FEM IMPLEMENTATION FOR NAVIER-STOKES EQUATIONS FOR INCOMPRESSIBLE STEADY FLOW USING HIERARCHICAL BASIS FUNCTIONS

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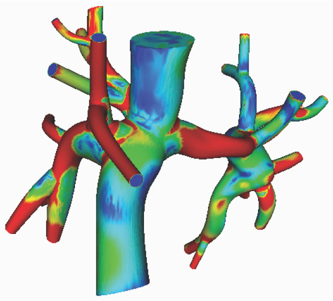
INTRODUCTION

Navier-Stokes equations permit to model:

- blood flow in the cardiovascular system
- air flow around airplane wings, rotor blades, vehicles, etc.
- ocean and atmospheric currents
- . . . and much more



Blood flow simulation
www.sinews.siam.org

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- blood flow in the cardiovascular system
- air flow around airplane wings, rotor blades, vehicles, etc.
- ocean and atmospheric currents
- . . . and much more

Flow around an aerofoil
`www.wikipedia.org`

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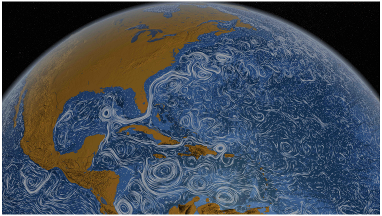


Visualisation of Golf Stream
NASA/Goddard Space Flight Center
Scientific Visualization Studio

PROBLEM STATEMENT

- Incompressible Newtonian fluid:

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + \mu\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathsf{T}}\right)$$

- Hydrostatic fluid pressure:

$$p = -\frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)$$

- Continuity equation:

$$\nabla \cdot \boldsymbol{u} = 0$$

- Balance of the momentum:

$$\underbrace{\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u}\right)}_{\text{inertia forces}} = \underbrace{\mu\nabla^2\boldsymbol{u}}_{\text{viscous forces}} - \nabla p + \boldsymbol{f}$$

$\boldsymbol{u}$ – velocity
ρ – density
μ – viscosity

Reynolds number:

$$\mathcal{R} = \frac{\text{inertia forces}}{\text{viscous forces}}$$

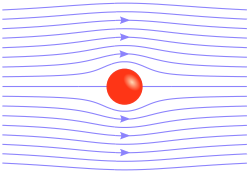
- Navier-Stokes and continuity equations:

$$\begin{cases} \rho\left(\boldsymbol{u}\cdot\nabla\right)\boldsymbol{u} - \mu\nabla^2\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega \\ \nabla\cdot\boldsymbol{u} = 0 \end{cases}$$

- Boundary conditions:

$$\begin{cases} \boldsymbol{u} = \boldsymbol{u}_D & \text{on } \Gamma_D \\ \boldsymbol{n}\cdot\left[-p\boldsymbol{I} + \mu\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\intercal}\right)\right] = \boldsymbol{g}_N & \text{on } \Gamma_N \end{cases}$$

- Stokes flow: $\mathcal{R} \ll 1$



Adopted from `www.wikipedia.org`

FINITE-ELEMENT IMPLEMENTATION

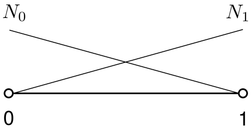
- Variational formulation:
  Find a vector field $\boldsymbol{u} \in \mathbf{V}$ and a scalar field $p \in \mathcal{P}$ s.t.
  $\forall\ \boldsymbol{v} = [v_1, v_2, v_3]^\intercal \in \mathbf{V}$ and $\forall\ q \in \mathcal{P}$ (test functions):

$$\int_\Omega \rho\,(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u}\cdot\boldsymbol{v}\,d\Omega + \int_\Omega \mu\,\nabla\boldsymbol{u} : \nabla\boldsymbol{v}\,d\Omega - \int_\Omega p\,\nabla\cdot\boldsymbol{v}\,d\Omega - \int_\Omega q\,\nabla\cdot\boldsymbol{u}\,d\Omega = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,d\Omega + \int_{\Gamma_N} \boldsymbol{g}_N\cdot\boldsymbol{v}\,d\Gamma_N$$

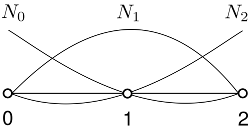
- Interpolation with shape functions:

$$u_i = \sum_{\alpha=1}^{n_{\boldsymbol{u}}} N_\alpha\, u_i^\alpha, \quad p = \sum_{\beta=1}^{n_p} \Phi_\beta\, p^\beta; \quad v_i = \sum_{\alpha=1}^{n_{\boldsymbol{u}}} N_\alpha\, v_i^\alpha, \quad q = \sum_{\beta=1}^{n_p} \Phi_\beta\, q^\beta$$

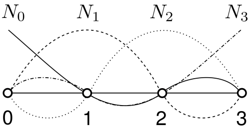
- Standard shape functions:



linear



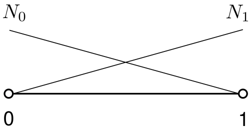
second order



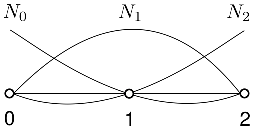
third order

HIERARCHICAL SHAPE FUNCTIONS (1D ELEMENT)

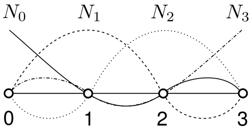
- Standard shape functions:



linear

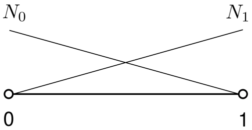


second order

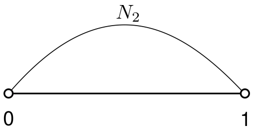


third order

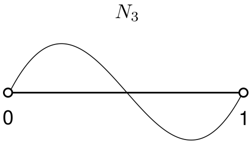
- Hierarchical shape functions:



linear



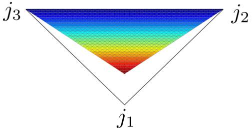
second order



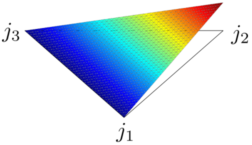
third order

HIERARCHICAL SHAPE FUNCTIONS (2D ELEMENT)

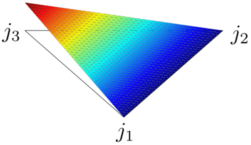
- Hierarchical shape functions for *vertices* (first order):



vertex j_1

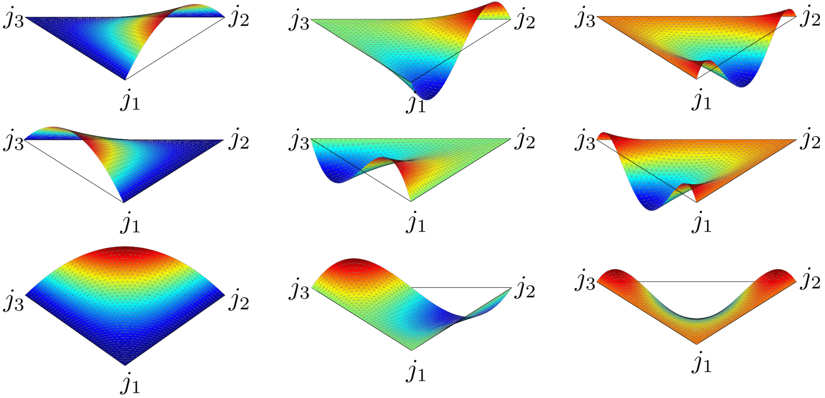


vertex j_2



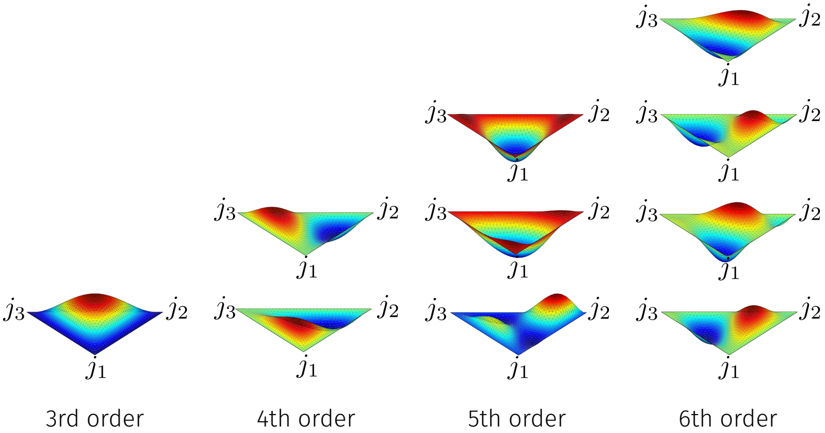
vertex j_3

- Hierarchical shape functions for *edges*:



2nd order 3rd order 4th order

- Hierarchical shape functions for *faces*:



- Decomposition of DOFs and shape functions vectors:

$$\boldsymbol{u}^{\mathrm{el}} = \left[\boldsymbol{u}^1, \dots \boldsymbol{u}^\alpha, \dots \boldsymbol{u}^{n_{\boldsymbol{u}}}\right]^{\mathsf{T}} = \left[\boldsymbol{u}^{\mathrm{ver}}, \boldsymbol{u}^{\mathrm{edge}}, \boldsymbol{u}^{\mathrm{face}}, \boldsymbol{u}^{\mathrm{vol}}\right]^{\mathsf{T}}$$

$$\boldsymbol{N}^{\mathrm{el}} = [N_1, \dots N_\alpha, \dots N_{n_{\boldsymbol{u}}}]^{\mathsf{T}} = [\boldsymbol{N}_{\mathrm{ver}}, \boldsymbol{N}_{\mathrm{edge}}, \boldsymbol{N}_{\mathrm{face}}, \boldsymbol{N}_{\mathrm{vol}}]^{\mathsf{T}}$$

- Decomposition of DOFs and shape functions vectors:

$$\boldsymbol{u}^{\text{el}} = \left[\boldsymbol{u}^{1}, \dots \boldsymbol{u}^{\alpha}, \dots \boldsymbol{u}^{n_u}\right]^{\mathsf{T}} = \left[\boldsymbol{u}^{\text{ver}}, \boldsymbol{u}^{\text{edge}}, \boldsymbol{u}^{\text{face}}, \boldsymbol{u}^{\text{vol}}\right]^{\mathsf{T}}$$

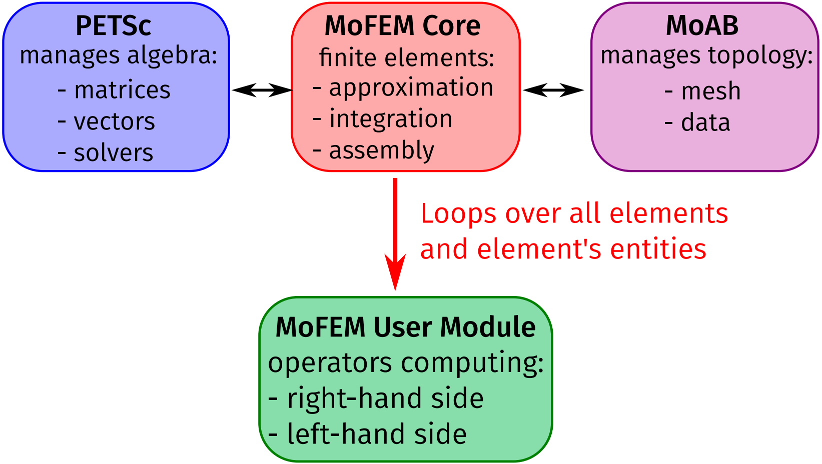
$$\boldsymbol{N}^{\text{el}} = [N_1, \dots N_{\alpha}, \dots N_{n_{\boldsymbol{u}}}]^{\mathsf{T}} = [\boldsymbol{N}_{\text{ver}}, \boldsymbol{N}_{\text{edge}}, \boldsymbol{N}_{\text{face}}, \boldsymbol{N}_{\text{vol}}]^{\mathsf{T}}$$

- Element stiffness matrix:

$$\mathbf{K}^{\text{el}} = \int\limits_{\Omega^{\text{el}}} \left(\nabla \mathbf{N}^{\text{el}}\right)^{\mathsf{T}} \nabla \mathbf{N}^{\text{el}} \text{d}\Omega^{\text{el}}$$

$$(\nabla \mathbf{N}^{\text{el}})^{\mathsf{T}} \nabla \mathbf{N}^{\text{el}} =$$

$$\begin{bmatrix} \nabla \mathbf{N}_{\text{ver}}{}^{\mathsf{T}} \nabla \mathbf{N}_{\text{ver}} & \nabla \mathbf{N}_{\text{ver}}{}^{\mathsf{T}} \nabla \mathbf{N}_{\text{edge}} & \nabla \mathbf{N}_{\text{ver}}{}^{\mathsf{T}} \nabla \mathbf{N}_{\text{face}} & \nabla \mathbf{N}_{\text{ver}}{}^{\mathsf{T}} \nabla \mathbf{N}_{\text{vol}} \\ \nabla \mathbf{N}_{\text{edge}}{}^{\mathsf{T}} \nabla \mathbf{N}_{\text{ver}} & \nabla \mathbf{N}_{\text{edge}}{}^{\mathsf{T}} \nabla \mathbf{N}_{\text{edge}} & \nabla \mathbf{N}_{\text{edge}}{}^{\mathsf{T}} \nabla \mathbf{N}_{\text{face}} & \nabla \mathbf{N}_{\text{edge}}{}^{\mathsf{T}} \nabla \mathbf{N}_{\text{vol}} \\ \nabla \mathbf{N}_{\text{face}}{}^{\mathsf{T}} \nabla \mathbf{N}_{\text{ver}} & \nabla \mathbf{N}_{\text{face}}{}^{\mathsf{T}} \nabla \mathbf{N}_{\text{edge}} & \nabla \mathbf{N}_{\text{face}}{}^{\mathsf{T}} \nabla \mathbf{N}_{\text{face}} & \nabla \mathbf{N}_{\text{face}}{}^{\mathsf{T}} \nabla \mathbf{N}_{\text{vol}} \\ \nabla \mathbf{N}_{\text{vol}}{}^{\mathsf{T}} \nabla \mathbf{N}_{\text{ver}} & \nabla \mathbf{N}_{\text{vol}}{}^{\mathsf{T}} \nabla \mathbf{N}_{\text{edge}} & \nabla \mathbf{N}_{\text{vol}}{}^{\mathsf{T}} \nabla \mathbf{N}_{\text{face}} & \nabla \mathbf{N}_{\text{vol}}{}^{\mathsf{T}} \nabla \mathbf{N}_{\text{vol}} \end{bmatrix}$$


PETSc
manages algebra:
- matrices
- vectors
- solvers
MoFEM Core
finite elements:
- approximation
- integration
- assembly
MoAB
manages topology:
- mesh
- data
Loops over all elements
and element's entities
MoFEM User Module
operators computing:
- right-hand side
- left-hand side

RIGHT-HAND SIDE OPERATORS (RESIDUAL VECTOR)

- Components corresponding to velocity DOFs $\boldsymbol{u}^{\alpha}, \alpha = \overline{1, n_{\boldsymbol{u}}}$

$$\mathbf{R}^{\text{el}}_{\boldsymbol{u}^{\alpha}} = \sum_{\beta,\gamma=1}^{n_{\boldsymbol{u}}} \int\limits_{\Omega^{\text{el}}} \rho N_{\alpha} \left(N_{\beta} \boldsymbol{u}^{\beta} \cdot \nabla N_{\gamma} \right) \boldsymbol{u}^{\gamma} \, d\Omega^{\text{el}} +$$

$$+ \sum_{\beta=1}^{n_{\boldsymbol{u}}} \int\limits_{\Omega^{\text{el}}} \mu \left(\nabla N_{\alpha} \cdot \nabla N_{\beta} \right) \boldsymbol{u}^{\beta} \, d\Omega^{\text{el}} - \sum_{\beta=1}^{n_p} \int\limits_{\Omega^{\text{el}}} \nabla N_{\alpha} \Phi_{\beta} p^{\beta} \, d\Omega^{\text{el}}$$

$$+ \int\limits_{\Omega^{\text{el}}} \boldsymbol{f} N_{\alpha} \, d\Omega^{\text{el}} + \int\limits_{\Gamma^{\text{el}}_{N}} \boldsymbol{g}_{\boldsymbol{N}} N_{\alpha} \, d\Gamma^{\text{el}}_{N}$$

- Components corresponding to velocity DOFs $\boldsymbol{u}^{\alpha}, \alpha = \overline{1, n_{\boldsymbol{u}}}$

$$\mathbf{R}^{\text{el}}_{\boldsymbol{u}^{\alpha}} = \overbrace{\sum_{\beta,\gamma=1}^{n_{\boldsymbol{u}}} \int\limits_{\Omega^{\text{el}}} \rho N_{\alpha} \left(N_{\beta} \boldsymbol{u}^{\beta} \cdot \nabla N_{\gamma} \right) \boldsymbol{u}^{\gamma} \, d\Omega^{\text{el}}}^{\text{non-linear part (Navier-Stokes)}} + \underbrace{+ \sum_{\beta=1}^{n_{\boldsymbol{u}}} \int\limits_{\Omega^{\text{el}}} \mu \left(\nabla N_{\alpha} \cdot \nabla N_{\beta} \right) \boldsymbol{u}^{\beta} \, d\Omega^{\text{el}} - \sum_{\beta=1}^{n_p} \int\limits_{\Omega^{\text{el}}} \nabla N_{\alpha} \Phi_{\beta} p^{\beta} \, d\Omega^{\text{el}}}_{\text{linear part (Stokes)}} + \int\limits_{\Omega^{\text{el}}} \boldsymbol{f} N_{\alpha} \, d\Omega^{\text{el}} + \int\limits_{\Gamma^{\text{el}}_{N}} \boldsymbol{g}_{\boldsymbol{N}} N_{\alpha} \, d\Gamma^{\text{el}}_{N}$$

- Components corresponding to velocity DOFs $\boldsymbol{u}^{\alpha}, \alpha = \overline{1, n_{\boldsymbol{u}}}$

$$\mathbf{R}^{\text{el}}_{\boldsymbol{u}^{\alpha}} = \overbrace{\sum_{\beta,\gamma=1}^{n_{\boldsymbol{u}}} \int\limits_{\Omega^{\text{el}}} \rho N_{\alpha} \left(N_{\beta} \boldsymbol{u}^{\beta} \cdot \nabla N_{\gamma}\right) \boldsymbol{u}^{\gamma} \, d\Omega^{\text{el}}}^{\text{non-linear part (Navier-Stokes)}} +$$

$$\underbrace{+ \sum_{\beta=1}^{n_{\boldsymbol{u}}} \int\limits_{\Omega^{\text{el}}} \mu \left(\nabla N_{\alpha} \cdot \nabla N_{\beta}\right) \boldsymbol{u}^{\beta} \, d\Omega^{\text{el}} - \sum_{\beta=1}^{n_{p}} \int\limits_{\Omega^{\text{el}}} \nabla N_{\alpha} \Phi_{\beta} p^{\beta} \, d\Omega^{\text{el}}}_{\text{linear part (Stokes)}}$$

$$+ \int\limits_{\Omega^{\text{el}}} \boldsymbol{f} N_{\alpha} \, d\Omega^{\text{el}} + \int\limits_{\Gamma^{\text{el}}_{N}} \boldsymbol{g}_{\boldsymbol{N}} N_{\alpha} \, d\Gamma^{\text{el}}_{N}$$

- Components corresponding to pressure DOFs $p^{\alpha}, \alpha = \overline{1, n_{p}}$

$$\mathbf{R}^{\text{el}}_{p^{\alpha}} = - \sum_{\beta=1}^{n_{\boldsymbol{u}}} \int\limits_{\Omega^{\text{el}}} \Phi_{\alpha} \nabla N_{\beta} \cdot \boldsymbol{u}^{\beta} \, d\Omega^{\text{el}}$$

LEFT-HAND SIDE OPERATORS (TANGENT MATRIX)

$$\mathbf{K}^{\mathrm{el}}_{\boldsymbol{u}^\alpha \boldsymbol{u}^\beta} = \overbrace{\sum_{\gamma=1}^{n_{\boldsymbol{u}}} \int\limits_{\Omega^{\mathrm{el}}} \rho N_\alpha \left[\left(N_\gamma \boldsymbol{u}^\gamma \cdot \nabla N_\beta \right) \boldsymbol{I} + N_\beta \nabla N_\gamma \otimes \boldsymbol{u}^\gamma \right] d\Omega^{\mathrm{el}} +}^{\text{non-symmetric (Navier-Stokes)}}$$

$$\underbrace{+ \int\limits_{\Omega^{\mathrm{el}}} \mu \left(\nabla N_\alpha \cdot \nabla N_\beta \right) \; \boldsymbol{I} \, d\Omega^{\mathrm{el}}}_{\text{symmetric (Stokes)}}$$

$$\mathbf{K}^{\mathrm{el}}_{\boldsymbol{u}^\alpha \boldsymbol{p}^\beta} = - \sum_{\beta=1}^{n_p} \int\limits_{\Omega^{\mathrm{el}}} \nabla N_\alpha \Phi_\beta \, d\Omega^{\mathrm{el}}$$

$$\mathbf{K}^{\mathrm{el}}_{\boldsymbol{p}^\alpha \boldsymbol{u}^\beta} = - \sum_{\beta=1}^{n_{\boldsymbol{u}}} \int\limits_{\Omega^{\mathrm{el}}} \Phi_\alpha \nabla N_\beta \, d\Omega^{\mathrm{el}}$$

GLOBAL SYSTEM OF EQUATIONS

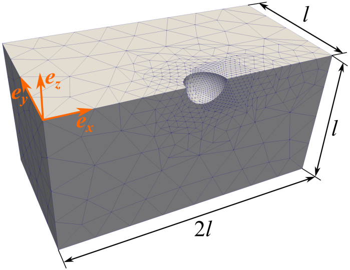
$$\begin{bmatrix} \mathbf{K}^{\mathrm{S}}_{\boldsymbol{uu}} + \mathbf{K}^{\mathrm{NS}}_{\boldsymbol{uu}}(\boldsymbol{u}) & \mathbf{K}^{\mathrm{S}}_{\boldsymbol{u}p} \\ \mathbf{K}^{\mathrm{S}}_{p\boldsymbol{u}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{u} \\ \Delta p \end{bmatrix} = \begin{bmatrix} \mathbf{R}^{\mathrm{S}}_{\boldsymbol{u}}(\boldsymbol{u}, p) + \mathbf{R}^{\mathrm{NS}}_{\boldsymbol{u}}(\boldsymbol{u}) \\ \mathbf{R}^{\mathrm{S}}_{p}(\boldsymbol{u}) \end{bmatrix}$$

- Nested matrix representation
- Use field-split preconditioner
- Need to invert only $\mathbf{K}^{\mathrm{NS}}_{\boldsymbol{uu}}$ on every iteration of Newton-Raphson method
- Use multi-grid method

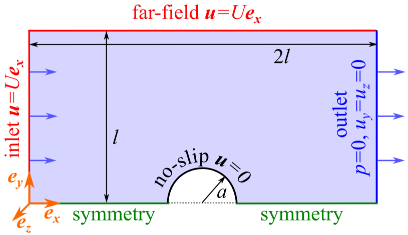
Elman H. et al, *Journal of Computational Physics* (2008)

NUMERICAL EXAMPLE

- Simulation of the fluid flow around a rigid sphere



FE mesh: $l = 10$ [m]



Sketch on the section $z = 0$:

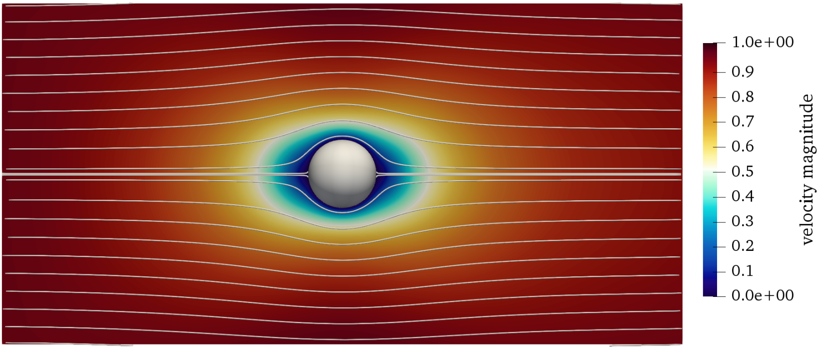
$a = 1$ [m] and $l = 10$ [m]

$U = 0.005 \dots 150$ [m/s]

$\mathcal{R} = 0.01 \dots 300$

- Reynolds number

$$\mathcal{R} = \frac{2a\,U\rho}{\mu} \left(= \frac{\text{inertia forces}}{\text{viscous forces}} \right)$$



$$Re = 0.01$$

Definitions of a vortex core [1]:

- maximal vorticity

$$\max ||\nabla \times \boldsymbol{u}||$$

- Q-criterion

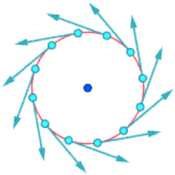
$$Q = ||\nabla^{\mathrm{A}} \boldsymbol{u}||^2 - ||\nabla^{\mathrm{S}} \boldsymbol{u}||^2 > 0$$

- λ_2-criterion

$$\lambda_2 < 0,$$

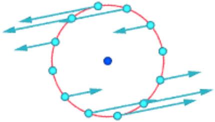
where $\lambda_1 \geq \lambda_2 \geq \lambda_3$ are

eigen values of $\left(\nabla^{\mathrm{S}} \boldsymbol{u}\right)^2 + \left(\nabla^{\mathrm{A}} \boldsymbol{u}\right)^2$



www.wikipedia.org

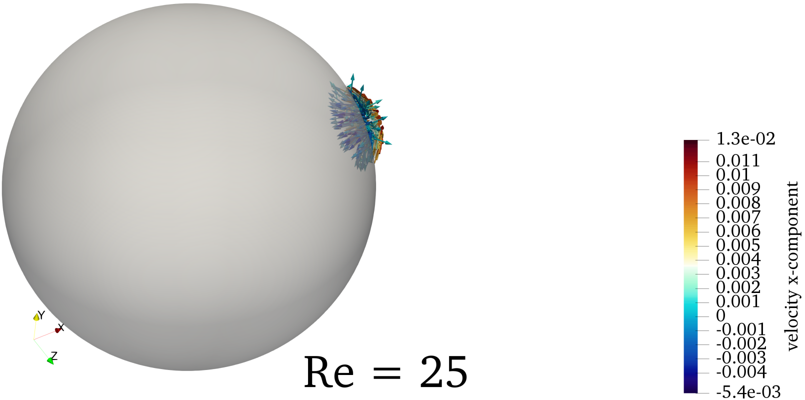
Rotation: $||\nabla \times \boldsymbol{u}|| > 0$

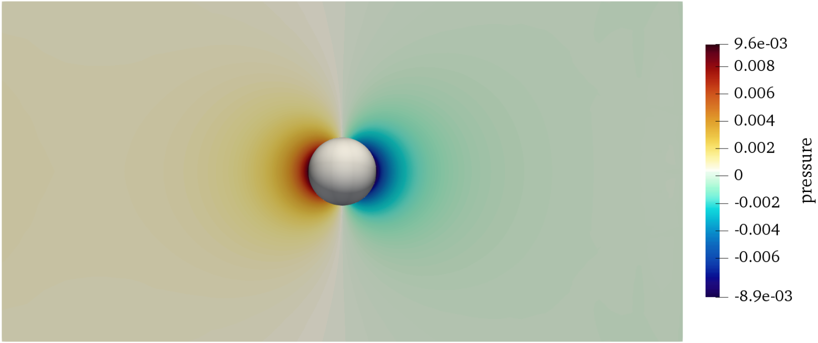


www.wikipedia.org

Shear: $||\nabla \times \boldsymbol{u}|| > 0$

[1] Jeong J, Hussain F, *Journal of Fluid Mechanics* (1995)


Y
X
Z
Re = 25
1.3e-02
0.011
0.01
0.009
0.008
0.007
0.006
0.005
0.004
0.003
0.002
0.001
0
-0.001
-0.002
-0.003
-0.004
-5.4e-03
velocity x-component



$$\mathrm{Re} = 0.01$$

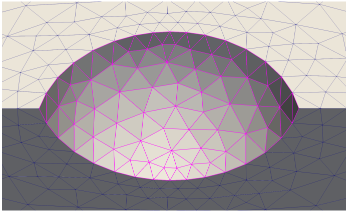
- Traction vector on the surface of the sphere Γ_S:

$$\boldsymbol{t} = -p\boldsymbol{I} + \mu\,(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\mathsf{T})$$

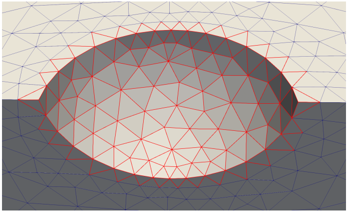
- Total drag force acting on the sphere:

$$F_\mathrm{D} = -\int_{\Gamma_\mathrm{S}} \boldsymbol{t}\; d\Gamma_\mathrm{S}$$

- **NB1:** $\nabla \boldsymbol{u}$ is computed using shape functions of the adjacent tetrahedra
- **NB2:** shape functions order can be increased in the boundary layer



Faces on surface of the sphere



Layer of adjacent tetrahedra

- Drag force in the Stokes flow (*Stokes formula* for $\mathcal{R} \ll 1$):

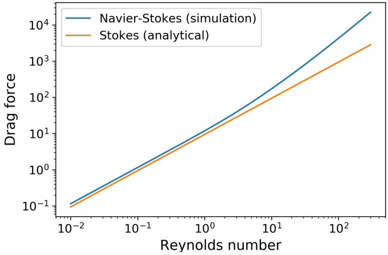
$$F_{\mathrm{D}} = 6\pi\mu a U$$

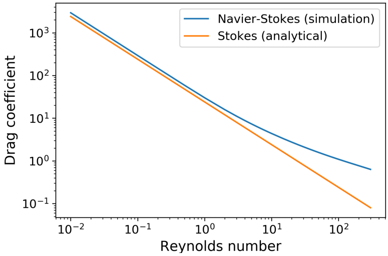
- Drag coefficient (dimensionless drag force):

$$C_{\mathrm{D}} = \frac{F_{\mathrm{D}}}{\frac{\rho U^2}{2}\pi a^2}$$

- For the Stokes flow:

$$C_{\mathrm{D}} = \frac{24}{\mathcal{R}}$$

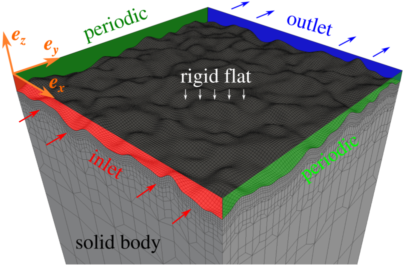




PERSPECTIVES

Simulation of:

- contact between deformable solid with a **rough surface** and a rigid flat
- with pressure-driven flow in the interface
- permits to study evolution of transmissivity



A. Shvarts, PhD Thesis, MINES ParisTech (2019)

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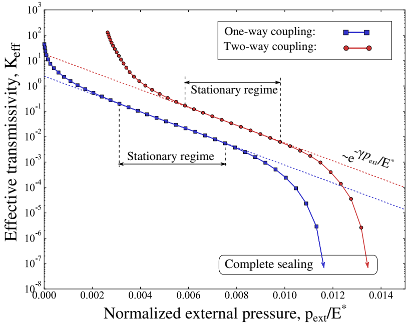
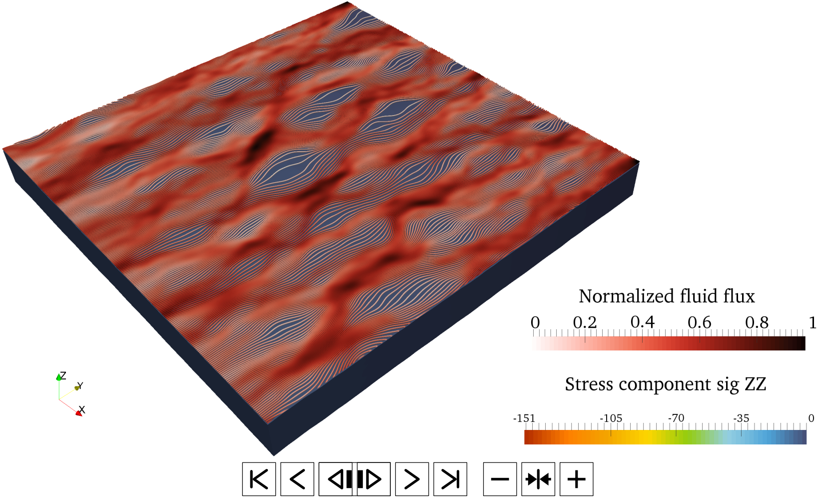


A. Shvarts, PhD Thesis, MINES ParisTech (2019)

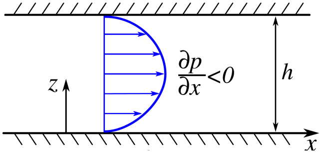


A. Shvarts, PhD Thesis, MINES ParisTech (2019)

Reynolds equation:

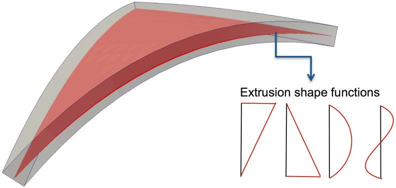
$$\nabla \cdot \left[\frac{h^3}{12\mu} \nabla p \right] = 0$$

- $h(x, y)$ is the film thickness
- p is constant across the film
- parabolic profile of the velocity:



Navier-Stokes equations:

- available simulations show difference in flow velocity
  S. Brown et al, *Geoph. Res. Lett. (1995)*
- Proposition: single layer of **prism elements** with hierarchical approximation across thickness



L. Kaczmarczyk et al., *Proceedings of 24th UKACM Conference (2016)*

CONCLUSIONS

- Implementation of Navier-Stokes equations in the finite-element framework using hierarchical basis approximation
- Validation on a problem of the fluid flow around a rigid sphere for different Reynolds numbers
- Perspective application to the study of fluid flow in contact interfaces between solids with rough surfaces

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Thank you for your attention!