Luminous flux through the inclined circle

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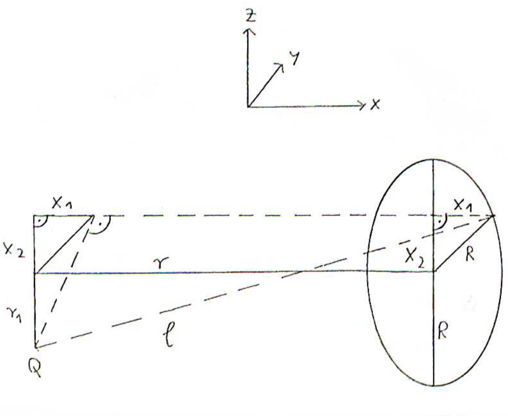
We look at the following situation in vacuum (see figure):



Q= point light source (with luminous intensity I), that is shifted with r_1 from the coordinate origin.
r= distance circle - coordinate origin

$$x_2 \in [-R, +R] \qquad x_1 \in [-\sqrt{R^2 - x_2^2}, +\sqrt{R^2 - x_2^2}]$$

r is vertical to x_2 and x_1.
We obtain the distance-vector:

$$\vec{l} = \begin{pmatrix} r \\ x_1 \\ r_1 + x_2 \end{pmatrix}$$

and the distance l:

$$l = |\vec{l}| \qquad l^2 = (r_1 + x_2)^2 + x_1^2 + r^2$$

For the angle of inclination α of the circle to the light source Q:

$$\tan\alpha = \frac{r_1}{r}$$

Then we have for the illumination E:

$$E(x_1, x_2) = \frac{I}{(r_1 + x_2)^2 + x_1^2 + r^2} \tag{1}$$

The angle of inclination $\beta(x_1, x_2)$ at the point (x_1, x_2) can be written:

$$\cos[\beta(x_1, x_2)] = \frac{r}{l} = \frac{r}{\sqrt{(r_1 + x_2)^2 + x_1^2 + r^2}} \tag{2}$$

At last we conclude for the luminous flux Φ through the circle:

$$\Phi = 2 \cdot \int_{-R}^{+R} \int_{0}^{\sqrt{R^2 - x_2^2}} E(x_1, x_2) \cdot \cos\left[\beta(x_1, x_2)\right] dx_1\, dx_2$$

$$= 2Ir \cdot \int_{-R}^{+R} \int_{0}^{\sqrt{R^2 - x_2^2}} \frac{dx_1\, dx_2}{[(r_1 + x_2)^2 + x_1^2 + r^2]^{\frac{3}{2}}} \tag{3}$$

$$r > 0$$

We reduce (3) to an one-dimensional integral:

$$\int_{0}^{\sqrt{R^2 - x_2^2}} \frac{dx_1}{\left(\sqrt{(r_1 + x_2)^2 + x_1^2 + r^2}\right)^3}$$

$$= \left[\frac{x_1}{(r^2 + (r_1 + x_2)^2) \cdot \sqrt{(r_1 + x_2)^2 + x_1^2 + r^2}}\right]_0^{\sqrt{R^2 - x_2^2}} \tag{4}$$

Proof with differentiation:

$$\frac{d}{dx_1}\left[\frac{x_1}{(r^2 + (r_1 + x_2)^2)} \cdot ((r_1 + x_2)^2 + r^2 + x_1^2)^{-\frac{1}{2}}\right] = \frac{1}{r^2 + (r_1 + x_2)^2}$$

$$\cdot \left[((r_1 + x_2)^2 + r^2 + x_1^2)^{-\frac{1}{2}} + x_1 \cdot -\frac{1}{2}((r_1 + x_2)^2 + r^2 + x_1^2)^{-\frac{3}{2}} \cdot 2x_1\right]$$

$$= \frac{((r_1+x_2)^2+r^2+x_1^2)-x_1^2}{((r_1+x_2)^2+r^2+x_1^2)^{\frac{3}{2}}} \cdot \frac{1}{r^2+(r_1+x_2)^2}$$

$$= \frac{1}{\left(\sqrt{(r_1+x_2)^2+r^2+x_1^2}\right)^3}$$

It follows with (4):

$$\int_0^{\sqrt{R^2-x_2^2}} \frac{dx_1}{((r_1+x_2)^2+x_1^2+r^2)^{\frac{3}{2}}}$$

$$= \frac{\sqrt{R^2-x_2^2}}{(r^2+(r_1+x_2)^2)\cdot\sqrt{(r_1+x_2)^2+R^2-x_2^2+r^2}}$$

and with (3):

$$\Phi = 2Ir \cdot \int_{-R}^{R} \frac{\sqrt{R^2-x_2^2}\,dx_2}{(r^2+(r_1+x_2)^2)\cdot\sqrt{(r_1+x_2)^2+R^2-x_2^2+r^2}} \tag{5}$$

To solid angle: $r > 0$

$$\Omega = 2\cdot r\cdot \int_{-R}^{R} \frac{\sqrt{R^2-x_2^2}\,dx_2}{(r^2+(r_1+x_2)^2)\cdot\sqrt{(r_1+x_2)^2+R^2-x_2^2+r^2}} \tag{6}$$

Now we view the same situation in pure air. The absorption coefficient m is spatial and temporal constant. With equation (3) we obtain:

$$\Phi = 2Ir \cdot \int_{-R}^{R} \int_0^{\sqrt{R^2-x_2^2}} \frac{e^{-m\sqrt{(r_1+x_2)^2+x_1^2+r^2}}}{((r_1+x_2)^2+x_1^2+r^2)^{\frac{3}{2}}}\,dx_1\,dx_2 \tag{7}$$

$$r > 0$$

One approximation:
Now we assume the distance is very large compared with the radius of the circle $r \gg R$.
Then it is valid:

$$\Omega \approx \frac{A\cos\alpha}{r^2}$$

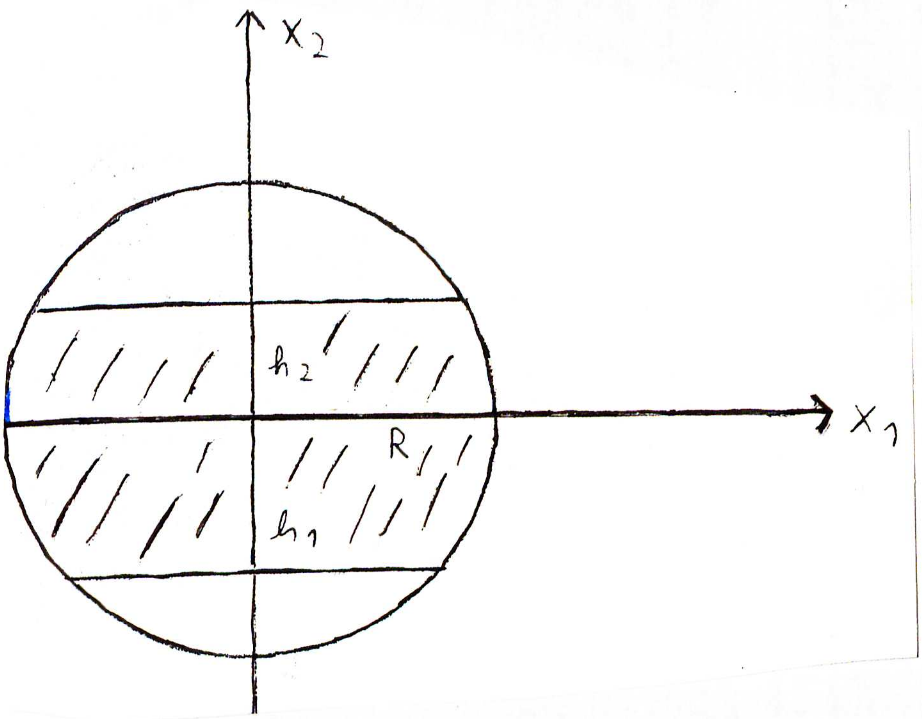
with $A = \pi R^2$ and $\tan\alpha = \frac{r_1}{r}$, at which α is the angle of inclination of the circle. The light source has the luminous intensity I.The luminous flux follows im medium(pure air).

$$\Phi \approx \frac{\pi R^2 I\cos\alpha}{r^2}\cdot e^{-mr} \tag{8}$$

without medium ($m = 0$):

$$\Phi \approx \frac{\pi R^2 I \cos\alpha}{r^2} \tag{9}$$

The formulas (3), (5), (6) und (7) can be generalized with integrals from h_1 to h_2.
$h_1, h_2 \in [-R, R]$
Then we have the luminous flux respectively the solid angle through a circular segment as in the figure.



Now we introduce a further non quadratic approximation for the luminous flux in vacuum:

$$\Phi \approx \pi \cdot I \cdot \left(1 - \frac{r}{\sqrt{r^2 + R^2}}\right) \cdot \cos\alpha \tag{10}$$

This follows from chapter 2 of [1]. We calculate the luminous flux with the one-dimensional integal with simpson's rule (50 steps). We neglect the absorption. The luminous intensity is 0,5 Candela:

First at $r = 1000m$ and $R = 1m$:

α in degree	Φ(Simpson's rule)	Φ with (10)	Φ with (9)
10	$1,54512 \cdot 10^{-6}$	$1,54690 \cdot 10^{-6}$	$1,54693 \cdot 10^{-6}$
20	$1,47434 \cdot 10^{-6}$	$1,47604 \cdot 10^{-6}$	$1,47607 \cdot 10^{-6}$
30	$1,35876 \cdot 10^{-6}$	$1,36032 \cdot 10^{-6}$	$1,36035 \cdot 10^{-6}$
40	$1,20189 \cdot 10^{-6}$	$1,20328 \cdot 10^{-6}$	$1,20330 \cdot 10^{-6}$
50	$1,00851 \cdot 10^{-6}$	$1,00967 \cdot 10^{-6}$	$1,00969 \cdot 10^{-6}$
60	$7,84479 \cdot 10^{-7}$	$7,85382 \cdot 10^{-7}$	$7,85398 \cdot 10^{-7}$
70	$5,36615 \cdot 10^{-7}$	$5,37233 \cdot 10^{-7}$	$5,37244 \cdot 10^{-7}$
80	$2,72447 \cdot 10^{-7}$	$2,72760 \cdot 10^{-7}$	$2,72766 \cdot 10^{-7}$

Here the exact formula corresponds well with the both approximations.

Now we turn to the case $r = 1m$ and $R = 0,1m$:

α in degree	Φ(Simpson's rule)	Φ with (10)	Φ with (9)
10	$1,53449 \cdot 10^{-2}$	$1,53543 \cdot 10^{-2}$	$1,54693 \cdot 10^{-2}$
20	$1,46654 \cdot 10^{-2}$	$1,46509 \cdot 10^{-2}$	$1,47607 \cdot 10^{-2}$
30	$1,35490 \cdot 10^{-2}$	$1,35023 \cdot 10^{-2}$	$1,36035 \cdot 10^{-2}$
40	$1,20212 \cdot 10^{-2}$	$1,19435 \cdot 10^{-2}$	$1,20330 \cdot 10^{-2}$
50	$1,01198 \cdot 10^{-2}$	$1,00218 \cdot 10^{-2}$	$1,00969 \cdot 10^{-2}$
60	$7,89602 \cdot 10^{-3}$	$7,79556 \cdot 10^{-3}$	$7,85398 \cdot 10^{-3}$
70	$5,41479 \cdot 10^{-3}$	$5,33248 \cdot 10^{-3}$	$5,37244 \cdot 10^{-3}$
80	$2,75369 \cdot 10^{-3}$	$2,70737 \cdot 10^{-3}$	$2,72766 \cdot 10^{-3}$

Here the both approximations correspond not so well with the exact formula.

Now we treat the case $r = 8m$ and $R = 4m$:

α in degree	Φ(Simpson's rule)	Φ with (10)	Φ with (9)
10	0,32959	0,32663	0,38673
20	0,32402	0,31166	0,36902
30	0,31357	0,28723	0,34009
40	0,29632	0,25407	0,30082
50	0,26929	0,21319	0,25242
60	0,22846	0,16583	0,19635
70	0,16968	0,11344	0,13431
80	0,09160	0,05759	0,06819

Here the both approximations correspond badly with the exact calculation.

At last we turn to the case $r = 1m$ and $R = 100m$:

α in degree	Φ(Simpson's rule)	Φ with (10)	Φ with (9)
10	5,87642	3,06293	15469,3
20	5,61406	2,92261	14760,7
30	5,18405	2,69349	13603,5
40	4,59723	2,38253	12033,0
50	3,86876	1,99918	10096,9
60	3,01823	1,55509	7854,0
70	2,06990	1,06374	5372,4
80	1,05277	0,54008	2727,7

The non quadratic approximation corresponds very badly (in order) and the quadratic approximation corresponds not at all with the exact formula.

The quality of correspondance is dependent from the proportion $\frac{R}{r}$.

References

[1] Harald Schröer, "Luminous Flux and Illumination",Wissenschaft & Technik Verlag,Berlin, 2001