

Relating quantization and time evolution in the presence of gauge symmetry

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Abstract

Up to now there is no definition or an example of a gauge-invariant quantum field theory in four-dimensional space-time, with mathematical rigor and in the absence of approximations. This is mainly due to three features of the theory: 1) the canonical quantization is not manifestly Lorentz invariant; 2) the Lagrangian is singular due to the gauge symmetry; and 3) the gaussian measure is not gauge invariant. At least one of these features becomes an obstacle to mathematical rigor in either the canonical or the path-integral quantization.

Using the fact that there is a wave-function associated to any probability distribution, we study a class of statistical field theories in four-dimensional space-time where the (classical) canonical coordinates when modified by the unitary time evolution, verify the canonical commutation relations. We show that these statistical field theories have all the features of a gauge-invariant quantum field theory in four-dimensional space-time. Thus the features 1), 2) and 3) are not obstacles to define the theory in our formalism.

1 Introduction

Schrödinger described quantization as the consequence of solving an eigenvalue problem for the Hamiltonian [1]: in an infinite-dimensional linear space of functions, continuous and discrete (i.e. quantized) energy spectra may coexist. Thus, from the very beginning there was a relation between the time evolution (defined by the Hamiltonian) and the notion of quantization.

There is no doubt that the best known description of the experimental data collected so far is based on a quantum theory [2]. However, the notion of quantization is not much clearer than it was in 1926 [3]. In this paper we will change this status, proposing a simple and mathematically meaningful definition of quantization. We start by addressing what quantization is not.

Quantization is not replacing Poisson bracket's by canonical commutation relations. The method of replacing Poisson bracket's by canonical commutation relations can always be applied (for analytical functions), it is called prequantization [4, 5]. However it doesn't lead by itself to useful results (hence the name prequantization). Of course, we can try to improve the method so that it leads to useful results (this is the geometrical quantization program [4]), however we

end up with a definition of quantization which is so complex and arbitrary that it is not useful in practice, in particular in the presence of gauge symmetries.

Quantization is also not second quantization (based on a Fock-space), which relates a quantum description of a single-particle system to a quantum description of a many-particle system [5]. We can only apply second quantization to a quantum theory, hence the name “second”.

Quantization is also not computing the Feynman’s path integral, since we know that the Feynman’s path integral does not have the property (sigma-additivity), which allows computation of the integral by approximating the integrand [6], and thus it is not an integral. Of course as in prequantization, we can try to improve the path integral [7], however we are very far from a consistent definition of path integral which is useful in practice.

Quantization is also not a perturbative expansion or a lattice regularization. These two different approximations are useful and have a clear definition, but since we know that they are complementary [6] then neither of them can be used to define quantization.

Note that there is enough experimental evidence to conclude that all the methods above mentioned —namely prequantization, second quantization, Feynman’s path integral, perturbative expansion, lattice regularization— are related to the quantum phenomena and thus they are necessarily related with the definition of quantization. But we insist that it is also clear that none of them by itself can be used to define quantization.

There is a big conceptual problem with the notion of quantization: we are trying to relate a deterministic theory (classical mechanics) with a non-deterministic theory (quantum mechanics). From the point of view of (classical) information theory [8], the root of probabilities (i.e. non-determinism) is the absence of information. Statistical methods are required whenever we lack complete information about a system, as so often occurs when the system is complex [9]. Thus we can convert a deterministic theory to a statistical theory unambiguously (using trivial probability distributions); but we cannot convert a statistical theory into a deterministic theory unambiguously since we need new information ¹.

On the other hand, the relation between quantum mechanics and a statistical theory (both are non-deterministic) is clear: the wave-function is a parametrization for any probability distribution [11]. It is a very useful parametrization because it allows us to represent a group of transformations using linear transformations on an hypersphere. Since these linear transformations have an intrinsically random nature, Quantum Mechanics is a generalization of Classical Mechanics (but not of probability theory, thanks to the wave-function’s collapse [11]).

Theories such as classical electrodynamics or more generally classical non-abelian gauge theories [12] involve a system of non-linear partial differential equations. It is a very hard problem to study in general the space of classical solutions of such systems². Even when a few solutions can be found, they may not be the ones that describe the physical system correctly. A consistent theory covering many cases only exists (at the moment) for systems of linear partial differential equations [14]. Thus to solve many non-linear deterministic theories we may not

¹E.g. the assumptions required by the deterministic models in reference [10] are new information.

²A well known example is the Navier-Stokes equation [13].

have better alternative (at the moment) than to consider them as a particular case of a statistical theory and apply linear quantum methods on its wave-function parametrization [15, 16]—then the building blocks of the overall deterministic theory are non-deterministic.

The non-commutativity of operators is thus intrinsic to any statistical theory. This saves us from the need to “deform” commutative algebras into non-commutative ones upon quantization. In our opinion, either the quantization of a classical theory or the classical limit of a quantum theory cannot go much beyond Koopman-von Neumann version of classical mechanics [15], i.e. a description of classical mechanics as a statistical theory (which is always possible, since a deterministic theory is a particular case of a statistical theory).

2 Statistical Source Field Theory

The method of quantization described in Section 3 is inspired by the Source formalism of Schwinger [17] which is itself both an alternative to and inspired by the Feynman’s path integral, where time-ordering [18, 19] plays a key role.

In this Section and in Section 3 we will consider fields defined in a one dimensional time, neglecting the space dimensions. The extension of the results of this section to fields defined in four dimensional space-time is discussed in Section 4.

We here use the term field meaning a function of time t . However, our fields are part of the phase-space of the theory: the state of the system is given by the functions of time t . Then, the time-evolution will modify the state of the system as a function of another parameter τ (which we also call time, the justification follows).

Therefore, our fields are best described as source fields and we are dealing with a statistical source field theory. Using a wave-function, we can parametrize the probability distribution for a source field in time. The linear space generated by all wave-functions is a Fock space [16]. The Fock space has the properties of a continuous tensor product of Fock-spaces corresponding to fields defined in infinitesimal time-intervals, i.e. $\varphi(t)dt$. The time-evolution will not only advance the time-intervals forward, but it will modify the wave functions corresponding to each time interval accordingly to an Hamiltonian which plays here the role of a covariant derivative. With abuse of language, we can describe the situation as a continuous tensor product of initial-value (i.e. Cauchy) problems, instead of just one initial value problem as in standard Quantum Mechanics.

We have the self-adjoint position $x(t)$ and momentum $p(t)$ operators, verifying the Weyl relations.

$$e^{i \int dt f(t)x(t)} e^{i \int ds g(s)p(s)} = e^{-i \int dt f(t)g(t)} e^{i \int ds g(s)p(s)} e^{i \int dt f(t)x(t)} \quad (1)$$

where f, g are real functions.

The Stone-von Neumann theorem implies that the Weyl relations uniquely define the unitary operators $e^{i \int dt f(t)x(t)}$ and $e^{i \int ds g(s)p(s)}$ up to a unitary transformation.

Thus, we can assume without loss of generality that the momentum and position operator satisfy the canonical commutation relations:

$$[p(t), x(\tau)] = i\delta(t - \tau) \quad (2)$$

We can define a unitary translation operator as $T(\tau)e^{i\int dt f(t)x(t)}T^\dagger(\tau) = e^{i\int dt f(t)x(t+\tau)}$ and acting on the momentum operator in an analogous way. We can express $T(\tau) = e^{i\frac{\tau}{2}\int dt p(t)\partial_t x(t) - x(t)\partial_t p(t)}$.

If the Hamiltonian is trivial (i.e. $H = p(t)\partial_t x(t) - x(t)\partial_t p(t)$), then the time-evolution is given by $T(a)$ which will merely advance forward the time-intervals of the Fock-space, without any other modification to the Fock space. Therefore, the parameter t from the phase-space and the parameter τ from the time-evolution are deeply related and thus we call both parameters time, although strictly speaking they play different roles in our framework.

But we may consider instead the Hamiltonian H and time-evolution U defined as:

$$H = \int dt \frac{1}{2}(p(t)\partial_t x(t) - x(t)\partial_t p(t)) + p^2(t) + V(x(t), t) \quad (3)$$

$$U(\tau) = e^{i\frac{\tau}{2}H} \quad (4)$$

Where $V(x(t), t)$ is a potential dependent on the position operator and possibly also time-dependent.

We will use now the Trotter exponential product approximation [20], verifying for small ϵ and A, B self-adjoint:

$$e^{i\epsilon A}e^{i\epsilon B} = e^{i\epsilon(A+B) - \frac{\epsilon^2}{2}[A, B] + i\mathcal{O}(\epsilon^3)} \quad (5)$$

This is a good approximation since it works for unbounded self-adjoint operators A, B .

Then the time evolution is:

$$U(\tau) = e^{i\int_0^\tau dz \int dt p^2(t) + V(x(t), t-z)}T(\tau) \quad (6)$$

Where the exponential above stands for the time-ordered (with parameter z) product. Thus the Fock-space parametrization of a statistical field theory allows us to implement the concept of time-ordering [18, 19] consistently.

If we relax the mathematical rigor for a moment and imagine a source field completely localized in one instant of time t , then the time-evolution (with time τ) of that source field could be described as a physical field function of time $\tau + t$ with initial conditions defined at time t . In this way we reproduce the formalism of Quantum Mechanics, both for a time-independent potential and also for a time-dependent potential.

The advantages of this more general formalism will be discussed in the next sections.

3 Quantization due to time evolution

We introduce now the procedure of quantization due to unitary time evolution. Time evolution transforms a sequence of time-ordered operators [18] (which commute algebraically but the time-ordering is non-commutative) into a sequence of (algebraically) non-commuting operators acting on a single slice of time of the wave-function. In particular, the canonical commutation relations of position and momentum are reproduced (strictly speaking, it is the Weyl relations that are reproduced, i.e. the exponentiated version of the canonical commutation relations).

We use again the Trotter exponential product approximation [20], verifying for small ϵ :

$$e^{i\epsilon A} e^{i\epsilon B} = e^{i\epsilon(A+B) - \frac{\epsilon^2}{2}[A,B] + i\mathcal{O}(\epsilon^3)} \quad (7)$$

This is a good approximation since it works for unbounded self-adjoint operators A, B .

Let now $\epsilon = \frac{1}{n}$ with n arbitrarily large. Then,

$$e^{i\epsilon A} e^{i\epsilon B} e^{-i\epsilon A} = (e^{i\epsilon A} e^{i\epsilon B} e^{-i\epsilon A})^n = e^{iB - \epsilon[A,B] + i\mathcal{O}(\epsilon^2)} \quad (8)$$

Therefore, for small enough ϵ

$$e^{i\epsilon \int d\tau p^2(\tau)} e^{i \int dt f(t)x(t)} e^{-i\epsilon \int d\tau p^2(\tau)} = e^{i \int dt f(t)(x(t) + \epsilon p(t))} \quad (9)$$

$$T(\epsilon) e^{i\epsilon \int d\tau p^2(\tau)} e^{i \int dt f(t)x(t-\epsilon)} e^{-i\epsilon \int d\tau p^2(\tau)} T^\dagger(\epsilon) = e^{i \int dt f(t)(x(t) + \epsilon p(t))} \quad (10)$$

Now we need a definition of covariant derivative in time of the position operator x , consistent with the fact that only $F(a) = U(a) e^{i \int dt f(t)x(t-a)} U^\dagger(a)$ (where $U(a)$ is the time-evolution) is bounded while x is unbounded. If we would be dealing with a commutative algebra, then the natural definition would be

$$\lim_{\epsilon \rightarrow 0} F^{\frac{1}{\epsilon}}(0) (F^{\frac{1}{\epsilon}}(\epsilon))^\dagger \quad (11)$$

For a trivial parallel transport, we would get as required:

$$\lim_{\epsilon \rightarrow 0} F^{\frac{1}{\epsilon}}(0) (F^{\frac{1}{\epsilon}}(\epsilon))^\dagger = \lim_{\epsilon \rightarrow 0} e^{i \int dt f(t) \frac{x(t) - x(t-\epsilon)}{\epsilon}} \quad (12)$$

But since we are dealing with a non-commutative algebra, we need to use the Trotter exponential product approximation formula, to define the exponential version of the covariant derivative:

$$\lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} (F^{\frac{1}{n\epsilon}}(0) (F^{\frac{1}{n\epsilon}}(\epsilon))^\dagger)^n \quad (13)$$

And so for the parallel transport $U(\epsilon) = T(\epsilon)e^{i\epsilon \int d\tau p^2(\tau) + V(x(\tau))}$ where $V(x)$ is a potential only dependent on the position operator, the exponential version of the covariant derivative of the position operator x is:

$$\lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} (F^{\frac{1}{n\epsilon}}(0)(F^{\frac{1}{n\epsilon}}(\epsilon))^{\dagger})^n = \quad (14)$$

$$\lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} = (e^{i\frac{1}{n\epsilon} \int dt f(t)x(t)} U(\epsilon) e^{-i\frac{1}{n\epsilon} \int dt f(t)x(t-\epsilon)} U^{\dagger}(\epsilon))^n = e^{i \int dt f(t)p(t)} \quad (15)$$

The result is the exponential of the momentum operator, which verifies the Weyl relations with respect to the exponential of the position operator. With some abuse of language, we can say that for this type of time-evolution (and thus for this type of Hamiltonian $p^2 + V(x, t)$, which is most common in Quantum Mechanics), the covariant derivative of the position operator is the momentum operator. Thus the quantization (i.e. the Weyl relations) may appear in a statistical field theory due to a particular time evolution.

4 Lorentz covariance

Concerning the relation between the quantization due to time evolution and special relativity, there are two kind of questions we can ask: 1) based on the space-time "philosophy", what about the concept of quantization due to translation in space? 2) is the quantization due to time evolution compatible with Lorentz symmetry? The first question is conceptual, while the second question is technical. We do not have an answer to the first question, which is expected given the difficulties with Lorentz-symmetry of other approaches to quantization [21]. But in this section we will answer explicitly and positively to the second question. In short, the fact that time evolution plays a special role allows us to use only Poincare representations with positive squared mass. Considering Poincare representations with positive squared mass is self-consistent and it is in no way in conflict with Lorentz symmetry.

A complete physical system is a free system. If we neglect gravity, the wave-function associated to the free system is a unitary representation of the Poincare group, regardless of the interactions occurring within the free system [22].

When the Hilbert space is the direct sum of irreducible representations of a symmetry group, then these representations will be defined by numbers (e.g. mass and spin) which are invariant under the symmetry group. Thus there will be a set of operators whose diagonal form corresponds to those invariants, we will call them Casimir operators. When the symmetry group is abelian and continuous (e.g. translation in time), then the generator of the group (e.g. the Hamiltonian) is a Casimir operator, and the invariant numbers defining the representations are called the constants of motion. Certainly, when we move from non-relativistic quantum mechanics and consider instead the Poincare group, then the Hamiltonian is no longer a Casimir operator and the notion of constants of motion needs to be reviewed. Nevertheless, the Casimir operators can be chosen arbitrarily (just like the Hamiltonian in non-relativistic

quantum mechanics). For a positive squared mass, the spin and the sign of the Energy are also Poincare invariants. The sign of the Energy times the modulus of the mass is the center-of-mass Energy, while the spin is the center-of-mass angular momentum. Thus, the Casimir operator whose eigenvalues are the center-of-mass Energy may have negative eigenvalues and it will be the analogous operator to the Hamiltonian of the non-relativistic formalism. As was seen in the previous section, such operator has the formal form of the Hamiltonian action (i.e. it is the difference between the generator of translations in time and the Hamiltonian operators). The Casimir operators necessarily commute with the momentum operator and thus they do not change the 3-momentum eigenstate. Thus we can solve the problem in a basis where the 3-momentum operator is diagonal.

In such a basis, the translations in space-time can be written as $T(x)\Psi(\gamma\vec{v}) = e^{iM\tau(\gamma\vec{v},x)}\Psi(\gamma\vec{v})$, where $\tau(\gamma\vec{v},x) = \gamma x_0 - \gamma\vec{v} \cdot \vec{x}$. Note that $\gamma = \sqrt{1 + \gamma^2\vec{v}^2}$ is a function of $\gamma\vec{v}$. Thus in a basis where the 3-momentum is diagonal, the translations in space-time have the same structure as the time-evolution in non-relativistic space-time, with M playing the role of the non-relativistic Hamiltonian and the numerical factor $\tau(\gamma\vec{v},x)$ playing the role of the time (it is indeed the proper-time).

For each 3-momentum eigenstate, there is a corresponding inertial referential where the 3-momentum is null, i.e. the referential of the center-of-mass. In such referential, the modulus of the energy is the invariant mass, the signal of the energy is also a Lorentz invariant and the angular momentum is the spin. Thus, the eigenvalues of the Hamiltonian and angular momentum operator in the center-of-mass define three Lorentz invariants which define the Poincare representation completely.

Despite we do not know a priori the diagonal form of the Hamiltonian, we know that it is either continuous or discrete in the neighborhood of the eigenvalue 0 (in the referential of the center-of-mass). If it is continuous then the zero energy has null measure. If it is discrete, we can modify the Hamiltonian adding an appropriate constant such that the zero energy is not one of the eigenvalues (this is equivalent to adding to the system a free massive particle with null 3-momentum relative to the system). In any case, we can assume without loss of generality that our system is a quantum superposition of massive free systems with null 3-momentum. Then, the Lorentz transformations become known and are given by the Wigner irreducible massive representations of the Poincare group [22]. If the Hamiltonian is bounded from below then the vacuum state is not Lorentz invariant, as it was already suggested [21].

In the center-of-mass, the relevant group is not the Poincare group, but the little group of spatial rotations and the translation in time [22]. Thus the spatial and time coordinates of space-time, become separated. The fields are no longer representations of the Lorentz group, but only of the rotation group and the canonical commutation relations in no way are in conflict with the little group of spatial rotations.

Therefore and unlike what it is often claimed in the literature, it is false that quantization is incompatible with Lorentz covariance. The only restriction is that we need to consider

representations with positive squared mass, then the dynamics determined by the Hamiltonian becomes linked with the time coordinate [23]. The question why only positive squared masses are relevant is a reformulation of question 1) which will be left open in this paper. Similar assumptions concerning the energy-momentum of the full system are also done in the Källén-Lehmann representation of a non-perturbative two-point correlation function, where it is assumed that the eigenvalues of the 3-momentum squared are not larger than those of the squared energy [24][p203].

Note that a formalism based on a Lorentz scalar (such as the path integral or the Peierls bracket [25] instead of an Hamiltonian) is already problematic at the classical level, since the dynamics is Lorentz covariant but not invariant [26]. For instance, numerical simulations of (classical) general relativity use an Hamiltonian formalism [26]. Moreover, the phenomenologically successful (but ill defined) path integral formalism based on the Lagrangian is in fact equivalent to a path integral based on the Hamiltonian [27].

5 Gauge-variant gaussian measure

One of the features of the Feynman's path integral is that it has a constant (i.e. Lebesgue like) measure which is thus gauge-invariant. Yet, it is proved that in rigor such infinite-dimensional Lebesgue measure cannot exist. Thus, when following the path-integral approach, the notion of gauge-invariant vacuum state is inconsistent, because such state requires a gaussian measure which is gauge-variant.

But is there any fundamental need of a gauge-invariant state, in the first place? Usually, the need for a gauge-invariant state arises from the constraint equations which arise from the Dirac method of quantization [28–30], starting from a classical Lagrangian. However by following the quantization due to time-evolution, we start (in a self-consistent way) from a quantum Hamiltonian and not from a classical Lagrangian. Then there isn't a need of gauge-invariant wave-functions (even if the Hamiltonian is gauge-invariant), because there is no gauge anomaly due to the quantization procedure and there is no spontaneous gauge symmetry breaking (see also Section 7).

First, we need to review the concept of gauge-invariance, in the context of a statistical field theory. We need to consider the fact that a probability theory can be defined as a particular case of a statistical theory where there is a (possibly non-commutative [11]) algebra of operators and an expectation functional [31]. The algebra of operators may have a symmetry that the expectation functional does not have. In fact, there are some symmetries which necessarily the expectation functional cannot have, since the expectation functional is a trace-class operator (the expectation of the operator 1 is 1) and its dual-space is bigger (the space of bounded operators).

For instance, consider an infinite-dimensional discrete basis $\{e_k\}$ of an Hilbert space (indexed by the integer numbers k) and the symmetry group generated by the transformation $e_k \rightarrow$

e_{k+1} (translation). There is no normalized wave-function (and thus no expectation functional) which is translation-invariant, while there is a translation-invariant algebra of bounded operators (starting with the identity operator).

In the above example, what prevents us from enlarging the algebra of operators and argue that there is in fact no translation symmetry at all? The answer is that the Hilbert space may be merely an auxiliary space used to define a measure (the expectation functional) on a manifold with an intrinsic symmetry. Because of this, probability theory is merely a particular case of a statistical theory: when the algebra of operators is necessarily constrained by a symmetry we have expectation values but probabilities are ill-defined.

In the case of gauge symmetry, the gauge potentials can be fully reconstructed from the algebra of gauge-invariant operators [32]. Moreover, the Fock space (defined on a 4-dimensional space-time) produces a well-defined expectation functional for the gauge-invariant operators [33]. The expectation-values of the gauge-invariant operators fully define the statistical gauge field theory (since the gauge potentials can be fully reconstructed [32]), thus the gauge-variant operators can be neglected. Of course, gauge-variant operators can act on the Fock-space, but the link between these operators and the underlying manifold of gauge potentials vanishes since the expectation-value is not gauge-invariant.

Since only (fully) gauge-invariant operators are allowed and the wave-functions necessarily break the gauge-symmetry, in scattering theory we always need to work in the in-in formalism. Of course, we can use the more common in-out formalism in intermediate steps and then build gauge-invariant operators from linear combinations of projections to different final-states. Note that in the in-in formalism, we can characterize an initial state with a non-null gauge-charge using only gauge-invariant operators (e.g. the charge operator in an abelian gauge theory is gauge-invariant).

It is reassuring that for non-abelian gauge theories, the charge operator is not an observable because it is not gauge-invariant, which is consistent with the experimental results. Nevertheless, the absence of a gauge-invariant charge operator by itself is not a solution to the confinement problem.

6 Relation with the BRST formalism

We are working from the start with a self-consistent statistical field theory. If two hermitian operators do not commute, they cannot be diagonalized in a common basis and therefore the phase-space is not determined by both operators simultaneously.

The same does not happen with Poisson brackets in a classical field theory, where two variables with non-vanishing Poisson bracket both belong to the phase-space (e.g. position and momentum).

This difference affects the relation between a classical gauge field theory and a quantum gauge field theory (in the Hamiltonian formalism).

The gauge-invariance constraint and the corresponding gauge-fixing cannot be both imposed on the wave-function, because the corresponding operators do not commute. Since the definition of an expectation-value is related to gauge-fixing, then the algebra of operators is gauge-invariant, while the expectation functional may be affected by the gauge-fixing (if gauge-fixing is imposed) but it is not gauge-invariant.

The classical BRST cohomology applies to a commuting algebra of operators: the algebra of operators is enlarged from gauge-invariant operators to BRST-invariant (not necessarily gauge-invariant) operators. These BRST-invariant operators are divided into equivalence classes. The BRST cohomology maps (in a bijective way) each equivalence class with a corresponding gauge-invariant operator. When performing algebraic manipulations in a gauge-invariant operator, we can convert the gauge-invariant operator into BRST-invariant, then do the manipulations in the space of BRST-invariant operators and then convert the resulting BRST-invariant operator again into a gauge-invariant operator. In the quantization due to time-evolution, we can apply the BRST cohomology (as if it were the classical version) to the algebra of operators with the only difference that the algebra of operators is non-commutative. Since the BRST cohomology is well-defined by construction and it is based on the vectorial properties of the algebra of operators, it applies to non-commutative algebras as long as we manage to construct the BRST cohomology for the specific algebra (this is straightforward in our case, where the operators are bounded and act in a Hilbert space). Crucially, the BRST cohomology merely simplifies the expression defining of a gauge-invariant operator into another equivalent expression, it does *not* affect the gauge-variant wave-function and therefore the Gribov problem does not arise.

The advantage of doing manipulations with BRST-invariant operators is that we can have a well-defined time-evolution of all fields of a 3-dimensional Fock space (corresponding to the 3-dimensions of physical space), unlike a Quantum gauge theory where the time-evolution can only be well-defined for fields of a 4-dimensional Fock space (corresponding to space-time). Thus given both gauge-invariant Hamiltonian and operator at an initial time, we can use the BRST cohomology to modify the Hamiltonian such that the gauge-invariant operator at a final time is easier to calculate (without affecting the result).

There is a well-known subtlety with the BRST cohomology that we need to address: the BRST cohomology is itself gauge-invariant and mathematically well-defined, but it is merely a dispensable auxiliary step in a calculation performed in the context of a quantum formalism. If the quantum formalism is mathematically inconsistent, if the formulation of the calculation crucially depends on the BRST-invariant algebra (not our case, but it is the case of the path integral), surprises are possible. In particular, if all the details of the calculation are only known for the BRST-invariant algebra and not before for the gauge-invariant operators (not our case, but it is the case of the quantum BRST formalism), then the gauge-invariance of the BRST cohomology does not imply that the calculation would be the same in all gauges or that the quantum formalism is logically consistent (and in fact it is not due to the Gribov problem).

"Being gauge invariant, the BFV-PI necessarily reduces to an integral over modular space,

irrespective of the gauge fixing choice. Nevertheless, which domain and integration measure over modular space are thereby induced are function of the choice of gauge fixing conditions. The BFV-PI is not totally independent of the choice of gauge fixing fermion Ψ ." [34]

In the (our) case of the quantization due to time-evolution, the quantum formalism is mathematically well-defined and all details of the calculations are known, regardless of whether we apply the BRST cohomology or not. Since we use the BRST cohomology to merely simplify the expression defining of a gauge-invariant operator into another equivalent expression, the Gribov problem does not affect us.

7 Spontaneous symmetry breaking

There is no mathematical difference between spontaneous symmetry breaking and anomaly: in both cases there is a failure of a symmetry of the wave-function to be restored in the limit in which a symmetry-breaking parameter goes to zero. The difference is about the origin of the symmetry-breaking parameter: if it arises due to a physical process (e.g. a probe field in an experimental setting); or due to the mathematical consistency of the theory. We only consider symmetries of the Hamiltonian as candidate symmetries of the wave-function, since only these are respected by the time-evolution.

We address here the question of gauge symmetry breakdown in our formalism. As it was discussed in the previous sections, the full gauge symmetry is necessarily broken by the wave-function. But the spontaneous breaking of the global gauge symmetry (remnant of the full gauge symmetry) is still possible.

The analysis of a gauge symmetry is more subtle (wrt a non-gauge symmetry), because the algebra of operators is necessarily gauge-invariant. In a strict sense, since there are no symmetry-breaking observables then there is no symmetry breaking of the gauge symmetry. However, precisely because the algebra of operators is constrained by the gauge-symmetry, we can still have a situation where the wave-function is *necessarily* such that the expectation-value of a gauge-variant operator is discontinuous with respect to a parameter. If no continuous wave-function exists, we know that there is a relevant phenomenon occurring which is very similar to the spontaneous symmetry breaking of a non-gauge symmetry with the only difference that the asymmetric operators are not observable now.

More relevant is the fact that, since the wave-function is gauge-variant in our formalism, the gauge-variant asymptotic states are perfectly fine as intermediate calculations, even at the non-perturbative level. But as it was discussed in a previous section, despite the fact that the wave-function is gauge-variant the algebra of operators is still gauge-invariant and so the perturbative calculations still have to be algebraically combined to produce fully gauge-invariant observables: does this leads to surprises?

Our answer is no. The gauge-variant operators can be composed with destruction operators of the Higgs doublet to form gauge-invariant operators. These gauge-invariant operators can

then be left-multiplied by its transposed operator to form gauge-invariant self-adjoint operators. The destruction/creation operators of the Higgs doublet reduce to the Higgs vacuum expectation value when they act on the wave-function, accomplishing the Higgs mechanism. The only detail missing is that these operators are not bounded, but the projection-valued measure corresponding to these self-adjoint operators can be considered instead, and so we obtain bounded operators.

Note that the above does not imply that the mean-field approximation (frequently used to deal with spontaneous symmetry breaking in a perturbative framework) always works, but the non-perturbative problems of the mean-field approximation are not exclusive to the local gauge-symmetry (e.g. the mean-field approximation also breaks global symmetries in the two-Higgs-doublet model).

8 Renormalization group

We couldn't propose a mathematical definition of a Quantum gauge theory without commenting on the renormalizability of the theory. Without perturbative expansions, our expectation values are finite for bounded operators. On the other hand, since we are working on the continuum, there is no cutoff scale (usual in a theory on the Lattice). Thus the renormalization group in our theory cannot be distinguished from a regular background symmetry group: that is, a symmetry group that acts not just on the fields but also on the parameters of the theory, leaving the observables invariant. The renormalization group plays no fundamental role in a Quantum Gauge Theory and in principle we can formulate mathematically consistent theories where the renormalization group is broken (although such an exercise is beyond the scope of our work).

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