

Time in the Time-Independent Schrodinger Equation

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In classical mechanics, the spatial density $d(x)$ of a particle moving under a conservation of energy law $\frac{1}{2}mv(x)v(x) + V(x) = E$ is proportional to $1/v(x)$. This is obtained from $d(x) = dt/T$, i.e. the probability to be at x equals the time spent at x divided by the cycle time T . Given that $dx = vdt$ to first order, density is proportional to $1/v(x)$. Now, $d(x)$ is supposed to match the time independent quantum mechanical density $W^*(x)W(x)$, where $W(x)$ is the wavefunction, for high energy levels. Thus, one would expect the time a p wave (plane wave) spends at a point should be part of time independent quantum mechanics, but it does not seem to be overtly present. Furthermore, there is the question of T for the quantum mechanical problem. It is known that even for a time-independent solution $W(x)$ has a factor $\exp(iEt)$. Thus, it seems time is present, even if it disappears in the density $W^*(x)W(x)$. It is argued that $W^*(x)W(x) = P(x)$ where P is probability, but $W(x)$ is related to conditional probabilities such as $P(x/p)$ or $P(p/x)$. Thus, these conditional probabilities should be related to time. In an earlier note, we tried to show the time-independent Schrodinger equation implies all p plane waves are cycling with the same frequency E . In this note, we try to find a quantum mechanical representation for $P(x,p) = dt/T$ where $dt(x,p)$ is the time spent at x by a plane wave p and also try to evaluate T .

Classical Mechanics

At first, it seems that describing a classical mechanical point particle governed by a conservation law $p^2/2m + V(x) = E$ and following a cycle (e.g. a spring) with a spatial density $d(x)$ means that there is no time in the picture. One could imagine an ensemble of springs. At time t , one could find the number of cases with the particle at x_1, x_2 etc. This would give $P(x)$ with time absent. One may think of $P(x)$, however, as the time spent by the particle at point x during its cycle of time length T . Then $P(x) = dt/T$, and with $dx = vdt$ to first order $dt = dx/v$ with dx being constant. Alternatively, one may imagine making measurements of the particle moving through a constant dx , with the measurement time taking a fixed Δt . A fast particle will move through dx in dt so the average density is proportional to: $[1 \text{ dt} + 0 (\Delta t - dt)] / \Delta t$. In either case, time is critical to the understanding of this density. Given that this density, proportional to $1/v(x)$, is supposed to match roughly (the envelope of) the time independent Schrodinger density $W^*(x)W(x)$ for high energy eigenvalues, it seems time should play an important role in the time-independent Schrodinger equation as well.

Quantum Mechanics

In quantum mechanics, one usually deals with non-conditional probabilities $P(x)=W(x)*W(x)$ and $P(p) = f_p*f_p$ where f_p is the Fourier transform of $W(x)$. Thus, if a wavefunction has an $\exp(iEt)$ factor, it disappears when calculating density and it seems that time is not present.

We argue that $W(x)$ should not be considered the square root of probability. It happens to be the square root of the $P(x)$ for a bound system, but is also part of a conditional probability. It seems that time-independent Schrodinger equation is based on a conditional probability, namely $P(p/x)$ because quantum particles are described by $\exp(ipx)$ (they have internal cycling) and so p and x are intertwined. Thus, we try to define:

$$P(p/x) = f_p \cos(px) / W(x) \quad \text{where } W(x) = \sum f_p \cos(px) \quad ((1))$$

Actually, one could use $\exp(ipx)$ instead of $\cos(px)$. Here we have summed over p and $-p$ with $f_p = f_{-p}$. Then, the time-independent Schrodinger equation is:

$$\sum \text{over } p \quad p^2/2m P(p/x) + V(x) = E \quad ((2))$$

Such an equation can only be satisfied for special "E" values.

One may also argue that there should be a conditional probability: $P(x/p)$, i.e. given a momentum, it has different probabilities to be at different x points. This is key, because this should be the spatial density of the p wave. Averaging over p , should give the overall spatial density. Thus:

$$\sum \text{over } p \quad P(p) P(x/p) = P(x) = W(x)W(x) \quad ((3))$$

We suggest, in analogy to ((1)) that :

$$P(x/p) = W(x)/f_p \cos(px) \quad ((4))$$

With $P(p)=f_p*f_p$, ((3)) is satisfied.

If one compares with classical mechanics, we suggest that:

$$P(x/p) = dt(x,p) / T \quad ((5))$$

Here T is an unknown cycle time in the problem. Averaging over p then yields the spatial density. Thus, quantum mechanics with conditional probabilities seems to have a direct link to classical mechanics, but one has an average over p waves.

The difficulty is now to find T .

In quantum mechanics, it is suggested that instead of point particles, one has a kind of zitterbewegung or internal cycling modeled by $\exp(ipx-iEt)$. This is also the form of a wave.

Thus, it seems there are uncertainties between p and x and E and t . We list the energy uncertainty relation used in quantum mechanics: $\Delta E \Delta t \geq \hbar/2$. Note: This relationship does not follow from noncommuting operators like P and X . If, however, one believes that there is such an energy time uncertainty, it may be possible to use it to determine T in the quantum mechanical case.

Consider: $E W(x)W(x)$ where E is the total energy and $W(x)W(x)$ the density. Then:

$$E W(x)W(x) = \sum_p E P(p) P(x/p) = \sum_p E P(p) (1/T) d t(x,p) \quad ((6))$$

Consider integrating ((4)) over dx :

$$\int dx P(x/p) = \int dx [W(x)/f_p] \cos(px) \quad ((7))$$

If one considers $f_p = \int W(x) \cos(px) = f_p$ then ((7)) is constant for different p values. Thus:

$$\int dx P(x/p) = \int dx d t(x,p)/T = \text{constant} \quad ((8))$$

Thus, the integrated time for any p plane wave is equal to the same constant. This may imply that T is a universal cycling time in the quantum bound system. The question is now to determine T . Using energy uncertainty with equality pertaining and no $1/2$, one may try to argue that:

$$E \int dx d t(x,p) = E (\text{time for any } p \text{ wave}) = E (\text{time}) \approx 1 \quad (\hbar=1)$$

Then, using $\sum_x P(x/p) = 1 = (1/T) \sum_x d(x,p) = (1/T) (1/E)$ so $T=1/E$.

The above argument is "hand-wavy", but does seem to bring in a local time for each x for each p wave. It also argues that each p wave has the same overall integrated time which is $1/E$. This seems to imply a time resonance in a quantum bound system.

In a previous note (1), we tried to argue quantum mechanics is a time resonance involving cycling over different p values. We argued one could write the time-independent Schrodinger equation as:

$p^2/2m f_p + \sum_{k+j=p} V_k f_k = E f_p$ where V_k is the k th term of the Fourier transform of $V(x)$. Thus, it seems each plane wave p combines with other waves interacting with $V(x)$ to form $E f_p$ with E being the frequency, i.e. $id/dt f_p = \exp(iEt)$. E is the same for different p 's indicating a resonance.

Conclusion

In conclusion, a classical mechanical particle following $.5mv(x)v(x) + V(x) = E$ and undergoing a cycle (such as a spring) has a density proportional to $1/v(x)$ or dt/T where T is the cycle time and $dx=vdt$ to first order. This density is supposed to match quantum mechanics for high energy eigenvalues. Thus, we argue such a time picture should also be valid in time-independent quantum mechanics. We suggest that $P(x/p)$, a conditional probability, given by $W(x)/\int p \cos(px)$ represents dt/T for the quantum system, but T is unknown. We use the energy time uncertainty principle to try to argue that $T=1/E$ for all p waves. Thus, it seems that a bound state quantum system is a time resonance.

Reference

1. Ruggeri, Francesco R. The 2x2 Matrix Model and the Schrodinger Equation (preprint, zenodo, 2018)