Compact perturbative expressions for oscillations with sterile neutrinos in matter

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December 4, 2018

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Neutrino oscillations in vacuum

In a scheme with N sterile neutrinos, the oscillation probabilities in vacuum are

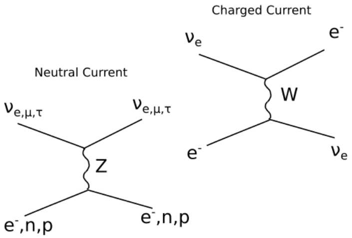
$$P(\nu_\alpha \to \nu_\beta) = \Big| \sum_{j=1}^{3+N} U^*_{\alpha j} U_{\beta j} \, e^{i \Delta m^2_{j1}/2E} \Big|^2$$

U is the PMNS matrix which converts the energy eigenstates to the flavor eigenstates.

$$H_{\text{vacuum}} = \frac{1}{2E} \, U \begin{pmatrix} 0 & & & & \\ & \Delta m^2_{21} & & & \\ & & \Delta m^2_{31} & & \\ & & & \ddots & \\ & & & & \Delta m^2_{(3+N)\,1} \end{pmatrix} U^\dagger$$

Matter effect

In matters, propagation of the neutrinos will be altered by the L. Wolfenstein matter effect.



$$V_{NC} = \mp\sqrt{2}G_F N_n/2 \qquad V_{CC} = \pm\sqrt{2}G_F N_e$$

N_n and N_e are the number densities of the neutrons and electrons, respectively, when $N_n \simeq N_e$, we have $V_{NC} \simeq -V_{CC}/2$. The sterile neutrinos will not be engaged in the matter effects.

Hamiltonian in matter

The Hamiltonian in the flavor basis becomes (free to add a multiple of the identity)

$$H = H_{\text{vacuum}} + \frac{1}{2E}\begin{pmatrix} a & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & a/2 & & \\ & & & & \ddots & \\ & & & & & a/2 \end{pmatrix},$$

where $a = 2\sqrt{2}G_F N_e E$.

Now the PMNS matrix in vacuum U can no longer diagonalize the Hamiltonian, the energy eigenstates and eigenvalues are altered by the matter effect.

Solve the eigensystem in matter

$$H = \frac{1}{2E} V^\dagger \begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \lambda_3 & & & \\ & & & \lambda_4 & & \\ & & & & \ddots & \\ & & & & & \lambda_{3+N} \end{pmatrix} V$$

Solve for V and λ_i.

Analytic solutions

- Only possible for 3+1 model 1808.03985

Perturbation expansions

- degeneracy of the zeroth order eigenvalues

Rotations+Perturbation expansions

The rotations can do...

- Disentangle the crossings of the 0^{th} order eigenvalues
- Diminish off-diagonal elements of the Hamiltonian
- Give 0^{th} order eigenvalues and mixing parameters (angles and phases)

Expansion parameter

Define $\Delta m^2_{ee} \equiv \Delta m^2_{31} - s^2_{12}\Delta m^2_{21}$**,** $\epsilon \equiv \Delta m^2_{21}/\Delta m^2_{ee} \simeq 0.03$**.**

Orders of some important parameters

- weak mixing with sterile neutrinos, $\sin\theta_{i\,(3+n)} \sim \mathcal{O}(\sqrt{\epsilon})$
- heavy sterile neutrinos, $\Delta m^2_{ee}/\Delta m^2_{(3+n)\,1} \lesssim \mathcal{O}(\epsilon)$.
- not extremely strong matter effect, $a \sim \Delta m^2_{ee}$, so $a/\Delta m^2_{(3+n)\,1} \lesssim \mathcal{O}(\epsilon)$

Step 0: Convention of the vacuum PMNS matrix

A usual convention to define the PMNS matrix in vacuum, rotations mixing with the sterile neutrinos come after the ones in the active neutrino space

$$U = U_{\text{sterile}}\, U_{23}\, U_{13}\, U_{12}$$

A different convention to define the PMNS matrix

$$U = U_{23}\, U_{\text{sterile}}\, U_{13}\, U_{12}$$

The matter potential term in the Hamiltonian is invariant under a transformation in the (2-3) sector. If U_{23} is the last rotation, the following rotations process will be simplified.

Step 1: Vacuum U_{23} rotation

$$H \Rightarrow U_{23}^{\dagger}(\theta_{23}, \delta_{23})\, H\, U_{23}(\theta_{23}, \delta_{23})$$

$$= U_{23}^{\dagger}(\theta_{23}, \delta_{23})\, H_{\text{vacuum}}\, U_{23}(\theta_{23}, \delta_{23}) + \frac{1}{2E} \begin{pmatrix} a & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & a/2 & & \\ & & & & \ddots & \\ & & & & & a/2 \end{pmatrix}$$

θ_{23} and δ_{23} are in vacuum.

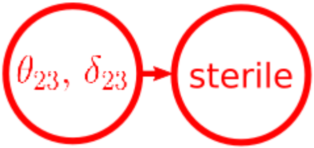
θ_{23}, δ_{23}

Step 2: Vacuum U_{sterile} rotations.

$$U_{23}^{\dagger}(\theta_{23}, \delta_{23})\, H\, U_{23}(\theta_{23}, \delta_{23})$$

$$\Rightarrow \tilde{H} \equiv U_{\text{sterile}}^{\dagger} U_{23}^{\dagger}(\theta_{23}, \delta_{23})\, H\, U_{23}(\theta_{23}, \delta_{23}) U_{\text{sterile}}$$

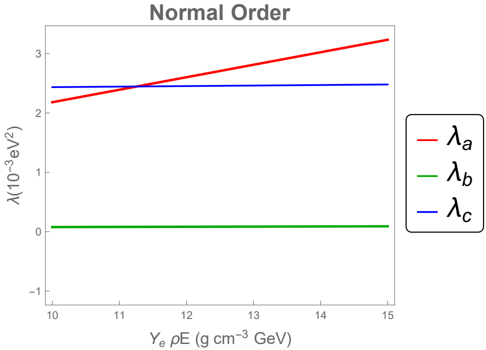
Rotations parameter (angles and phases) in U_{sterile} are still in vacuum



Step 3: U_{13} rotation, explicit derivation in the 3+1 scheme

$$\tilde{H} = \frac{1}{2E}\begin{pmatrix} \lambda_a & \cdots & (\tilde{H})_{13} & \cdots \\ \vdots & \lambda_b & \vdots & \\ (\tilde{H})^*_{13} & \cdots & \lambda_c & \\ \vdots & & & \ddots \end{pmatrix}$$

- Kill $(\tilde{H})_{13}$
- Resolve the crossing of λ_a and λ_c at $a \simeq \frac{\cos 2\theta_{13}}{c_{14}^2}\Delta m^2_{ee}$.



Step 3: Continued

$$\lambda_a = (s_{13}^2 + \epsilon s_{12}^2)\Delta m_{ee}^2 + (c_{14}^2 + \frac{\epsilon}{2} k_{11} c_{24}^2 c_{34}^2) a \qquad \underline{k_{ij} = \frac{s_{i4} s_{j4}}{\epsilon} \sim \mathcal{O}(1)}$$

$$\lambda_b = \epsilon c_{12}^2 \Delta m_{ee}^2 + \frac{\epsilon}{2} k_{22} c_{34}^2 a$$

$$\lambda_c = (c_{13}^2 + \epsilon s_{12}^2)\Delta m_{ee}^2 + \frac{\epsilon}{2} k_{33} a$$

$$(\tilde{H})_{13} = s_{13} c_{13} \Delta m_{ee}^2 + \frac{\epsilon}{2} a k_{13} c_{24} c_{34} e^{-i\delta_{13}}$$

Step 3: Continued

Diagonalize the (1-3) sector of $\tilde{H}$ by implementing a complex rotation $U_{13}(\tilde{\theta}_{13}, \alpha_{13})$

$$\tilde{H} \Rightarrow \hat{H} \equiv U_{13}^{\dagger}(\tilde{\theta}_{13}, \alpha_{13})\, \tilde{H}\, U_{13}(\tilde{\theta}_{13}, \alpha_{13})$$

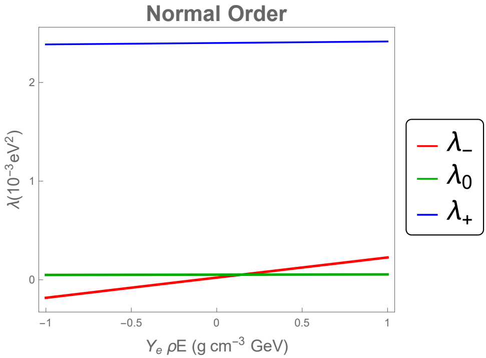
$$\tilde{\theta}_{13} = \frac{1}{2} \arccos \frac{\lambda_c - \lambda_a}{\sqrt{|\lambda_c - \lambda_a|^2 + 4|s_{13} c_{13} \Delta m^2_{ee} + \frac{\epsilon}{2} a k_{13} c_{24} c_{34} e^{-i\delta_{34}}|^2}}$$

$$\alpha_{13} = \mathrm{Arg}\left[s_{13} c_{13} \Delta m^2_{ee} + \frac{\epsilon}{2} a k_{13} c_{24} c_{34} e^{-i\delta_{34}}\right]$$

Step 4: U_{12} rotation, explicit derivation in the 3+1 scheme

$$\hat{H} = \frac{1}{2E}\begin{pmatrix} \lambda_- & (\hat{H})_{12} & 0 & \cdots \\ (\hat{H})^*_{12} & \lambda_0 & (\hat{H})_{23} & \\ 0 & (\hat{H})^*_{23} & \lambda_+ & \\ \vdots & & & \ddots \end{pmatrix}$$

- Kill $(\hat{H})_{12}$
- Resolve the crossing of λ_- and λ_0 at the solar resonance.



Step 4: Continued

$$\lambda_- = \frac{1}{2}\Big[(\lambda_a + \lambda_c) - \text{sign}(\Delta m^2_{ee})\sqrt{(\lambda_a - \lambda_c)^2 + 4\Big|s_{13}c_{13}\Delta m^2_{ee} + \frac{\epsilon}{2}ak_{13}c_{24}c_{34}e^{-i\delta_{34}}\Big|^2}\Big]$$

$$\lambda_0 = \lambda_b = \epsilon c^2_{12}\Delta m^2_{ee} + \frac{\epsilon}{2}k_{22}c^2_{34}a$$

$$\lambda_+ = \frac{1}{2}\Big[(\lambda_a + \lambda_c) + \text{sign}(\Delta m^2_{ee})\sqrt{(\lambda_a - \lambda_c)^2 + 4\Big|s_{13}c_{13}\Delta m^2_{ee} + \frac{\epsilon}{2}ak_{13}c_{24}c_{34}e^{-i\delta_{34}}\Big|^2}\Big]$$

$$(\hat{H})_{12} = \epsilon\Big\{s_{12}c_{12}(c_{13}\tilde{c}_{13} + s_{13}\tilde{s}_{13}e^{-i\alpha_{13}})\Delta m^2_{ee} + \frac{a}{2}\Big[k_{12}c_{24}c^2_{34}\tilde{c}_{13} - k_{23}c_{34}\tilde{s}_{13}e^{i(\delta_{34}+\alpha_{13})}\Big]e^{-i\delta_{24}}\Big\}$$

$$(\hat{H})_{23} = \epsilon\Big\{s_{12}c_{12}(-s_{13}\tilde{c}_{13} + c_{13}\tilde{s}_{13}e^{i\alpha_{13}})\Delta m^2_{ee} + \frac{a}{2}\Big[k_{12}c_{24}c^2_{34}\tilde{s}_{13}e^{i\alpha_{13}} + k_{23}c_{34}\tilde{c}_{13}e^{i\delta_{34}}\Big]e^{i\delta_{24}}\Big\}$$

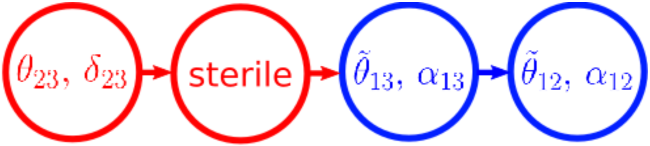
Step 4: Continued

Diagonalize the (1-2) sector of $\hat{H}$ by implementing a complex rotation $U_{12}(\tilde{\theta}_{12}, \alpha_{12})$

$$\hat{H} \Rightarrow \check{H} \equiv U_{12}^{\dagger}(\tilde{\theta}_{12}, \alpha_{12})\, \hat{H}\, U_{12}(\tilde{\theta}_{12}, \alpha_{12})$$

$$\tilde{\theta}_{12} = \frac{1}{2} \arccos \frac{\lambda_0 - \lambda_-}{\sqrt{|\lambda_0 - \lambda_-|^2 + 4|(\hat{H})_{12}|^2}}$$

$$\alpha_{12} = \mathrm{Arg}\left[(\hat{H})_{12}\right]$$



0^{th} order PMNS matrix and Hamiltonian

In the 3+1 scheme

$$V^{(0)} = U_{23} U_{34} U_{24} U_{14} U_{13} U_{12}$$

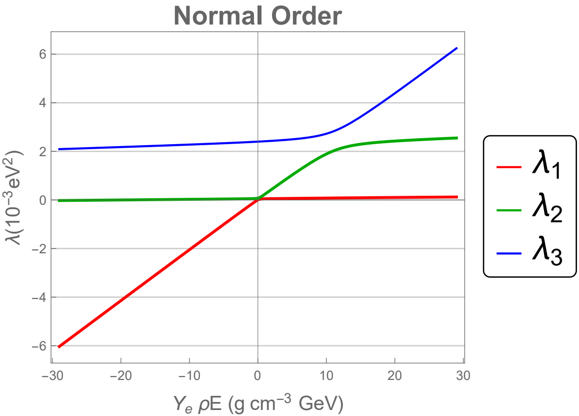
$$\check{H} = \underbrace{\frac{1}{2E} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{pmatrix}}_{\check{H}_0} + \check{H}_1$$

All diagonal elements of $\check{H}$ have been absorbed into the 0^{th} order Hamiltonian $\check{H}_0$

0^{th} order eigenvalues in the active neutrino space

$$\lambda_{1,2} = \frac{1}{2}\left[(\lambda_- + \lambda_0) \mp \sqrt{(\lambda_- - \lambda_0)^2 + 4|(\hat{H})_{12}|^2}\right]$$

$$\lambda_3 = \lambda_+$$



Active sectors of the perturbative Hamiltonian

All the diagonal elements of $\check{H}_1$ vanish, the off-diagonal elements in the sector of the active neutrinos (first three rows and columns) are

$$(\check{H}_1)_{12} = 0$$
$$(\check{H}_1)_{13} = -\tilde{s}_{12}(\hat{H})_{23}\, e^{-i\alpha_{12}}$$
$$(\check{H}_1)_{23} = \tilde{c}_{12}(\hat{H})_{23}$$

Since $(\hat{H})_{23} \sim \mathcal{O}(\epsilon)$, sectors of the active neutrinos $\sim \mathcal{O}(\epsilon)$

Sectors of the sterile neutrino

Crossings of the eigenvalues to λ_4

$$\lambda_4 = \Delta m_{41}^2 - \frac{a}{2} c_{14}^2 c_{24}^2 c_{34}^2 \gg \Delta m_{ee}^2 \sim a$$

Crossings to λ_4 only happen with very high neutrino energy ($E \gg 10 GeV$), we are not interested in this energy scale.

4th row and column of $\check{H}_1$

Elements in the 4th row and column of the perturbative Hamiltonian

$$(\check{H}_1)_{i4} \propto \frac{a\, s_{i4}}{2E} \sim \mathcal{O}(\sqrt{\epsilon}),\ i = 1, 2, 3$$

However, they are not going to give $\mathcal{O}(\sqrt{\epsilon})$ corrections, because in perturbative expressions they will be divided by λ_4.

Important special cases

Back to exact values in vacuum

In vacuum, $a = 0$, the 0^{th} order approximations will give exact vacuum values, i.e. $\tilde{\theta}_{13,12} = \theta_{13,12}$, $\alpha_{13,12} = 0$, $\lambda_i = \Delta m_{i1}^2$ and $\check{H}_1 = 0$.

Related to the Standard Model

When $U_{\text{sterile}} = \mathbb{1}$, i.e. $s_{i4} = 0$ in the 3+1 scheme, the results go to the DMP for the SM.

Perturbative expansion: Corrections to the eigenvalues

$$\lambda_i^{(\text{ex})} = \lambda_i + \lambda_i^{(1)} + \lambda_i^{(2)} + \cdots$$

$\lambda_i^{(n)}$ are the n^{th} order corrections.

$$\lambda_i^{(1)} = 2E(\check{H}_1)_{ii}$$

Since $\check{H}_1$ has zero diagonal elements, the first order corrections are trivial.

$$\lambda_i^{(2)} = \sum_{k \neq i} \frac{|2E(\check{H}_1)_{ik}|^2}{\lambda_i - \lambda_k}$$

If $i, k \in \{1, 2, 3\}$, $|(\check{H}_1)_{ik}|^2$ will be zero or in scale of ϵ^2. Otherwise either λ_i or λ_k will be λ_4, then the denominator will be $\gtrsim \epsilon^{-1}$, moreover, since $(\check{H}_1)_{i4} \sim \sqrt{\epsilon}$, the square in the numerator provides another necessary ϵ.

Perturbative expansion: Corrections to the eigenstates

$$V^{(ex)} = V^{(0)}(\mathbb{1} + W_1 + W_2 + \cdots)$$

W_n are n^{th} order corrections.

$$(W_1)_{ij} = \begin{cases} 0, & i = j \\ -\frac{2E(\check{H}_1)_{ij}}{\lambda_i - \lambda_j}, & i \neq j \end{cases}$$

Again if $i, k \in \{1, 2, 3\}$, $(\check{H}_1)_{ik}$ will be zero or in scale of ϵ, otherwise either λ_i or λ_k will be λ_4, then the denominator will be $\gtrsim \epsilon^{-1}$.

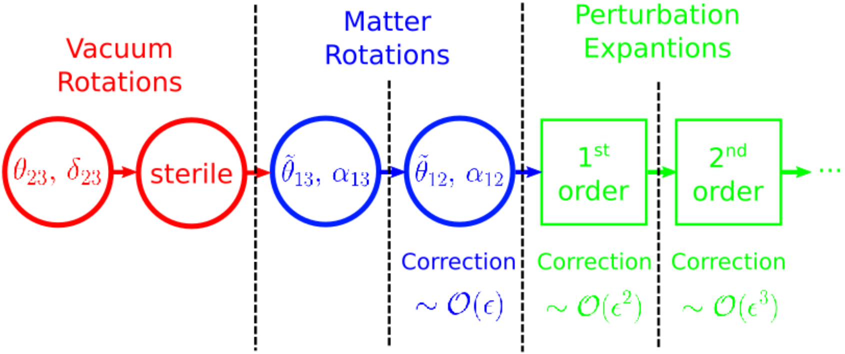
Perturbative expansion: eigenstates continued

$$(W_2)_{ij} = \begin{cases} -\frac{1}{2}\sum_{k\neq i}\frac{|2E(\check{H}_1)_{ik}|^2}{(\lambda_i-\lambda_k)^2}, & i=j \\ \frac{1}{\lambda_i-\lambda_j}\sum_{k\neq i,j}\frac{2E(\check{H}_1)_{ik}\,2E(\check{H}_1)_{kj}}{\lambda_k-\lambda_j}, & i\neq j \end{cases}$$

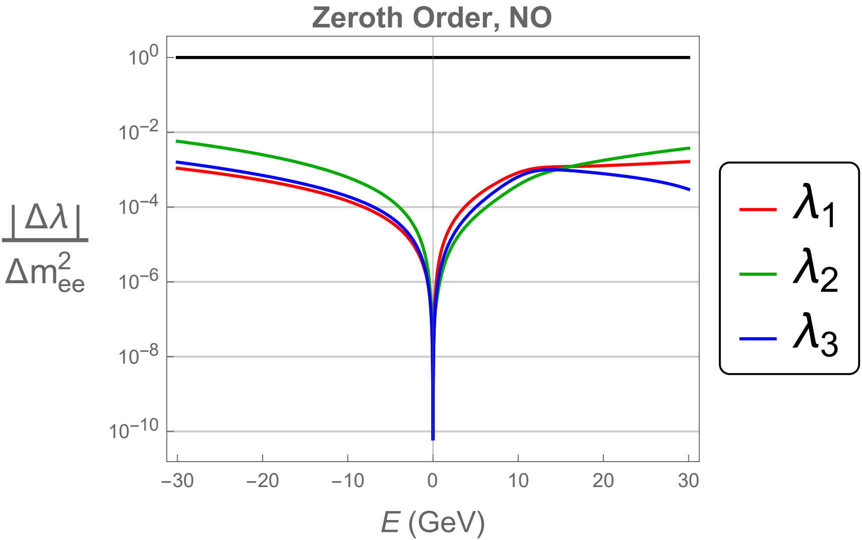
It is a little more complicated to confirm the scale of W_2.

- $i=j$ if $i=4$, the denominator will be $\gtrsim \epsilon^{-2}$; if $i=j\neq 4$ and $k\neq 4$ the numerator will be $\sim \epsilon^2$; if $i=j\neq 4$ and $k=4$, the denominator will be $\gtrsim \epsilon^{-2}$;
- $i\neq j$ if $i,j,k\in\{1,2,3\}$, the numerator will be $\sim \epsilon^2$; if $i=4$ or $j=4$, the denominator will be $\gtrsim \epsilon^{-1}$ and the numerator will be $\sim \epsilon^{3/2}$; if $k=4$, the denominator will be $\gtrsim \epsilon^{-1}$ and the numerator will be $\sim \epsilon$

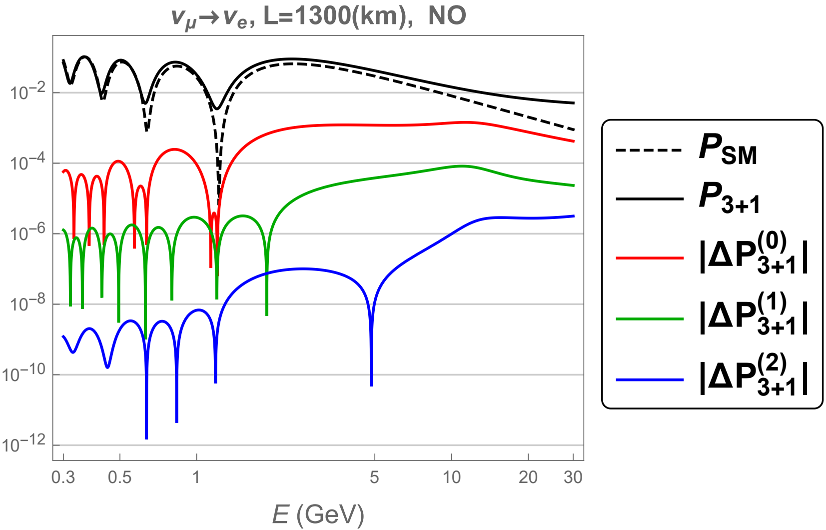
Review of the calculation process



Precision test: active eigenvalues

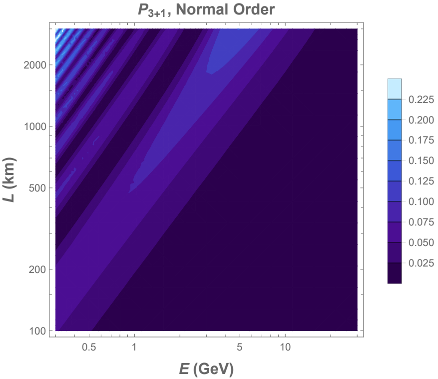


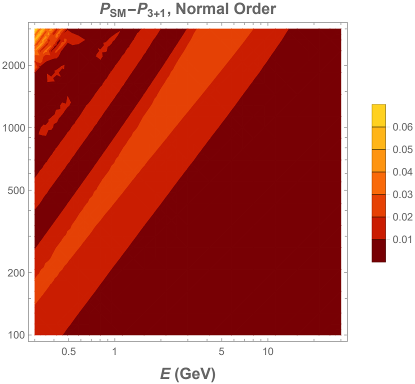
Presion test: oscillation possibilities



Possiblities shift from the Standard Model

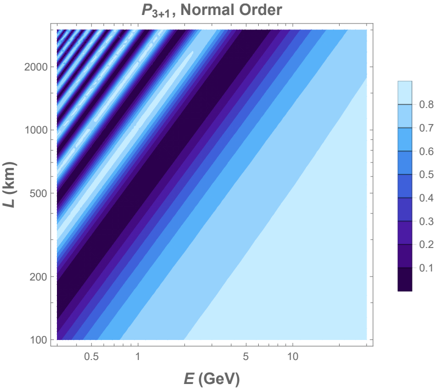
$$\nu_\mu \rightarrow \nu_e$$

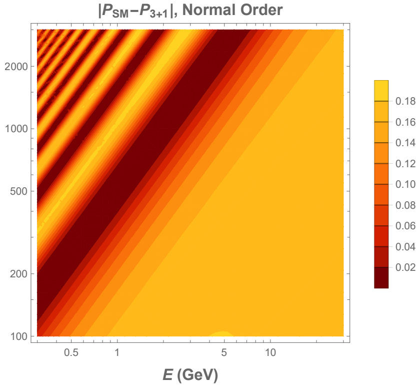




Possiblities shift from the Standard Model (Continued)

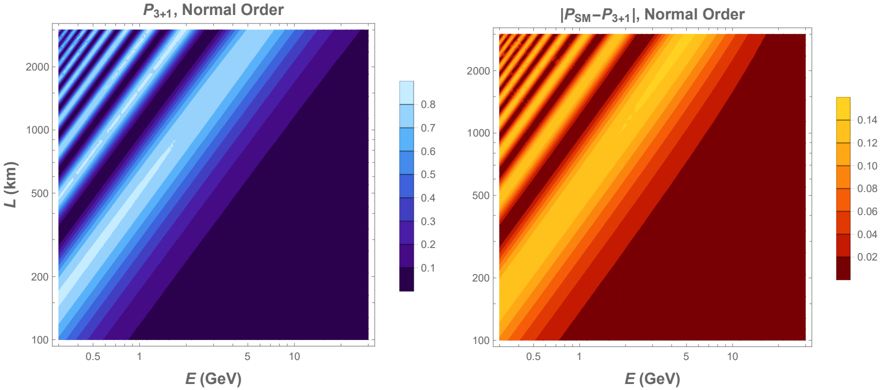
$$\nu_\mu \to \nu_\mu$$





Possiblities shift from the Standard Model (Continued)

$$\nu_\mu \rightarrow \nu_\tau$$



Summary

Resolve crossings of the 0^{th} order eigenvalues in the active neutrino space (crossings to sterile eigenvalues require very high neutrino energy)

Exact in vacuum

Accurate enough for current/future experiment