table can be constructed and a section for a scratchpad.

---

[55] An interview with Killian by me is Colignatus (2012c).

[56] http://map.mathshell.org/lessons.php?unit=8315&collection=8

[57] http://map.mathshell.org/lessons.php?unit=9325&collection=8

[58] https://en.wikipedia.org/wiki/Van_Hiele_model

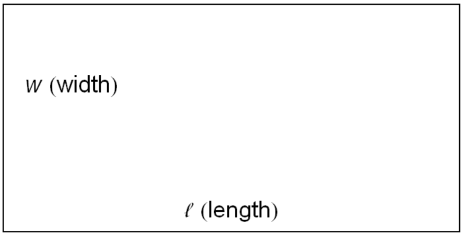
[59] (a) See footnote 51 on page 85. (b) Van Hiele levels are normally seen as applying to the whole age range (4-18), concerning geometry in general. For this particular topic and stylized case I found it useful to apply the notion of levels to a single class event. Normally, with three Van Hiele levels (moments), there would be two learning phases (periods) inbetween. For this stylized case, I found it useful to associate the level with the period, so that pupils are said to work at a particular level.

## Level 1. Concrete. Rekindling what already is known

The lesson opens by telling a bit about Pythagoras (c. 570-495 BC). [60] It may help to recall that a generation might be 25 years, so he lived 100 generations ago. The introduction closes by a statement: Pythagoras did not discover the theorem itself, but the *proof* of the theorem is ascribed to him.

Figure 7 sets the stage.

**Figure 7. A rectangle with length and width**




Recall the following formulas:

$c$ (circumference) $= \ell + w + \ell + w = 2 \times \ell + 2 \times w = 2 \times (\ell + w)$

$s$ (surface) $= \ell \times w$

Example values are in Table 6. Length is measured horizontally and width vertically, for such oriented rectangles. (Otherwise the longest side would be the length.)

Give only values for $\ell$ and *w,* and let pupils draw the rectangles and calculate the values for circumference and surface. Pupils who have some time left over, before others are finished, can create an additional own rectangle. Check that *c* and *s* have the same outcomes for all.

**Table 6. Example values for length and width**

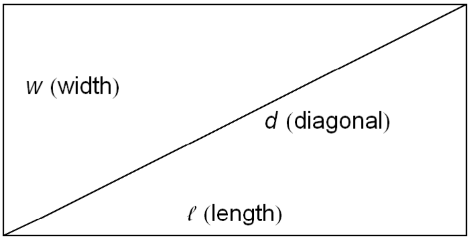
| $\ell$ | $w$ | $c$ | $s$ |
|---|---|---|---|
| 3 | 4 | 14 | 12 |
| 5 | 12 | 34 | 60 |

[60] https://en.wikipedia.org/wiki/Pythagoras

# Level 1. Concrete. The news

The news is the Pythagorean Theorem. This concerns the diagonal of a rectangle. Figure 8 shows a lower left to upper right) diagonal. Questions: is this the only one ? Are they equal in length ? Why ? (Yes, because of symmetry.)

**Figure 8. A rectangle with diagonal**




The news consists of the Pythagorean Theorem: the formula for the length of the diagonal:

Diagonal *d* has the formula: $d \times d = \ell \times \ell + w \times w$

The first example of Table 6 can be calculated jointly with the class, the second example can be done by the pupils themselves. With the ruler they check the values on the rectangles that they have drawn. This gives Table 7.

**Table 7. Example values for the diagonal**

| $\ell$ | *w* | *c* | *s* | $\ell \times \ell$ | $w \times w$ | $d \times d$ | *d* | *ruler* |
|---|---|---|---|---|---|---|---|---|
| 3 | 4 | 14 | 3 | 9 | 16 | 25 | 5 | ... |
| 5 | 12 | 34 | 5 | 25 | 144 | 169 | 13 | ... |

Obviously, many pupils who measure the diagonal of their drawn rectangles by using the ruler, may observe differences. This requires the discussion of measurement error.

Collect some measurements and calculate an average, and show that the average is closer to the calculated value.

Overall, a key lesson can be drawn now: There is error in ℓ, there is error in *w,* and there is error in *d.* The Pythagorean Theorem is valuable since it allows to reduce overall error.

PM 1. One observation is that elementary schools have lost the focus on drawing neatly. This exercise shows that there is good value in restoring this.

PM 2. Pupils might accept calculated values as outcomes of their measurements. In that case it might be a learning goal for another lesson to better read off results from a ruler.

# Level 2. Sorting. Using approximations

The blackboard can contain Table 7, and new lines can be included. Here on paper it suffices to focus on the new two examples of rectangles, of which the diagonals are no integers. This gives Table 8.

The example $d = \sqrt{5}$ is done jointly in class, and the example $\sqrt{41}$ is done individually.

The pupils are given the length and width, and are asked to draw the rectangles, measure the diagonals with the ruler, and calculate the values from the Pythagorean formula.

It may take too much time to calculate an average for the measurements, but it is always feasible to ask a result from a single pupil.

**Table 8. Diagonals with square root values**

| $\ell$ | $w$ | $\ell \times \ell$ | $w \times w$ | $d \times d$ | $d$ | *appr. d* | *ruler* |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 1 | 4 | 5 | $\sqrt{5}$ | 2.23607... | ... |
| 4 | 5 | 16 | 25 | 41 | $\sqrt{41}$ | 6.40312... | ... |

At this moment it is not a learning objective to deal with the difference between perfect numbers [61] and decimal approximation. It distinction can be just mentioned:

- A lesson is: $\sqrt{5}$ and $\sqrt{41}$ are *perfect numbers* like the integers or fractions like 0.25. The numbers are perfect in the sense that they perfectly tell what the value is without approximation or the need to include an ellipsis (lingering dots). [62]
- For many perfect numbers like $\sqrt{5}$ and $\sqrt{41}$ the decimal expansion creates an infinite number of digits. Cutting of this string – chopping or truncation – always causes an approximation. If you aspire at perfection then you simply write $\sqrt{5}$ and $\sqrt{41}$.

[61] A common phrase is *exact numbers* but *Elegance with Substance* (2009, 2015) explains that this phrase can be confusing. Number theory for https://en.wikipedia.org/wiki/Perfect_number must recode to "ancient Greek perfect number".
[62] See the former footnote again.

# Level 2. Sorting. Reversion

Another quesion is: when *d* and ℓ have been given, find *w.*

Pupils are given the option: either first draw this, or first calculate an outcome and then draw it. They will work on this individually.

They will discover that they have to guess the angle of the diagonal, so that it makes more sense to first calculate *w.* (Some may be so smart to use a pair of compasses.)

Calculating the approximate value of the diagonal is not really necessary. It is mentioned here only to size up the number.

**Table 9. Reverse calculation**

| *ℓ* | *w* | *ℓ × ℓ* | *w × w* | *d × d* | *d* | *appr. d* | *ruler* |
|---|---|---|---|---|---|---|---|
| 1 | | | | | 6 | | |
| 4 | | | | | √20 | 4.47214... | |

Filling in the blanks gives Table 10.

**Table 10. Reverse calculation (full table)**

| *ℓ* | *w* | *ℓ × ℓ* | *w × w* | *d × d* | *d* |
|---|---|---|---|---|---|
| 1 | √35 ≈ 5.91608... | 1 | 35 | 36 | 6 |
| 4 | 2 | 16 | 4 | 20 | √20 |

# Level 2. Sorting. Overview

Lessons about the Pythagorean Theorem are:

(1) It is a welcome addition to the formulas for circumference and surface.

(2) It allows to find one value when two are known.

(3) It helps to check on measurement errors on ℓ, *w* and *d.*

(4) It causes the distinction between perfect √-numbers and their approximations.

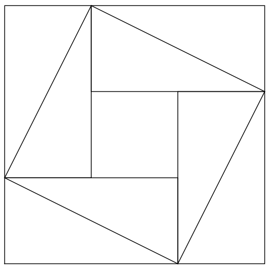
(5) The formula *a* × *a* occurs so often that a shorthand notation is $a^2$, the square of *a.* There is no need to emphasize exponentiation since this may only distract.

(6) Cutting a rectangle along a diagonal gives a *right triangle.* The theorem holds for such right triangles: for you can always extend it into a rectangle again.

The next step is to announce that we will now prove that the Pythagorean Theorem holds for any rectangle: $d \times d = \ell \times \ell + w \times w$. It is called a *theorem* because there is a proof.

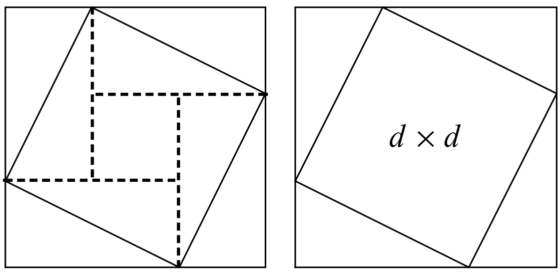
The first step in the proof is to draw four rectangles to create a square as in Figure 9. It is observed in discussion in class that the big square has sides $\ell + w$. Pupils are invited to copy this, each using his or her own rectangle, so that we can later check that the theorem holds for any rectangle that has been drawn today.

**Figure 9. Four rectangles with their diagonals**



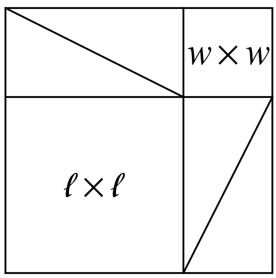
The second step is to erase the lines in the middle, and to recognise the tilted square in the center, see Figure 10. (We skip the proof that it is a square indeed – this can be done in later years.) We indicate its surface.

**Figure 10. Erase the lines in the middle, and recognise the center square**




The final step is to shift the triangles so that they form two rectangles again, see Figure 11. The pupils will have to draw the new big square again, and are invited to colour or shade the triangles to identify them. (Using labels *A, B, C, D* generates too much text.)

**Figure 11. Shifting the triangles**



The pupils are now asked:

> *When you look at the last two diagrams, can you give a good reason why d × d must be equal to ℓ × ℓ + w × w ? When you have given a good reason then you have proven the Pythagorean Theorem.*

The pupils will get time to think this over themselves individually. Who has found a good reason may raise a hand, and the teacher can come over to check.

When there are a few verified proofs, or after seven to ten minutes, depending upon progress and prodding, one can start a discussion in class. The objective is to create a list of reasons and to check how convincing they are. One would start with pupils who have not found a proof, and ask for what possible reasons they came up with, and why these indeed are not useful. Eventually the pupils who found the proof are invited up front and asked to explain it to the others.

It is not guaranteed that this order can be kept, since some pupils who have found the proof may be too enthousiastic to be silent about it.

An acceptable proof is:

> In the last two diagrams the areas of the squares with sides ℓ + *w* are the same. We take out the areas of two rectangles. In the first square we are left with *d* × *d*. In the final square we are left with ℓ × ℓ + *w* × *w*. We have taken out the same areas and thus the remainders must be the same too.

## Level 3. Analysis. Surplus

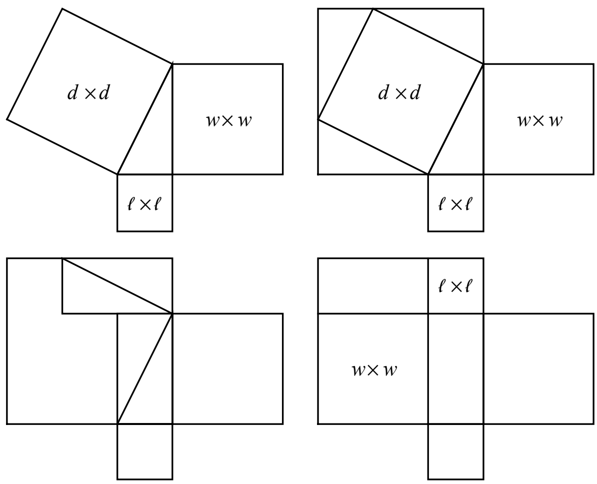
If there is time, or in a later session, one may return to the issue and let the finding sink in deeper by some exercises.

(1) Prove for squares that *d* × *d* = 2 × ℓ × ℓ.

(2) Algebra supported by the diagrams is: (ℓ + *w*) × (ℓ + *w*) = ℓ × ℓ + *w* × *w* + 2 × ℓ × *w*.

(3) Prove that the result holds for right triangles, starting from the drawing on the top left of Figure 12 (and let them find the other diagrams).

**Figure 12. Pythagoras for right triangles**



# Vectors in elementary school 2

We can take up the story at the point where we left it. Pupils can now calculate the distance that the soda can travelled in 10 seconds along the vector from {0, 0} to {67, 7}: 67.36 meters.

For us, this causes a moment of reflection. For design of the curriculum at elementary school and the creation of lesson plans, there are two major steps:

(1) Above outline of a lesson plan for the Pythagorean Theorem shows that this theorem can be presented and that many pupils will find the proof themselves.

(2) The introduction to vectors shows that the concept is simple and that there are useful basic applications.

Thus, Pierre van Hiele was right that vectors can be presented in elementary school.

It is not quite an issue how to proceed and what lesson plans can be developed. The real issue is that decision makers, both on the *curriculum* and the *education of elementary school teachers,* have to decide that these topics better be included.

# Circles and measurement of angles

The Californian implementation of the *Common Core* has for Grade 4 (ages 9-10):

> "Geometric measurement: understand concepts of angle and measure angles.
>
> 5. Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement:
>    a. An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through 1/360 of a circle is called a "one-degree angle," and can be used to measure angles.
>    b. An angle that turns through *n* one-degree angles is said to have an angle measure of *n* degrees.
> 6. Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure.
> 7. Recognize angle measure as additive. When an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real-world and mathematical problems, e.g., by using an equation with a symbol for the unknown angle measure." [63]

A better measure for angles is the plane itself, with unit 1.

A right angle would be $4^H$ = ¼ = 25% of the plane. While notation $4^H$ shifts understanding of fractions from division to multiplication, it may still be easier for pupils to work with integers than (such) fractions, so that 25% of the plane may be an easier measure for a right angle. Pupils might even appreciate the 250 promille measure.

The original proposal for this is in *Trig rerigged* (Colignatus (2008)). The issue translates directly to elementary school. When pupils in Grade 4 already must handle the protractor to measure angles on a scale of 360 degrees – while this is an illogical number w.r.t. the unity of the plane that they are taking sections of – then the clarity provided by *Trig rerigged* for highschool will surely be relevant for primary education too. I am at risk repeating the issue too much. *Trig rerigged* has been replaced by *Elegance with Substance* (2009, 2015) with principles, and *Conquest of the Plane* (2011) with details.

Pupils in primary education should also know that the angles of a triangle add up to half a plane. This discussion [64] is not targetted at their level but perhaps a version is feasible.

Observe the calculatory overload in the common programme. To understand an angle of 60 degrees for example, a pupil must calculate 60 / 360 to find $6^H$ = 1 / 6 of a circle. Thus calculation precedes understanding. The fractional form 1 / 6 invites one to continue with the calculator as well, and perhaps needlessly. To imagine what this might be, it may be transformed to 0.166... in decimal form, or 16.7% in common approximation. Instead, when the plane itself is used as the unit, then the angle $6^H$ stands by itself. Given the identity that 6 $6^H$ = 1 it would be easier to see that $6^H$ can indeed stand by itself as "(one) per six". A transformation into decimals might not be necessary, since 16.7% is not necessarily informative. If such transformation is desired, to compare with 25%, then such a calculation cannot be avoided. Still: it is not required to do a calculation 60 / 360 to understand that 60 degrees is $6^H$ plane, and this would further understanding.

[63] p32 of http://www.cde.ca.gov/be/st/ss/documents/ccssmathstandardaug2013.pdf
[64] https://boycottholland.wordpress.com/2014/06/29/euclids-fifth-postulate/

Please observe the circus: Given the Sumerian 360, there is a convention to consider special angles like 30 and 60 degrees. These have the supposedly "nice" property that sin[30] = cos[60] = ½. There is nothing particularly "nice" about this however. Don't blame the Sumerians. Blame generations of mathematicians who have been telling each other and us that this is "special". [65] But there is nothing nice or special about it. In our case we might say that sin[$12^H$ plane] = cos[$6^H$ plane] = $2^H$, and then it is immediately clear that there is nothing special here indeed. If pupils are trained on the decimal system then it may make more sense to focus on 5%, 10% and 25% of the circle. Note that sin[5% plane] = sin[18 degree] = (-1 + √5) $4^H$. Now, isn't that *special* ? [66]

[65] http://www.themathpage.com/atrig/30-60-90-triangle.htm
[66] https://en.wikibooks.org/wiki/Trigonometry/The_sine_of_18_degrees

# A key insight in the didactics of mathematics

Adapted from §15.2 of *Conquest of the Plane* (2011)
Also included in *Elegance with Substance* (2015)

## Introduction

It was an option to start the composition of this book with an introduction to didactics. In that case the reader could see how the subsequent parts fit the didactics. This would have been a top-down approach.

However, it seems more likely that the reader would not have understood this introduction to didactics. It is better to work bottom-up. Reading the book, the reader meets various arguments that argue for a particular approach. The arguments should make sense at the particular points when they are presented. Only in hindsight it appears that there is a method underlying it all.

The method is: that pupils first must become familiar with information at the bottom, before they can make the conceptual leap up to a higher level.

There is more to it: this didactic approach closely relates to thinking itself.

This chapter has been adapted from §15.2 of *Conquest of the Plane* (COTP). COTP is a primer for highschool and first year of higher education. Readers interested in elementary education will not quickly read COTP. But the discussion is relevant for all education.

## The didactic approach

Learning goals are generally knowledge, skills and attitude. The didactics are guided by the Van Hiele levels: *concrete, sorting, analysis,* or, with the latter split w.r.t. formality:

> Level 0: visualization and intuition
> Level 1: description, sorting, classification
> Level 2: informal deduction
> Level 3: formal deduction

Importantly, at each level the *same* words may be used but with *different* intentions, complicating mutual understanding.

Van Hiele (1973:177) gives the following example, and (1973:179) explains: "At each level we are explicitly busy with internally arranging the former level." (my translations):

(0) An isosceles triangle is recognized like an oak or mouse are recognized.
(1) It is recognized that this triangle has the property of at least two equal sides or angles.
(2) Relations between properties are recognized: at least two equal sides *if and only if* at least two equal sides.
(3) The logical reasons for these relations are considered: why *if*, and what does it mean to reverse an implication ?

Van Hiele (1973:179) on geometry:

> "At the base level we consider space like it appears to us; we can call this *spatial sense* (like *common sense*). At the first level we have the geometric spatial

sense. (E.g. measuring degrees of an angle / TC.) At the second level we have mathematical geometric sense; there we study what geometric sense involves. At the next level we study the mathematical logical sense; it then concerns the question why geometric manners of thought belong to mathematics."

The levels do not provide information about the boundaries of topics, and they are not strong when it comes to finalizing a topic and switching to a next one (that builds upon the earlier). In this book we mostly look at Level 1 and 2, and there are some patches that peek into possibilities for Level 3. The reader should be able to identify the spots.

In moving from one level to the next, Van Hiele (1973:149+) identifies phases:

(1) intake of information (examples)
(2) bounded orientation (direct instructions)
(3) explicitation (making explicit, verbalization in own words of what is known)
(4) free orientation (extending the relationships in the network)
(5) integration (summarizing and compacting what has been learned, often old fashioned learning).

Van Dormolen recognizes similar stages: Orientation, Sorting, Abstraction, Explicitation, Processing & Internalisation (OSAEP/I).

We reject Freudenthal's "realistic mathematics education" (RME) in its more extreme interpretation. This is best discussed in separate paragraphs.

## It hinges on what counts as experience

Van Hiele and Freudenthal overlap in the starting point in experience. The question remains what kind of experience we choose:

- Working in the plane itself is seen by Freudenthal as too abstract
- while Van Hiele in principle allows the notion that it might be experience too. Mental thought is an abstract process by nature and we can have experience in that.

Modern research on the brain clarifies many aspects of mental processes. Operational definitions of thinking and consciousness however cannot replace the definition of thinking as experienced by the conscious self. When we look for a definition of what thought is, in that experience of being conscious, then we quickly arrive at a Platonic version of ideas. In the mind's eye a triangle has a purity about it that is not caught in any drawing. Also mudd becomes perfect mudd. There is no difference between an image of a triangle and an image of mudd, or even an image of a sunset, in the sense that they are constructed out of the same mental elements that can only be pure. It are these mental ideas that education deals with, and experience in reality is only a tool to reach them. This does not mean that we have to be full Platonists in assigning an indestructible and immortal quality to these ideas. Thought and thinking, consciousness and awareness, are primitive notions for the thinking intellect itself, and up to this day and age of human history they do not generate any additional information for more conclusions than their very experience. [67]

The paradox – seeming contradiction – is that Freudenthal was an abstract thinking mathematician who developed an abstract notion of "realism", while Van Hiele was a practicing school teacher who was open to the relevance of abstraction itself.

There is a difference between:

[67] 2015: Lee Smolin (2015) also presents the naturalist view that shows that Platonism is not necessary, and can be eliminated with Occam's razor.

- designing a mathematical model, as in applied mathematics, by someone who already has a command of mathematical concepts, with the aim to match properties,
- learning to understand and developing a command of those mathematical concepts.

## What we can assume and build upon

Students and puplis have sufficient experience with the plane since making drawings in kindergarten. When they think about a triangle it is as abstract as it can get because such thought is abstract by nature. We can draw many triangles on paper but the notion of a triangle in the mind is an entirely different matter, and when the student or pupil thinks about a triangle then it is that notion that is in the mind and not the drawing on the piece of paper. What counts are the lingering notions in their abstract imagination that have to be activated. When we put labels to angles on paper and draw supporting lines then we use paper images to enter new concepts into the mind. It remains an essentially abstract activity, with pen and paper only tools for communication. It distracts and confuses when mental clarification is mixed with the application to reality. Application to reality is relevant but should be dosed wisely.

## Finding the proper dose and perspective

My book *A Logic of Exceptions* maintains that the force of logic derives from reality. If a truck approaches and if you do not jump aside then it will hit you. Mimicking this, *A Logic of Exceptions* starts with electrical switches to clarify the constants of propositional logic. In this case we do not need to explain these constants since we presume that students already know them. We only help making them explicit. The empirical examples are only intended to highlight the properties and to pave the road towards formalization. Here the electrical switches do not distract since the case is not presented as an exercise in building electrical circuits. The examples help to focus on the logical properties. Electrical switches are as good an example as language, and in a way a better example since the focus in logic is already so much on language that it helps to provide another angle.

For analytic geometry it may be argued that a bucket and a faucet that adds a liter per minute would be a similar good starting point. This is dubious however. If the objective is to distinguish linear processes from other processes then indeed examples in reality are the stepping stones, but that is another issue than linking up with geometry. The example distracts from the very abstract notion that we want to establish. “Realistic math” might require a student to spend a sizeable part of the lesson time on realistic examples trying to figure out what is the point. When supporters of “realistic math” argue that students of geometry do not understand a linear process without such examples as the bucket, then the reply is that those teachers have not spent sufficient effort in providing the abstract tools to perform the mental process.

It are different mental processes: imagining a bucket and faucet and imaging a graph of a linear function.

- The bucket and faucet have been learned in kindergarten.
- The graph and its geometric interpretation first have to be learned before they can be imagined and linked up to the bucket and faucet.

Once we have the graph then it is OK to say, and indeed we ought to say, that the bucket and faucet are an interpretation and application, and only then there can be that flash of understanding that shows that the link has been achieved. Once an aspect of the plane has been conquered then abstract understanding can be easier related to those other cases from reality, which means that those other examples are relevant for the Van Dormolen Processing & Internalisation stages. ***But first we must develop the geometry of that graph, using the mental images of geometry itself.***

## The challenge

There is a challenge though. Eudlid's *Elements* and his axiomatics have been the standard for more than two millenia. They are at Van Hiele's highest level. Perhaps 12-year olds can deal with those abstractions, as they actually are rather simple. But it becomes a bit different when we try to incorporate the advances in analytic geometry and calculus. Here are concepts that better be developed at a lower level and Van Hiele then wins from Euclid. Here Freudenthal steps in and resorts to the richness of reality, and at first that seems like a golden solution. Indeed, axiomatic geometry is at Level 3 and not at Level 0 ! However, as explained Freudenthal's approach is not convincing since it neglects that thought is abstract by nature. Rather than going sideways into reality we should focus more on the processes of thought and thinking itself.

## A missing link

We should provide for an abundance of words and concepts in the abstract plane, so that the student has enough to hold on to for visualization and intuition. A key observation is:

*A missing link in geometry appears to be that those anchors are rather absent.*

When you visit a new city then you tend to like it when the streets already have names. Suppose that you would be forced to invent your own labels, like "that crooked street with the blue shop" and then hope that other people understand you. Current textbooks on geometry send out students to conquer the plane but present it as a verbal desert, without conceptual guidance other than the $x$ and $y$ co-ordinates. The Van Hiele Level 0 requires them to visualize and to activate their intuition, yet that also requires a richness of words and concepts – that currently are lacking. Euclidean geometry has a poverty of points and lines that can intersect, be parallel or overlap: and though it is a great exercise in logic it must be admitted that Freudenthal has a point that Euclid's approach is not so appealing to the average student over the last two millenia. Conventional analytic geometry is an improvement since drawings are supported with formulas, and vice versa, yet again, its richness is only developed over time, and at the Level 0 and 1 there still isn't much to visualize and intuite and verbalize.

In particular, it will be useful to extend the plane with a nomenclature of *"named lines*". Chapter 4 of *Conquest of the Plane* opens with them and then builds up – see there to check what this means. A quick reply will be that we already have names, such as $x = 1$, $x = 2$, .... for vertical lines for example. Those names derive from a formal development however. Instead we rather first create standard names that fit the experience with the plane. This will provide the fertile ground, where the coin can drop when experience is morphed into abstract understanding.

It may be argued that it is fairly simple to draw a line and determine the starting value on the vertical axis and its slope. Exercises and realistic examples then provide for learning. However, experience shows that students later have difficulty with the horizontal and vertical lines. Why a line works as it does tends to remain elusive for them. A conclusion is that it is better to start with named horizontals en verticals and then awaken the motivation that a general formula will be useful.

Thus the didactic suggestion here is that the notion of *"named lines"* can be the missing link that resolves the issues in the choice between dropping Euclid and moving towards analytic geometry and calculus (and not just Descartes but along the lines of Van Hiele). The notion of these named lines caused the very layout of Chapter 4 on lines and subsequently from there the layout of the whole *Conquest of the Plane.*

## Co-ordinates and vectors

Pierre Marie van Hiele argued most of his life (May 4 1909 - November 1 2010) in favour of the use of vectors already in elementary school. Though he has been greatly valued for his ideas on the didactics of mathematics, he never succeeded in overcoming the opposing views. Vectors even appear late in highschools. The missing link suggested here of *named lines* is hopefully helpful. Logically, if this is indeed the missing link that has been provided only now, then teachers seem to have been right in resisting Van Hiele's suggestion, since the picture is complete only now. Alternatively, the suggestion of *named lines* is not really a missing link and only one of the possible bridges, and we are underestimating the capabilities of pupils and students all over the board.

Clearly, the proof of the pudding is in the eating, and only empirical testing will show whether students indeed learn faster following the didactic approach presented here. If this book would be mistaken, and "realistic mathematics education" would still be needed to propel the more practically minded students, then, the lame argument becomes, it would suffice to include it in this book as well, and the advantage of this book would remain to be its logical order and novel concepts.

## The importance of motivation

A final point of note is that I do not have clear ideas about what would motivate a pupil in elementary school to be interested in arithmetic and geometry, or a 12 or 14 year old kid to be interested in analytic geometry and calculus. Van Hiele (1973) rightly remarks that students and pupils hardly can be motivated for what they learn since they do not know yet what they will learn. A common ground is that man is a curious ape and cherishes the flashes of insight. Pupils recognise the moments when they grow in competence. Mathematics is a language and it can be fun to learn a new language and a new world. Paul Goodman (1962, 1973) *Compulsory miseducation* remains sobering though. While my books on mathematics education concentrate on knowledge the didactic setting naturally is a complex whole, in which motivation plays a key role, and it is mandatory to keep that in focus too.

# Relating to the Common Core (USA, California)

The USA in 2010 installed the *Common Core.* Implementations can differ per State, and my frame of reference is California, given my attendance at Burbank Highschool in 1972-73. The Common Core CA (henceforth CCC) is here. [68]

This book relates to the Common Core at various points.

## Decimal positonal system

The information for Grade 1 is ambiguous about the decimal positional system. The discussion of CCC:14 does not mention it, but the Overview of CCC:15 explicitly states:

> "Understand place value. Use place value understanding and properties of operations to add and subtract."

My impression is that CCC:15 presents the ambition but that CCC:14 presents the reality that addition and subtraction are regarded as more important than the awareness of the structure of the number system. The suggestion of this book on page 15 above is that a better pronunciation of the numbers will allow to make progress.

Grade 2 is more ambitious on the decimal positional system:

> "Students extend their understanding of the base-ten system. This includes ideas of counting in fives, tens, and multiples of hundreds, tens, and ones, as well as number relationships involving these units, including comparing. Students understand multi-digit numbers (up to 1000) written in base-ten notation, recognizing that the digits in each place represent amounts of thousands, hundreds, tens, or ones (e.g., 853 is 8 hundreds + 5 tens + 3 ones)." (CCC:18)

My suggestion is that they learn instead that 853 is 8 × hundred + 5 × ten + 3 × one. The multiples "hundreds", "tens" and "ones" are confusing. This invents some baby-language as if this would help. A teacher better asks "how many groups of a hundred are there ?", by which the notion of *grouping* (multiplication) is emphasized.

If Grade 1 succeeds in understanding 99 as 9 × ten + 9 × one then Grade 2 will quickly see that it is mere repetition to include hundred or thousand.

## Co-ordinates and vectors

If pupils can count to 1000 in Grade 2 then they will also understand yardsticks and number lines, a city grid, a chess board, and thus also the system of co-ordinates. There is no need to wait till Grade 5 as happens now.

In Grade 3 (age 8-9):

> "By decomposing rectangles into rectangular arrays of squares, students connect area to multiplication, and justify using multiplication to determine the area of a rectangle." (CCC:23)

Perhaps Grade 3 but then certainly Grade 4 (age 9-10) would be able to understand and likely prove the Pythagorean Theorem, if presented in the manner by Killian, see page 89 above.

[68] http://www.cde.ca.gov/be/st/ss/documents/ccssmathstandardaug2013.pdf

# Geometry and angles

On the measurement of angles in Grade 4, see page 99.

One can do a bit more geometry once the Pythagorean Theorem is known, see page 89.

Remarkably, only Grade 6 learns how to calculate the area of a triangle (CCC:40):

> "They find areas of right triangles, other triangles, and special quadrilaterals by decomposing these shapes, rearranging or removing pieces, and relating the shapes to rectangles."

One contributing reason for this delay is that texts on geometry tend to be rather prim by presenting special formulas for special forms. The formula for the triangle uses *height* while this is a 3D and not planar term. However, there is also a general formula, see Table 11. (This is an idea of Killian too.) The didactic set-up then would be:

- develop the notion of an *average,* also an average length of a trapezoid: $(\ell + k) \times 2^H$
- develop the algebraic skills to work with the general formula
- discuss each particular formula but also the relation to the general formula
- suggest that pupils only need to remember the general formula.

**Table 11. The general formula using the average length (base *ℓ*, across *k*)**

| | *length* | *width* | *surface s* | $s = 2^H \times (\ell + k) \times w$ |
|---|---|---|---|---|
| square | $\ell$ | $w = \ell$ | $\ell \times \ell$ | $k = w = \ell$ |
| rectangle | $\ell$ | $w$ | $\ell \times w$ | $k = \ell$ |
| triangle | $\ell$ | $w$ | $\ell \times w \times 2^H$ | $k = 0$ |
| parallelogram | $\ell$ | $w$ | $\ell \times w$ | $k = \ell$ |
| trapezoid | $\ell$, $k$ | $w$ | $(\ell + k) \times w \times 2^H$ | also when $0 \neq k \neq \ell$ |

# Fractions

Professor Wu from Berkeley gives much attention to the CCC. Consider his text on fractions, that is advised reading for anyone looking at arithmetic in primary education. [69] His objective is to accurately present the traditional approach. This differs from my objective to find possible sources of confusion in that traditional approach.

Wu:9 quickly moves to the number line, but this causes a rather complex discussion that slows down again, taking space till page 15. I would rather first introduce 2D co-ordinates on whole numbers, and then introduce Proportion Space (see COTP) so that the number line and equivalent fractions are immediately clear.

Wu:18 repeats CCC goal 4.NF 4: "Understand a fraction *a*/*b* as a multiple of 1/*b*." When we write this as $a\ b^H$ then it is clear that you can only understand the expression as a multiple of $b^H$. The 4.NF learning goal is provoked by a particular notation. What one should learn is that the notation is awkward. One might consider that $a \div b$ is an operation and *a*/*b* is a number, but this is awkward too since these are all numbers. The only thing of interest for $a\ b^H$ is what its decimal expansion is, for a location on the number line.

[69] https://math.berkeley.edu/~wu/CCSS-Fractions_1.pdf

Wu:26 states without hesitation on mixed numbers:

> There is a **notational convention** associated with mixed numbers: the + sign is omitted. Thus we write $5\frac{3}{4}$ in place of $5 + \frac{3}{4}$ and $7\frac{1}{5}$ in place of $7 + \frac{1}{5}$.

On p83 above we have mentioned that it is confusing to omit the plus-sign. This namely conflicts with the notation of multiplication. (Consider handwriting, not prints.)

These are but a few comments on the traditional programme on fractions. It is useful that Wu discussed the CCC programme so extensively, and also gave his own critiques (e.g. that a pizza is not a good learning example). But we should hope for change.

## Higher mathematics standards: the notion of proof

CCC:58+ discuss the higher mathematics standards. Let me refer to *Elegance with Substance* (2009, 2015), *Conquest of the Plane* (2011) and *Foundations of Mathematics. A Neoclasssical Approach to Infinity* (2015) for discussion of this area. There will be other consequences for primary education, but for now it suffices what has been said already.

A major point of course is that when the notion of *proof* is established in primary education – see above p81 on *H* and p94 on the Pythagorean Theorem – then this will be greatly advantageous for mathematics education and mathematical competence in general.

Logic is only mentioned for Grade 8 in CCC:52, and set theory has a vague existence between middle and high school. It is more logical to introduce these in primary education, avoiding the New Math disaster of 1960-70 of course.

## International standards, TIMSS and PISA

The Common Core programme has been based upon international standards too.

> "In mathematics, the standards draw on conclusions from the Trends in International Mathematics and Science Study (TIMSS) and other studies of high-performing countries that found the traditional U.S. mathematics curriculum needed to become substantially more coherent and focused in order to improve student achievement, addressing the problem of a curriculum that is "a mile wide and an inch deep."" [70]

I have only superficial understanding of TIMSS [71] and PISA [72] and draw a blank here. [73]

However, a critical comment is possible from the angle that we have mentioned the difference between the Van Hiele and Freudenthal approaches. The Freudenthal approach was institutionalised in the Freudenthal Institute in Utrecht, and its director Jan de Lange has been chair of the math working group of PISA. One of the issues is whether "arithmetic sums" are really arithmetic, and whether they are not "exercises in reading well". The Dutch position on the PISA list might reflect that Dutch mathematics education is an early adapter to the Freudenthal mold, and it might not reflect mathematics competence per se. Thus a general warning to be critical is no luxury.

The most relevant remark that I can make is to mention the website by Ben Wilbrink, a psychologist specialised in testing. [74] Apparently he values this paper [75] and he warns

---

[70] http://www.corestandards.org/about-the-standards/myths-vs-facts/
[71] http://timssandpirls.bc.edu/timss2011/downloads/T11_IR_Mathematics_FullBook.pdf
[72] https://www.tes.co.uk/article.aspx?storycode=6360708
[73] http://www.mathunion.org/ICM/ICM2006.3/Main/icm2006.3.1663.1672.ocr.pdf

about the involvement by big corporations like Microsoft, Cisco and Intel in a project like "Assessment and Teaching of the 21st Century Skills (ATC21S)". [76]

Let me refer to *Elegance with Substance* (2009, 2015) for the political economy of the mathematics industry. The proposal there is that nations create national institutes on mathematics education, under democratic control, with involvement of participants. For the US, such institutes at the State level might work well too.

## The Dijsselbloem confusion

The Common Core approach tends to folow the distinction that was also adopted by a Parliamentary committee in Holland, led by Jeroen Dijsselbloem, now President of the Eurozone ministers of Finance. This is the distinction between *what* and *how.* The idea is that policy makers (Parliament) decide what subjects shall be taught in education, for example arithmetic and geometry, and that teachers decide how it shall be taught. This seems fine for subjects like geography and biology (that I am not qualified for). However, for mathematics we run into the problem that mathematicians sell as "mathematics" which really is not very much of mathematics, when we look at it from the angle of didactics. For example, 2½ is rather crummy when it should be at least 2 + ½ and at best $2 + 2^{H}$. The list of errors is huge, including the major mishap that Freudenthal breached scientific integrity w.r.t. Van Hiele. Thus the *what* and *how* distinction doesn't work for mathematics, and nations need parliamentary investigations into mathematics education to sort out the mess and make funds available to re-engineer not only the dust of ages but also a culture that works against didactics. [77]

[74] http://benwilbrink.nl/projecten/pisa.htm
[75] http://www.utwente.nl/bms/omd/medewerkers/artikelen/vdLinden/IJER%201998%2C%20569-577-1.pdf
[76] http://blogs.msdn.com/b/microsoftuseducation/archive/2012/01/10/the-importance-on-assessing-students-21st-century-skills-not-just-math-science-and-reading.aspx
[77] https://boycottholland.wordpress.com/2013/03/27/jeroen-dijsselbloem-on-money-and-math/

# Conclusions

The Prologue stated that education is a mess, referring to *Elegance with Substance* (EWS) and *Conquest of the Plane* (COTP) as the evidence. This present look at primary education does not invalidate a similar impression. I am not qualified to judge in this particular field but offer the following conclusions as prospective.

The mathematical structure of arithmetic and geometry is fine, and computing devices and computer algebra programmes are wonders of technical advancement, but something goes seriously wrong between mathematical abstraction on one hand and educational empirics on the other hand.

- Number sense and understanding are hindered and obstructed by taking the English pronunciation of numbers as the norm, while English is a historically grown and clearly confusing dialect of mathematics. Counting with fingers blocks at 10, while it is easy to construct an alternative system to use signs with place values too. In fractions there is abuse of rank order names and an awkward switch in plus / times, compare 2½ with 2 + ½, while fractions might also be abolished with $2 + 2^H$. Subtraction doesn't use the decimal positional system to its full potential yet, and, by not doing so, creates confusion about it, while enlarging the fear for negative numbers.
- Algebraic sense and competence rely upon arithmetic, and thus are hindered and obstructed when arithmetic isn't developed well. Compare current 2½ × 3¼ with proper $(2 + 2^H) \times (3 + 4^H)$.
- Spatial sense and understanding are hindered and obstructed by the absence of the missing link of *named lines,* by not discussing vectors and the Pythagorean Theorem, and by adhering to the Sumerian 360 degrees instead of taking the plane as the unit itself.
- Logical sense and competence in reasoning are hindered and obstructed by above confusions and cumbersomeness, by the witholding of logic and set theory till middle school or later, and by not explicitly developing the notion of proof.

This evidence does not contradict the earlier conclusions of *Elegance with Substance*. To repeat those:

What is seen as mathematics appears to be illogical and/or undidactic. Hence it has to be redesigned. It is no use to improve on the didactics of bad material, it better is replaced. We also considered only a number of topics, a selection of ideas that this author found interesting to develop a bit. More can be found. We should allow for the possibility that teachers have more comments and suggestions themselves (though our critique is that either they don't have them or don't follow up on them). The situation is wanting.

This book looks at the result rather than at how this situation could have come about. Still, if the result is inadequate, the conclusion is warranted that some cause is wrong.

One of the most important human characteristics is the preference for what is known and familiar – and mathematicians are only human. They adapt to new developments and are are critical and self-critical, not only with respect to what is discussed but also on how things will change. Nevertheless, key issues got stuck, and the industry as a whole is incapable of freeing itself from grown patterns. New entrants in the industry are conditioned to the blind spots, and pupils and students suffer from them.

The situation is not such that there are no mathematicians to improve on content and that we lack researchers in didactics to improve on that angle. This book will hopefully be read by some in both groups and contribute to improvements. But it would be wrong for governments to think that it would suffice to leave the matter to the industry, and possibly give more subsidies for more of the same. More funds may well mean more outgrowth of awkwardness, cumbersomeness, irrationality. A call for more teaching hours may well mean more hours to mentally torture the students even more. Given this whole industry and the inadequate result the conclusion is rather that the whole industry is to be tackled.

Indeed, it sounds so well. Mathematicians will hold that only they are capable of deciding what is ‘mathematics’. Researchers in the education of ‘mathematics’ will hold that they do the research and nobody else. Will they regard this book as ‘research in the education in mathematics’ ? *Quis custodet custodes ?* It will be a mis-judgement to provide the industry with more funds without serious reorganization.

In sum, we have considered the work of men and found them to be men. It is a joy to see all these issues that can be improved upon. Let us hope that mathematicians proceed in this direction indeed. Let economists and the other professions support them.

2015: In Holland the State Secretary on Education Sander Dekker has observed that arithmetic skills are below requirements. He avoids a diagnosis on the Freudenthal "Realistic Mathematics Education" (RME) and thus he doesn't require a reschooling of the 150,000 elementary school teachers. Instead he shifts the burden to the 4,000 teachers of mathematics in secondary education, by requiring an additional arithmetic test for highschool graduation. Apparently he is not aware that creation of arithmetic competence in elementary school is required for later algebraic competence in secondary education. I am sorry to report that there is a breach in the integrity of science in the mathematics education research, so that Mr. Dekker does not get scientifically warranted information. At KNAW there are some abstract thinking mathematicians who think that they know more about mathematics education than empirical scientists, and they don't care about the evidence to the contrary. [78]

Check out the weblog or the book website for developments. [79]

## Final conclusion

My final conclusion definitely applies to Holland. I tend not to judge about other countries. But the same cumbersome and illogical issues can also be seen internationally. There is a structure to it. It is part of the economics of regulation. Didactics require a mindset sensitive to empirical observation which is not what mathematicians are trained for. Tradition and culture condition mathematicians to see what they are conditioned to see. The industy cannot handle its responsibility. This must hold internationally, country by country. A parliamentary enquiry is advisable, country by country.

Parents are advised to write their representative – and not only those parents who pay for extra private lessons. The professional associations of mathematics and economics are advised to write their parliament in support of that enquiry.

Here ends what I repeated from *Elegance with Substance*.

[78] https://boycottholland.wordpress.com/2014/07/16/integrity-of-science-in-dutch-research-in-didactics-of-mathematics/

[79] http://thomascool.eu/Papers/Math/Index.html

# Appendix A. What is new in this book ?

It is generally useful to specify what is new in a book. This overview may repeat some points from my own work that I already presented elsewhere.

(1) Identification of mathematics as the proper language and English as a dialect, so that there is proper perspective and focus on didactics and learning goals (see p15).

(2) Greater awareness that the positional system is under-utilised for its support in counting and arithmetic. Proper use might allow multiplication in First Grade. (This fits Gladwell's comment that "the necessary equation is right there, embedded in the sentence"; but now looks systematically how the positional system can be employed to support education.) (See p16.)

(3) The latter also concerns the use of the positional system for subtraction (see p80).

(4) Pronounciation of numbers with ten, like 19 = *ten·nine* and 23 = *two·ten·three.* (From Gladwell (2008) and Cantonese, but with *ten* instead of *tens* and middle dot instead of hyphen, and for Dutch "tig" instead of "tien".) (See also Appendix B.)

(5) The article *"Marcus learns counting and arithmetic with ten"* (p19) that combines these ideas in a draft lesson plan: (a) pronunciation, (b) calling the dialect what it is, (c) sums that relate to the positional system, (d) tables of addition and multiplication, (e) powers.

(6) Abolition of fractions by using $x^H = 1 / x$, pronounced as "per *x*", with $H = -1$ the Harremoës operator, pronounced as "eta" (see p81). (Van Hiele (1973) already proposed abolition, en Harremoës (2000) has a symbol for –1 (and much more). New is the choice of pronunciation of "per *x*" instead of the abuse of rank order names (like *"a fifth"*), and of suitable *H* that somewhat looks like "-1" and that gives a half turn on a circle in the complex plane.)

(7) Suggestions for gestures or signs that satisfy the positional system, for base 10 (p67) and base 6 (p121). Design principles that fit elementary school.

(8) Identification of the difference in the approaches by Van Hiele (right) and Freudenthal (erroneous), with the distinction between handling abstraction (Van Hiele) and applied mathematical modeling (Freudenthal). Identification of the *missing link* in the standard approaches to geometry: the *named lines.* (This is a copy of §15.2 from *Conquest of the Plane*.) (See p101.)

(9) (a) Identification of Killian's (2006) (2012) treatment of the Pythagorean Theorem in elementary school as a *key supporting step* for Pierre van Hiele's suggestion that vectors can already be presented in elementary school. (b) Presentation of this argument, by sandwiching the topics: (i) presentation of co-ordinates and vectors, (ii) derivation of the theorem, (iii) using the theorem to calculate the lengths of vectors. (c) Demonstration that the presentation of the theorem perfectly fits the Van Hiele didactic approach, with the Van Hiele levels *concrete, sorting, analysis.* (See p89.)

(10) Observation that an earlier analysis on angles and trigonometry is also highly relevant for primary education (see p99).

(11) Explicitation of the relevance for the USA *Common Core* programme, for the implementation given in California (see p107).

## *Comment w.r.t. Barrow (1993) "Pi in the sky"*

I had read Barrow (1993) somewhere before 2009 and referred to him in EWS 2009. Apparently I forgot some details, and in 2012 around my son *M's* 6th birthday it was Gladwell (2008:228) who set me thinking about the number system, as explained above on page 15. Following the hint by Gladwell I wrote the text on *Marcus learns counting and arithmetic with ten,* i.e. fully writing out all pronunciations in the format *two·ten·one.* Now in 2015 I just completed a 2nd edition of EWS 2015 as well, and this got me trying to remember what Barrow (1993) had to say about these number issues. His book has the subtitle *Counting, thinking and being,* and I did recall that he discussed the history of number systems. Thus, I decided to reread his book.

To my surprise I find a core of my suggestions just stated by Barrow. On his page 35:

> "This method of counting is called the '2-system'. One should compare it to that which we use today which is founded upon the base 10, so that we have distinct words for numbers up to and including ten and then we compose ten-one (which we term 'eleven'), ten-two (twelve), ten-three (thirteen), ten-four (fourteen), up to ten-nine (nineteen) and ten-ten (twenty), before continuing with ten-ten-one which we cal 'twenty-one'."

Barrow indeed uses *ten* rather Galwell's *tens.* Distinctions with my suggestion are:

- to use middle dots rather than hyphens
- to use two·ten·one for 21 and ten·ten for 100 (but Barrow p68 may intend this too).

Barrow also suggests that *eleven* and *twelve* derive from *one left over* and *two left over,* once you have counted to ten. *Twenty* would not derive from two·ten but from *twin of tens.*

Barrow's page 58 mentions a system that uses two hands, structurally the same as my paper *Numbers in base six in First Grade ?,* except that I propose to use base six while Barrow describes base five. Observe that base six is better to learn and handle the positional shift. There is also a paradox: The Bombay system works left to right (in Western style) while my proposed system works right to left (as the numbers are supposed to come from India) ...

> "Elsewhere in India, amongst some traders in the Bombay region, there are still traces of an early base-5 method of counting which uses finger counting in a novel and powerful fashion, enabling much larger number to be dealt with without taxing the memory unduly. The left hand is used in the normal way counting off the fingers from 1 to five, starting with the thumb. But when five is reached this is recorded by raising the thumb of the right hand whereupon counting to the next five begins again with the left hand until ten is reached, then the next finger of the right hand is raised, and so on. This system enables the finger-counter to count to thirty very easily, so that even if he is interrupted or distracted he can determine at a glance where the count has reached."

It is useful that Barrow agrees that the method is powerful. Base 5 might link up easier with base 10 later on. A decision on this however is quickly resolved by the signs in base 10, see page 67 above.

# Appendix B. Number sense and sensical numbers

## Introduction 2015

The discussion below was the section on *Number Sense* in EWS 2009. It caused the creation of the draft booklet *A child wants nice and not mean numbers* (2012) – now replaced by this whole book. The first point is that English is a dialect of mathematics. There is a coherent way to pronounce numbers, to start with in elementary school. The second point is that the positional system is underutilised, and that its proper use would allow a great improvement in arithmetic. Algebra in highschool depends upon arithmetic skills learned in primary education.

Other news is: (i) For negative numbers and subtraction, see p 77. (ii) Later on, I realised that fractions abuse the rank order names: e.g. rank order *fifth* is abused for *a fifth.* There is now the proposal to use $1 / x = x^H$, and pronounce this as "per-*x*". (iii) For an overview of pronunciation, addition, subtraction, multiplication, division, see p 73. The following can be retained as a rough introduction into all of this. It has been edited to fit this book.

## Brain, language, sounds and pictures

There is already some remarkable EBE on arithmetic. Gladwell (2008:228):

> "(…) we store digits in a memory loop that runs for about two seconds."

English numbers are cumbersome to store. Gladwell quotes Stanislas Dehaene:

> "(…) the prize for efficacy goes to the Cantonese dialect of Chinese, whose brevity grants residents of Hong Kong a rocketing memory span of about 10 digits."

[PM. Apparently fractions in Chinese are clearer too. Instead of *two-fifths* it would use *two-out of-five*. First creating *fifths* indeed is an additional operation. Perhaps the West is too prim on the distinction between the ratio 2:5 and the number 2/5. Perhaps it does really not make a difference except in terms of pure theory – the verb of considering the ratio and the noun of the result (called "number" when primly formalized in number theory). But fractions are not the topic of present discussion.]

Gladwell on addition:

> "Ask an English-speaking seven-year-old to add *thirty·seven* plus *twenty·two* in her head, and she has to convert the words to numbers (37 + 22). Only then can she do the math: 2 plus 7 is 9 and 30 plus 20 is 50, which makes 59. Ask an Asian child to add *three·tens·seven* and *two·tens·two*, and then the necessary equation is right there, embedded in the sentence. No number translation is necessary: It's *five·tens·nine*." (Hyphens replaced by middle dots.)

I am not quite convinced by the latter. *Thirty·seven* can be quickly translated into *three·ten·seven,* and *twenty·two* in *two·ten·two*. (Use position *ten* rather than quantity *tens.*) The "thir" and "ty" are liguistic reductions of "three" and "ten". There is no need to create the digital image of the numbers. I can imagine two tracks: pupils who learn to mentally code *thirty* (sound, and mental code too) as *three·ten* (brain meaning) and pupils who follow the longer route via the digits. That said, the Western way is a bit more complicated.

The problem has a quick fix: Use the Cantonese system and sounds for numbers. It would be good EBE to determine whether this would be feasible for an English speaking environment (for starters, perhaps begin in Hong Kong).

## Writing left to right, speaking right to left

A deeper issue is that the West reads and writes text from the left to the right while Indian-Arabic or rather Indian numbers are from the right to the left. Thus *fourteen* is 14.

English already adapted a bit, with *twenty-one* and 21. Dutch still has *een-en-twintig* up to *negen-en-negentig*. From hundreds onwards Dutch follows the Indian too, for example *vijf-honderd-een-en-twintig* (521). French of course still has the special *quatre-vingt* for 80 and *quatre-vingt-treize* (80 + 13) for 93.

## Sounds and pictures

There is a bit more to it, though. In Gladwell's case the pupils apparently are given a sum via verbal communication. This differs from a written test question. There are two ways to consider a number. 37 can be seen as a series of digits only and pronounced as *three·seven* – like specifying a telephone number – or it can be weighed as *thirty·seven* or *three·ten·seven*. We have to distinguish math from the human mind.

(a) For the mathematical algorithm of addition mentioning only the digits suffices since the order already carries the weights. The mathematically neat way starts with the singles, as indeed Gladwell first mentions 2 plus 7 is 9.

(b) But a human mind tends to have different priorities. It is interested in size. The mind tends to use the weights and to focus on the most important digit. Witness *"nine thousand four hundred twenty-six"*. In a written question this tendency is easier to suppress. In a verbal question the tendency is stimulated. Depending upon the circumstances there can be more focus on the size. (The actual algorithm / heuristic that a pupil uses can actually be anything, like first adding up the place values at thousand, then at hundred, ten, one, and then resolve the overflow. The Asian child might indeed start with *three plus two is five*.)

Counting in traditional / verbal manner follows the second approach, and uses the infixes ten, hundred, thousand, ten thousand etcetera to indicate the place and the unit of account. The weight infixes are intended for communicating size and would be redundant for merely transmitting the number, though redundancy can help for checking.

Expressions with weights still can be ambiguous though. With 100 million = 100 *times* 10^6 as the format, it follows that 23 pronounced as *twenty-three* might be understood as 20 *times* 3 giving 60. Dutch has prim *drie-en-twintig* thus with the *plus.* A proper use of weights should fully specify the sum $a \times 1000 + b \times 100 + c \times 10 + d$ for number *abcd.*

## Eye, ear, mouth & hand co-ordination

There are two key properties of the Indian order with a Western text direction:

- The mental advantage is that the most important digit is mentioned first.
- The disadvantage is that addition and multiplication work in the opposite direction from reading. It goes against the (over-) flow. For example 17 + 36 = 53 has overflow 7 + 6 = 13 and this has to be processed from the right to the left.

The requirement on eye, ear, mouth & hand co-ordination again shows the importance of Kindergarten – see the work by economist Heckman, e.g. his Tinbergen Lecture, who confirms what Kindergarten teachers have been telling since ages.

## History of the decimal system and the zero

Note that the West often speaks about Arabic digits but according to Van der Waerden (1975:58) the Arabs speak about Indian digits so we better follow them:

> "Our digits derive from the Gobar digits which were used in Moorish Spain. The East-Arabic digits are still in use to-day in Turkey, Arabia and Egypt; they are called "Indian digits". It is clear that both were derived from the Brahmi-digits."

A short excursion on the history of the decimal system and zero is useful. Barrow (1993:85) mentions that the Babylonians of 300-200 BC already had a symbol to indicate a blank spot. Possibly Freudenthal 1946 was the first to recover the most likely story on what happened next. It can be observed that Ptolemy in 150 AD wrote whole numbers with Roman numerals but fractions sexagesimally following the Babylonians – and in this positional system he wrote "o" for "ουδεν" ("nothing") when a position was blank. Apparently the Indians became familiar with Greek astronomy from 200 AD onwards. The Indians already had a decimal positional system though of some complexity. They used rhymes and verses to remember long numerical tables, but blank places apparently broke the rhythm and it would have come as an idea that those places could be filled with sounds too. Van der Waerden (1975:57) summarizes:

> "Along with Greek astronomy, the Hindus became acquainted with the sexagesimal system and the zero. They amalgamated this positional system with their own; to their own Brahmin digits 1 – 9, they adjoined the Greek o and they adopted the Greek-Babylonian order. It is quite possible that things went in this way. This detracts in no way from the honor due to the Hindus; it is they who developed the most perfect notation for numbers, known to us."

Clearly, when the zero arrived in Europe again via the Moors in Spain, it helped that astronomers were already used to it. The impact however came from the package deal with the decimal notation in general, that appeared very useful in commerce.

Interestingly, with respect to our discussion of the order of the digits, the Indian system originally had the order from low to high but switched due to the influence of the Greek-Babylonian order. Van der Waerden (1975:55):

> "Bhaskara I, a pupil of Aryabhata, introduced an improved system, which is positional and has zero; it has the further advantage of leaving the poet greater freedom in the choice of syllables and thus enabling him better to meet metrical requirements. According to Datta and Singh, this Bhaskara lived around 520. Like Aryabhata, he begins with the units, followed by the tens, etc., (…) The first to reverse the order (as far as we know) was Jinabhadra Gani, who lived about 537, according to Datta and Singh."

Thus the writing order of Indian numerals may have little to do with the writing order of the Arabic language but rather with the writing order of old Sumer numerals.

[PM. Van der Waerden observes that Sumerian and Chinese results on the Pythagorean Theorem are too similar to be parallel inventions and hence concludes that there must be some common ancestor civilization point where the original invention had been made. We may wonder whether such a point would have to be a very developed civilization. Possibly the basic choice would be to construct houses in rectangular instead of circular form, which is much simpler than what Van der Waerden discusses on celestial events.]

## Scope for redesign

The reader might as well skip this subsection on the scope for redesign, since the conclusion will be that we will not quickly change the Indian digits and number order. But some diehards might press on, and it might be relevant for developing more didactics for kids who have problems in co-ordinating eye, ear, mouth and hand.

## Overflow

The overflow problem is a bit awkward. It would be interesting – when we are considering changing to Cantonese – to see whether it can be solved at the same time. Thus, can we write numbers in the opposite way ? Let us use the word ***Novel*** when we write <123> for the Indian number 321 (and try not to get confused). To distinguish the Novel from the Indian it will be most useful to write the digits in mirror image (perhaps as they are intended to be read if you change the reading order). Thus 19 becomes 91. It does not take much time to get used to and **Table 12** contains the first practice.

**Table 12. Novel versus Indian notation and addition**

| | |
|---|---|
| 4321 | 1234 |
| 765 | 567 |
| 98 | 89 |
| 0981 | 1890 |

Overflow in Novel is processed neatly in the reading direction. This is straightforward.

Thus, to repeat: the mathematical algorithms for addition and multiplication basically work on the digits and not on how the whole numbers are pronounced. When we work silently on paper, or only pronounce the digits in stated order (with text from left to right) without pronouncing the whole number, and compare Indian and Novel:

- Addition in Indian 17 + 36 = 53 works with the digits as “*one·seven* plus *three·six* gives *five·three*”. The order of the digits conflicts with handling overflow.
- Addition in Novel works with digits as "*neves·eno* plus *xis·eerth* is *eerth·evif*". Or *"<seven·one>* plus *<six·three>* is *<three·five>"*. The order of the digits supports handling overflow.

## The problem is pronunciation of the whole number

Let us now pronounce the whole number. Something strange happens: the need to size up the number appears to interfere *always* with the reading and writing order.

Consider Indian 5,310,000. The eye traverses first from the left to the right to determine how many digits there are. The pupil deduces that 7 digits are millions, then either calls out the number from memory or the eye goes back, from the right to the left to the beginning, and then the pupil reads it off. Possibly there are parallel processes, as the eye picks out words rather than letters. What remains though is that to say *“five million, three hundred ten thousand”* is not exactly following the reading order since there is a jump somewhere. The *Jump* is unavoidable since the number of digits has to be counted. As the mind focusses on the most important digit, the speaking order will reflect the order in the mind – which is independent of the reading order.

Thus, where we had the distinction between the *mathematical algorithm* and the *human mind* we now see a parallel distinction between *reading order* and *order of pronunciation*. The tricky issue appears to be pronunciation of the whole number. Digit-wise pronunciation, provided that convention is in place, either Indian or Novel, is feasible. Pronunciation only causes problems when a number is communicated (verbally) with weights. Even a written question may carry this problem if the number is not merely processed in an algorithm but subvocalised. Subvocalisation tends to happen as part of the process of understanding, when the mind wonders what the number means. (The algorithm implicitly assigns weights when the working order implies how overflow is handled. The problem remains in pronunciation: this repeatedly burdens memory with (redundant) information about the weight of the digits.)

The true question is how we would pronounce these Novel numbers as a whole and how pronunciation with size interferes with the neat algorithms. If we follow the Novel reading and writing order (i.e. numbers just like text from left to right), then our mind still wants to pronounce a number starting with the most important digit. In that case the speaking order is opposite to the reading order again.

**Table 13** gives the four options: writing Indian / Novel and pronoucing leftward / rightward. The current situation in the West is that the number is written Indian, and spoken differently depending upon size. The cell "India / Arabia" is hypothetical and not relevant since it loses the advantage of pronoucing the most important digit first, without any benefit. Let us consider the two options for Novel.

**Table 13. Writing and speaking order of numbers**

| | | **Writing order of numbers** | |
|---|---|---|---|
| | | *Left to right - Novel* | *Right to left - Indian* |
| **Speaking order of numbers** | *Left to right* | Novel-L (no) | West (larger sizes) |
| | *Right to left* | Novel-R (yes) | India / Arabia<br>English (13-19) |

**Pronouncing Novel from the left to the right (Novel-L)**

In this case 91 is pronounced *nine·one·ten*. We stick to the text direction and the linguistic translation of numbers essentially mentions the digits as they appear, while adding the weight. This approach has the drawback that the largest value appears at the end.

There are some epi-phenomena here. People may have a tendency to drop infixes and this may cause ambiguity. <*One·two·three·hundred*> that drops the *ten* might perhaps also be understood as <*one·two·three*> <*hundred*>, which then would be 32100. One option is to first mention the base, as in “million 5.31”.

**Pronouncing Novel from the right to the left (Novel-R)**

The other possibility is to write 000,013,5 and still say *“five million, three hundred (and) ten thousand”,* i.e. temporarily reading from right to left. (The pronunciation order changes because the number writing order has reversed.) This would combine the Novel notation (so that addition and multiplication follow *text* reading order) with starting pronunciation with the biggest position. There would be a small added advantage in that you first count the digits and then have the option to say *“about 5 million”* if that is adequate, without resorting to reading it wholly in reverse direction. Writing from dictation would be more involved, requiring the dictator to either start with the lowest digit or stating the number of places in advance. It seems like a do-able system. Thus: *pronounce the same, write in mirror script.*

# Conclusion

We will not quickly drop the Indian digits and number order. But EBE on these aspects will help. The need to size up the number for speaking conflicts with any number order.

# Appendix C. Numbers in base six in First Grade ?

2012

## Introduction

A sense of number is natural to many mammals and at least humans, see Piazza & Dehaene (2004). We teach children to use their fingers to count to 10. Milikowski (2010): “Kaufmann concludes: a brain doing arithmetic needs the fingers for a long while for support. They apparently help to build a bridge from the concrete to the abstract. In other words: the use of the fingers helps the brain to learn the meaning of the digits.”

There is one *however.* We can wonder: might this not be misplaced concreteness ? Are we perhaps distracted by those ten fingers while mathematical insight can lead to a much better approach ? Might finger counting to 10 not be an archaic simplism without didactic foundation ?

This question causes these subquestions:

(1) Might counting to 10 not be too complex an introduction and might counting with the base of 5 or 6 not be sufficient to achieve insight in the meaning of number and the structure in arithmetic ?

(2) When you use the right hand for units and the left hand to count the number of right hands then don’t you count from 0 to 5 again on the left hand ? Doesn't this satisfy the educational use of the fingers ? And doesn't this mean that we achieve a higher level of abstraction at the same time, since counting hands actually is multiplication ?

(3) Doesn't the complexity of using 10 show from the fact that we use artificial means for the numbers higher than 10, since there are no more fingers, with the well-known difficulty of the positional shift ? Isn't the positional shift easier to grasp when using two hands ?

(4) Isn't the use of the decimal system based upon a misunderstanding, and actually wrong, since with ten fingers we actually should use a system with base 11 (the undecimal system) ?

In a system with base six with two hands, the right hand for the units 0 to 6 and the left hand for the number of right hands, in the order of the Indian-Arabian positional system, we still use the fingers with their great educational value, and (a) we use a limited number of symbols with short calculations for the positional shift, (b) we still have the richness of 36 for serious work, and (c) we use the positional system so that we can achieve elementary insight in the structure of numbers and arithmetic, including multiplication. When we have this foundation in First Grade and lower then the later change to the decimal system seems a repetition of moves, relatively simple and enlightening. If there is insight in the basics of arithmetic then this could make it easier to change to the decimal system with its larger numbers. Perhaps there would be an overall improvement, on balance.

The issue remains tentative because it has not been researched. Few parents will submit their children to such experiments. But we can make the proposition as attractive as possible. The following is targetted at designing the best senary system that could be subjected to research.

A consideration is that students learning to become teachers at elementary school might use the following system to re-experience themselves what steps pupils must learn. This should cause for greater awareness of those steps. This re-experience is best done in a suitably developed system.

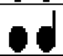
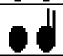
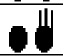
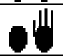
## New symbols and names

The senary system is not new, see wikipedia (2012). The examples show that using the same digits in a double role can be confusing. Thus we pick new symbols. We can use the same names ‘zero’ to ‘five’ as long as we use systematic pronunciation above six.

## The symbols and the first numbers

See Table 1 for the symbols (digits). The number of straight sides in the symbol gives the number (numerical value).

**Table 1. Symbols and the first six numbers**

| *Hands* | *Symbol* | *Pronunciation* | *Decimal* |
|---|---|---|---|
|  | ø | zero | 0 |
|  | I | one | 1 |
|  | Γ | two | 2 |
|  | Δ | three | 3 |
|  | □ | four | 4 |
|  | ⌂ | five | 5 |

**(i)** For 2 I also considered Λ (capital labda) but for dyslexia this is too similar to Δ (capital delta). V is already Roman 5. And > (larger than) better is reserved even though it may not be used at this level.

**(ii)** For arithmetic it is easier to look at your palm and check how the thumb holds down other fingers.

The question “How many fingers do you see ?” starts requiring a distinction between left and right. We might consider gloves or thimbles to indicate the different kinds of counting.

But the simple solution is: Counting the fingers on the back of the hand (with the thumbs in the middle) we use the decimal system, and, counting the fingers on the palm of the hand (with the thumbs sticking out) we use the senary system. [80]

**(iii)** One advantage of a new system is that we can choose the names systematically. The present decimal system has been stamped by tradition and we write 19 (from left to right, ‘ten·nine’) but pronounce ‘nineteen’ (from right to left). It is tempting to write and pronounce the new senary digits from the left to the right. However, the change to the present decimal system later on would become confusing. Hence we maintain the Indian-Arabian order from right to left. Since we are interested in the size of the number we also adopt the order of pronunciation from left to right.

[80] This Appendix C thus opposes former Appendix B.

# From five to hand

How do we go from five to six ? The best choice is that I on the left hand means, and is called, ‘(one) (right) hand’. Five fingers mean five when the fingers are spread out while there is one hand when all fingers are held together, see Table 2. The positional shift arises by the equality of a whole right hand with a single finger on the left, see Table 3.

**Table 2. From five to hand (value six)**

| *Five fingers, value five* | *One (right) hand, value six* |
|---|---|
|  |  |

**Table 3. Positional shift**

| *Equality* |
|---|
|  =  |

**(i)** It was an important step in this design to use ‘hand’ indeed instead of ‘six’, to signify the new unit of account. Six undoubtedly has a higher level of abstraction but in this phase we intend to support the process towards that abstraction. As a unit of account ‘right hand’ is too long but the use of ‘hand’ of course can be supported with the explanation that we are counting right hands.

**(ii)** It is a research question whether it is better to use base 5, where a right hand with five fingers is equal to a single left finger, and can be replaced by it. The positional shift might be coded as that five right fingers are equal to one left finger. However, this first advantage turns into a later disadvantage. Thinking continually in terms of equality and replacement will slow down the process of counting.

The positional shift can also be given form by letting five fingers be followed by a single finger on the left. In that manner the focus remains on the fingers. When asked what that single finger on the left stands for, one can say: a whole right hand but with fingers closed. The use of language requires accuracy: a hand has five fingers but as a whole it has the value of six fingers – and that whole is represented by holding the fingers together.

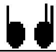
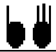
**(iii)** Advantages of the senary system are the number of divisors and the link to the hours of the day and the number of months. One can imagine adapted clock-faces.

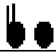
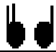
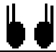
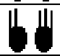
# Continued counting

See Table 4 for continued counting from six. Between the words we insert a middle dot that we do not pronounce. A dot is better than a hyphen since that can be confusing with minus.

**(i)** It is a research question whether we can also give the names of the decimal digits and numbers. Thus next to ‘hand·one’ also ‘seven’ and up to twelve. It might be confusing to have more names for a number but it might also help pupils to understand the strange words that their parents are speaking. Since children are apt at learning languages these other words need not be confusing, at least when the numbers maintain a system.

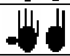
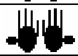
**Table 4. Combinations in hand and two·hand**

| *Hands* | *Symbol* | *Pronunciation* | *Decimal* |
|---|---|---|---|
|  | I ø | hand | 6 |
|  | I I | hand·one | 7 |
|  | I Γ | hand·two | 8 |
|  | I Δ | hand·three | 9 |
|  | I □ | hand·four | 10 |
|  | I ⌂ | hand·five | 11 |

| *Hands* | *Symbol* | *Pronunciation* | *Decimal* |
|---|---|---|---|
|  | Γ ø | two·hand | 12 |
|  | Γ I | two·hand·one | 13 |
|  | Γ Γ | two·hand·two | 14 |
|  | Γ Δ | two·hand·three | 15 |
|  | Γ □ | two·hand·four | 16 |
|  | Γ ⌂ | two·hand·five | 17 |

See Table 5 for how the two hands are exhausted at 36 so that we continue counting in Table 8 with lux = hand·hand (from the luxury of a third hand).

**Table 5. Combinations in five·hand**

| *Hands* | *Symbol* | *Pronunciation* | *Decimal* |
|---|---|---|---|
|  | ⌂ ø | five·hand | 30 |
|  | ⌂ I | five·hand·one | 31 |
|  | ⌂ Γ | five·hand·two | 32 |
|  | ⌂ Δ | five·hand·three | 33 |
|  | ⌂□ | five·hand·four | 34 |
|  | ⌂⌂ | five·hand·five | 35 |

## Positional shift at hand·hand

The positional shift at hand·hand is a repetition of the positional shift at hand, but it is useful to be explicit about this, so that pupils can verify that it is a repetition. Table 6 gives the step to hand·hand, but the use of a new sign and name as in Table 7 will allow us to count on. The sign for lux assumes a second pupil who uses the fingers of the right hand to start counting in that place value. Counting onwards gives Table 8.

**Table 6. From five·hand·five to hand·hand = lux**

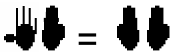
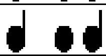
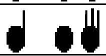
| *Five·hand·five* | *Five·hand + hand = Hand·hand = lux* |
|---|---|
|  |  |

**Table 7. Positional shift**

| *Equality* |
|---|
|  |

**Table 8. Combinations in lux = hand·hand**

| *Hands* | *Symbol* | *Pronunciation* | *Decimal* |
|---|---|---|---|
|  | I ø ø | lux | 36 |
|  | I ø I | lux·one | 37 |
|  | I ø Γ | lux·two | 38 |
|  | I ø Δ | lux·three | 39 |
|  | I ø □ | lux·four | 40 |
|  | I ø ⌂ | lux·five | 41 |

## Plus and times

The number of possible additions with result ⌂ is limited (I + □ en Γ + Δ, and both in reverse) so that there will quickly be a positional shift, that can be calculated easily as well.Table 9 shows a calculation in columns, that also might be done in larger jumps: Δ + Δ = Δ + Γ + I = I + ⌂ = Iø.

**Table 9. Addition in columns**

| | | | | |
|---|---|---|---|---|
| Number | Δ | □ | ⌂ | Iø |
| Plus | Δ | Γ | I | ø |
| Is | | | | Iø |

The advantage of having both few symbols and numbers till 36 means that we can consider the introduction of multiplication. For these pupils it seems better to speak about ‘times’ and ‘to time’ rather than the long terms ‘multiplication’ and ‘multiply’ (multi-plus).

In Holland, First Grade is limited to addition and subtraction with the numbers to 20 – a bit comparable to the US *Common Core*. This will be related to the positional shift, the illogical pronunciation of the numbers ('nineteen' instead of 'ten·nine' and 'twenty' instead of 'two·ten'), and the fact that multiplication may quickly give such awkward numbers. When we take a fresh look at the issue then we may agree that learning the numbers to 20 does not have a priority in itself. In the senary system we can count till 36 and this seems doable and clear. Unless research would show that First Grade can only grasp number size but not multiplication.

The curious point is:

> When pupils in First Grade can master above senary system then this itself shows that they can master elementary multiplication. Counting hands namely is multiplication by six. Can they multiply different numbers ?

Standard is Γ + Γ + Γ = ∆ × Γ = ıø. There is also ∆ + ∆ = Γ × ∆ = ıø. The discussion of a rectangle and its surface shows that *times* is commutative. Thus, the order of *times* does not matter.

When there are five cats with each two eyes then there are Γ + Γ + Γ + Γ + Γ = ⌂ × Γ = ı□ or *hand·four* eyes in total. With five cats you have five left eyes and five right eyes, thus ⌂ + ⌂ = Γ × ⌂ = ı□. Many pupils of age six could learn this. Would there be a sufficient number of them to introduce the approach in the general curriculum ?

Counting the number of Γs is a higher level of abstraction (the levels identified by Pierre van Hiele). Counting is the ticking-off of the elements of a set. It is a higher level of abstraction to group elements, see a set as a new unit of account, and then tick off the sets.

The following is an important insight with respect to *times*:

> A result like 5 × 2 = 10 is trivial for us but only since we learned this by heart.

Some authors argue that pupils need not learn the *table of times* by heart but must first feel their way. This runs against logic. If you don't learn the table of *times* by heart then you remain caught in the world of addition. This is very slow and does not contribute to understanding. Remember what *times* is:

(6) Taking a set of sets
(7) To know how you can count single elements but that it is faster to only count the border totals
(8) To know which table to use to look it up (namely × instead of +)
(9) And get your result faster because you know the table by heart
(10) To know all of this.

A calculation like ıΓ × ıı = ı∆Γ contains operations that seem doable at this level. Table 10 uses those higher numbers to make the issue nontrivial. First Grade will use lower numbers. How high can we go ? Nice is ıø × ıø = ıøø but ıı × ıı = ı Γı would remain instructive.

**Table 10. Calculating 7 times 8**

| | | |
|---|---|---|
| ıΓ | | 6 + 2 = 8 |
| ıı | × | 6 + 1 = 7 |
| ıΓ | | 6 + 2 = 8 |
| ıΓø | + | 36 + (2 × 6) = 48 |
| ı∆Γ | | 36 + (3 × 6) + 2 = 56 |

This calculation shows the advantage of knowing what *times* means. Who knows what it is can understand how the numbers are contructed, and can also understands what arithmetic is (the collection of the weights of the powers of the base number). For this reason it is didactically advantageous to have *times* available as quickly as possible.

## Tables

Table 11 gives the table of addition and Table 12 for *times*. Learning by heart is required for the decimal system but inadvisable for the senary system since you would later learn the decimal one. At this stage it suffices to be able to look up the result in the table, and to see how the tables hang together, and how they have some structure (e.g. the diagonals). For example, see in both tables how Δ + Δ + Δ = Δ × Δ.

**Table 11. Table of addition for 1 to 12**

| + | I | Γ | Δ | □ | ⌂ | Iø | II | IΓ | IΔ | I□ | I⌂ | Γø |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| I | Γ | Δ | □ | ⌂ | Iø | II | IΓ | IΔ | I□ | I⌂ | Γø | ΓI |
| Γ | Δ | □ | ⌂ | Iø | II | IΓ | IΔ | I□ | I⌂ | Γø | ΓI | ΓΓ |
| Δ | □ | ⌂ | Iø | II | IΓ | IΔ | I□ | I⌂ | Γø | ΓI | ΓΓ | ΓΔ |
| □ | ⌂ | Iø | II | IΓ | IΔ | I□ | I⌂ | Γø | ΓI | ΓΓ | ΓΔ | Γ□ |
| ⌂ | Iø | II | IΓ | IΔ | I□ | I⌂ | Γø | ΓI | ΓΓ | ΓΔ | Γ□ | Γ⌂ |
| Iø | II | IΓ | IΔ | I□ | I⌂ | Γø | ΓI | ΓΓ | ΓΔ | Γ□ | Γ⌂ | Δø |
| II | IΓ | IΔ | I□ | I⌂ | Γø | ΓI | ΓΓ | ΓΔ | Γ□ | Γ⌂ | Δø | ΔI |
| IΓ | IΔ | I□ | I⌂ | Γø | ΓI | ΓΓ | ΓΔ | Γ□ | Γ⌂ | Δø | ΔI | ΔΓ |
| IΔ | I□ | I⌂ | Γø | ΓI | ΓΓ | ΓΔ | Γ□ | Γ⌂ | Δø | ΔI | ΔΓ | ΔΔ |
| I□ | I⌂ | Γø | ΓI | ΓΓ | ΓΔ | Γ□ | Γ⌂ | Δø | ΔI | ΔΓ | ΔΔ | Δ□ |
| I⌂ | Γø | ΓI | ΓΓ | ΓΔ | Γ□ | Γ⌂ | Δø | ΔI | ΔΓ | ΔΔ | Δ□ | Δ⌂ |
| Γø | ΓI | ΓΓ | ΓΔ | Γ□ | Γ⌂ | Δø | ΔI | ΔΓ | ΔΔ | Δ□ | Δ⌂ | □ø |

**Table 12. Table of *times* for 1 to 12**

| × | I | Γ | Δ | □ | ⌂ | Iø | II | IΓ | IΔ | I□ | I⌂ | Γø |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| I | I | Γ | Δ | □ | ⌂ | Iø | II | IΓ | IΔ | I□ | I⌂ | Γø |
| Γ | Γ | □ | Iø | IΓ | I□ | Γø | ΓΓ | Γ□ | Δø | ΔΓ | Δ□ | □ø |
| Δ | Δ | Iø | IΔ | Γø | ΓΔ | Δø | ΔΔ | □ø | □Δ | ⌂ø | ⌂Δ | Iøø |
| □ | □ | IΓ | Γø | Γ□ | ΔΓ | □ø | □□ | ⌂Γ | Iøø | Iø□ | IIΓ | IΓø |
| ⌂ | ⌂ | I□ | ΓΔ | ΔΓ | □I | ⌂ø | ⌂⌂ | Iø□ | IIΔ | IΓΓ | IΔI | I□ø |
| Iø | Iø | Γø | Δø | □ø | ⌂ø | Iøø | IIø | IΓø | IΔø | I□ø | I⌂ø | Γøø |
| II | II | ΓΓ | ΔΔ | □□ | ⌂⌂ | IIø | IΓI | IΔΓ | I□Δ | I⌂□ | Γø⌂ | ΓΓø |
| IΓ | IΓ | Γ□ | □ø | ⌂Γ | Iø□ | IΓø | IΔΓ | I□□ | Γøø | ΓIΓ | ΓΓ□ | Γ□ø |
| IΔ | IΔ | Δø | □Δ | Iøø | IIΔ | IΔø | I□Δ | Γøø | ΓIΔ | ΓΔø | Γ□Δ | Δøø |
| I□ | I□ | ΔΓ | ⌂ø | Iø□ | IΓΓ | I□ø | I⌂□ | ΓIΓ | ΓΔø | Γ□□ | ΔøΓ | ΔΓø |
| I⌂ | I⌂ | Δ□ | ⌂Δ | IIΓ | IΔI | I⌂ø | Γø⌂ | ΓΓ□ | Γ□Δ | ΔøΓ | ΔΓI | Δ□ø |
| Γø | Γø | □ø | Iøø | IΓø | I□ø | Γøø | ΓΓø | Γ□ø | Δøø | ΔΓø | Δ□ø | □øø |

## Advantages and disadvantages

In summary, the advantages of the senary system for First Grade are:

(1) There is a small number of symbols that can be chosen for clarity

(2) The number of calculations for the positional shift is small and clear too

(3) Calculations can be done on two hands, with the right hand for units and the left hand for the number of right hands, in the same order (from right to left) as in the current decimal system

(4) Numbers have the same structure as the decimal system

(5) The pronunciation of the numbers is not dictated by tradition but can be chosen systematically

(6) *Times* can be introducted more quickly so that it allows earlier insight in the structure of number and arithmetic ($a + b$ × hand + $c$ × lux + …).

The disadvantages of a senary system are:

(1) It is intended for didactics only and not applied in practice

(2) A question like “How many fingers do you see ?” requires distinction between left and right, back and palm

(3) It may be confusing, on balance, in learning the decimal system later on.

## Conclusions

I doubt whether this system with base six will be used in First Grade indeed. Current problems in teaching arithmetic may have to do less with the number system itself, see for comparison the 1950s. In Holland since then there has been a curious move towards not learning the tables by heart, see Milikowski (2004). We may already see a big improvement when misunderstandings like these are resolved. That said, it still is a contribution to think about the number system and its relation to arithmetic.

Libraries have been filled on number and arithmetic but the present discussion seems to includes these useful points:

(1) Above senary system has an attractive form, both by its streamlining and by a minimum of confusion with the decimal system. If you use a senary system, the advice is to use this one. (This would be suitable for students who are learning to become teachers at elementary school.)

(2) This paper gives another perspective on the proposal to revise the names of the decimal numbers (with 11 = ten·one and so on).

(3) Research in both didactics and brains could look with priority whether First Grade can multiply. When pupils can learn above senary system then this already shows their elementary grasp of *times*. Counting hands is *times hand*. Γ × ıø + □ = Γ□ seems doable and shows the structure of numbers. Can pupils also multiply with other numbers ? Five cats with two eyes each gives five times two or hand·four or ten eyes. Seems doable as well. When a range of numbers can be found then this can be exploited to develop arithmetic.

(4) Above discussion may also help to beter target learning aims for Second Grade. Problems like 2 × 10 + 4 = 24 highlight the structure of number as well.

# Literature

PM 1. Colignatus is the name of Thomas Cool in science. See http://thomascool.eu.
PM 2. References in footnotes might not be repeated here.


Barrow, J. (1993), “Pi in the sky. Counting, thinking and being”, Penguin
Colignatus (1981 unpublished, 2007, 2011), "A Logic of Exceptions" (ALOE), http://thomascool.eu/Papers/ALOE/Index.html
Colignatus (2008), "Trig rerigged", **Legacy:** see EWS for principles and COTP for details
Colignatus (2009, 2015), “Elegance with Substance”, 1st edition Dutch University-Press, 2nd edition mijnbestseller.nl, http://thomascool.eu/Papers/Math/Index.html
Colignatus (2011), “Conquest of the Plane”, http://thomascool.eu/Papers/COTP/Index.html
Colignatus (2012a), “Een kind wil aardige en geen gemene getallen” (KWAG), http://thomascool.eu/Papers/AardigeGetallen/Index.html
Colignatus (2012b), "A child wants nice and not mean numbers", **Legacy** collection, parts of KWAG in English, no longer on the web, included and revised in this book (*not* becoming *no*)
Colignatus (2012c), "Pas je uitleg aan. Interview met Yvonne Killian", Euclides 88 (2) p 83-84
Colignatus (2014), "Pierre van Hiele and David Tall: Getting the facts right", http://arxiv.org/abs/1408.1930
Domahs, F., L. Kaufmann, M.H. Fischer (eds) (2012), "Handy numbers: finger counting and numerical cognition", Frontiers in Psychology
Dormolen, J. van (1976), "Didactiek van de wiskunde", tweede herziene druk, Bohn, Scheltema & Holkema
Ejersbo, L.R. & M. Misfeldt (2011), “Danish number names and number concepts”, http://pure.au.dk/portal/da/publications/danish-number-names-and-number-concepts(3f252eb9-a752-49a0-a973-e6011504cb55).html
Gamboa, J.M. (2011), “Book review. Conquest of the Plane, by Thomas Colignatus", http://www.euro-math-soc.eu/node/2081
Gill, R.D. (2012), “Book reviews. Thomas Colignatus. (1) Elegance with Substance, (2) Conquest of the Plane”, Nieuw Archief voor Wiskunde, http://www.nieuwarchief.nl/serie5/pdf/naw5-2012-13-1-064.pdf
Gladwell, M. (2008), “Outliers. The story of success”, Little Brown
Goodman, P. (1962, 1973) "Compulsory miseducation", Penguin Education Specials
Harremoës, P. (2000), "Talnotation", LMFK-bladet 5, No. 5, http://www.harremoes.dk/Peter/talnot.pdf
Hiele, P.M. van (1973), "Begrip en inzicht", Muusses
Killian, Y. (2006), "De stelling van Pythagoras voor de brugklas of groep 8", Euclides 81 (7), p 338
Killian, Y. (2012), "De stelling van Pythagoras voor de brugklas of groep 8. Deel 2. Praktijkervaring en een bijbehorend bewjs", Euclides 88 (5), p 250-251 (plus erratum)
Milikowski, M. (2004), “Pleidooi voor de tafels”, http://www.rekencentrale.nl/Recent/pleidooi_voor_de_tafels.pdf
Milikowski, M. (2010), “Goed zo, tel maar op je vingers”, balans magazine, oktober, p43, http://www.rekencentrale.nl/Recent/Vingers.pdf
Piazza, M. & S. Dehaene (2004), “From number neurons to mental arithmetic: the cognitive neuroscience of number sense”. In M. Gazzaniga (2004), “The Cognitive Neurosciences”, 3rd edition, http://www.unicog.org/publications/PiazzaDehaene_ChapGazzaniga.pdf
Smolin, L. (2015), "A naturalist account of the limited, and hence reasonable, effectiveness of mathematics in physics", http://fqxi.org/data/essay-contest-files/Smolin_FQXi_2015_math_final.pdf
Tall, D.O. (2002), “Three Worlds of Mathematics”, Bogota, Columbia, July 2–5, 2002, see http://www.warwick.ac.uk/staff/David.Tall/themes/procepts.html
Waerden, B.L. van der (1975), “Science Awakening I”, 4th edition, Kluwer
Waerden, B.L. van der (1983), “Geometry and algebra in ancient civilizations”, Springer
Waerden, B.L. van der (1985), “A history of algebra”, Springer
Wikipedia (2012), http://en.wikipedia.org/wiki/Senary, retrieved April 15
Wu, H.-H. (2011, 2014), "Teaching Fractions According to the Common Core Standards (For teachers of K-8 and educators) (August 5, 2011; revised February 8, 2014), https://math.berkeley.edu/~wu/CCSS-Fractions_1.pdf

# Index

## R



## S



## T



## U



## V



## W



## Z