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If only we knew the drift

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Abstract:

Based on a simple stochastic model, we illustrate that the market price of risk can hardly be estimated.

Key Words: Drift estimation, Volatility estimation, Uncertainty

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If only we knew the drift

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ABSTRACT. Based on a simple stochastic model, we illustrate that the market price of risk can hardly be estimated.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. In the Black-Scholes framework, the real-world dynamics of a price process are assumed to follow the stochastic differential equation

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \qquad S_0 = s_0$$

for some $\mu \in \mathbb{R}$, $\sigma, s_0 \in (0, \infty)$ and a Brownian motion $W = (W_t)_{t \geq 0}$. The stochastic differential equation can be solved explicitly by

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}.$$

For some risk-free interest rate level $r > 0$, the corresponding risk-neutral dynamics for pricing purposes are given by

$$dS_t = r S_t \, dt + \sigma S_t \, d\widetilde{W}_t, \qquad S_0 = s_0,$$

where, by Girsanov's theorem, the process $\widetilde{W}_t := W_t - \frac{r-\mu}{\sigma} \, dt$ forms a Brownian motion under the equivalent measure $\mathbb{Q} \approx \mathbb{P}$ with the density process

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{F}_t} = e^{-\frac{1}{2} \frac{(r-\mu)^2}{\sigma^2} t + \frac{r-\mu}{\sigma} W_t}.$$

The volatility adjusted excess return $\frac{\mu - r}{\sigma}$ is usually referred to as *market price of risk* or *Sharpe ratio*. Knowing this quantity would be utterly helpful in the risk management process of derivatives written on S. Furthermore, the optimal strategy in S that maximises the expected utility would be known explicitly for a wide range of utility functions. However, if one observed one (discrete) real world sample path under these idealised assumptions, it would take centuries to estimate the market price of risk to a satisfactory precision. The volatility σ can be estimated efficiently as standard deviation of the log-returns, whereas the actual drift μ remains obscured for a long time. The interest rate level r is typically approximated as average yield of liquid government bonds. Let

$$m_n = \frac{1}{n \, \Delta t} \sum_{k=1}^{n} \log \frac{S_{t_k}}{S_{t_{k-1}}} = \frac{1}{n \, \Delta t} \sum_{k=1}^{n} X_{t_k}, \qquad {s_n}^2 = \frac{1}{n \, \Delta t} \sum_{k=1}^{n} \big(X_{t_k} - m_n \, \Delta t \big)^2$$

for $t_k = k \, \Delta t = \frac{k}{252}$ and $k \in \mathbb{N}$ denote the maximum likelihood sample mean and sample variance respectively of the annualised log-returns. The series $(X_{t_k})_{k \in \mathbb{N}}$ are i.i.d. $\sim \mathcal{N}\big((\mu - \frac{1}{2}\sigma^2) \, \Delta t, \sigma^2 \, \Delta t\big)$-distributed. By Cochran's theorem, since $\frac{(n-1){s_n}^2}{\sigma^2}$ has a chi-square distribution with $n-1$ degrees of freedom, the $(1-\alpha)$-confidence interval for the sample variance has the boundaries

$$\left[\frac{(n-1){s_n}^2}{\chi^2_{n-1,1-\alpha/2}}, \frac{(n-1){s_n}^2}{\chi^2_{n-1,\alpha/2}} \right].$$

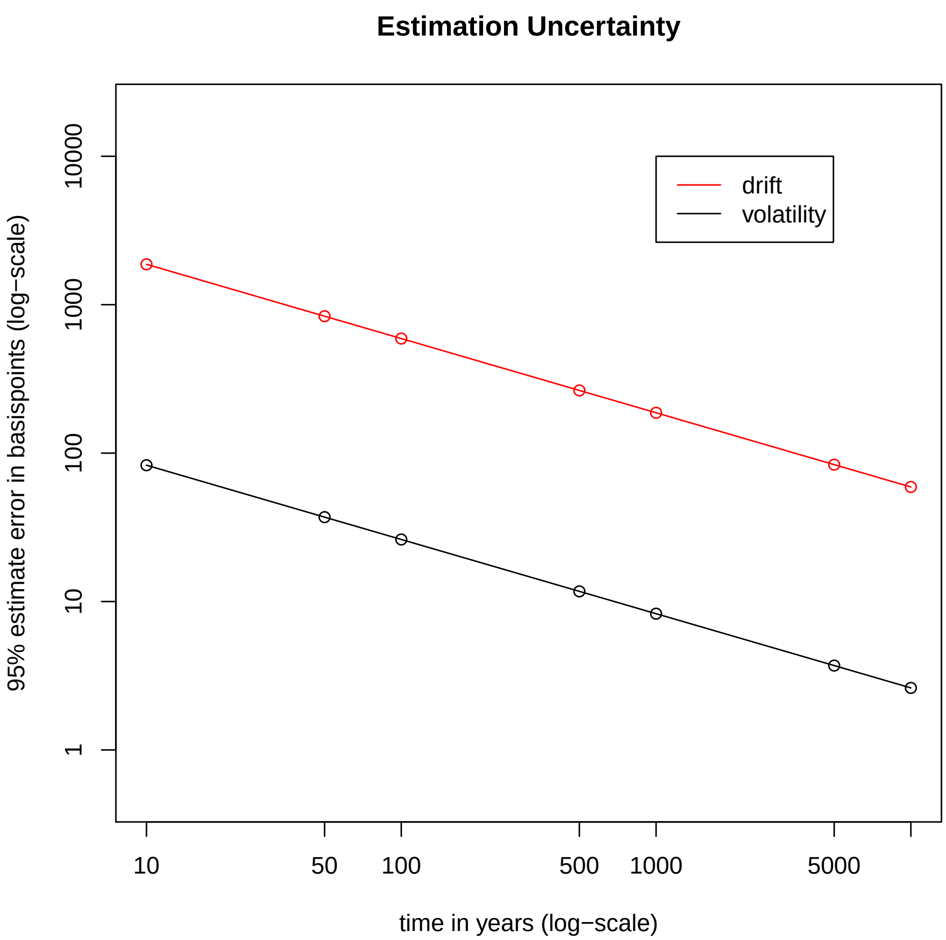
The 95%-confidence interval of the estimated volatility based on 252 daily log-returns amounts to roughly $\pm 10\%$ of s. If the true level of the volatility was $\sigma = 30\%$, then it could be estimated based on daily returns over 10 years with a likelihood of 95% up to a precision of roughly 80 basispoints. In contrast, it would take approximately 5 500 years of daily log returns to estimate the drift level with 95% confidence up to the same precision. As a matter of fact, since the volatility affects the mean of the log-returns, this statement requires additionally that we know σ accurately enough. Indeed, $\frac{m_n-(\mu-\frac{1}{2}\sigma^2)}{s_n/\sqrt{(n-1)\,\Delta t}}$ has a t-distribution with $n-1$ degrees of freedom, the $(1-\alpha)$-confidence interval for the sample mean has the boundaries

$$\left[m_n+\frac{1}{2}\sigma^2-\frac{s_n}{\sqrt{(n-1)\,\Delta t}}t_{n-1,1-\alpha/2}, m_n+\frac{1}{2}\sigma^2+\frac{s_n}{\sqrt{(n-1)\,\Delta t}}t_{n-1,1-\alpha/2}\right].$$

The uncertainty will shrink down to 10 basispoints only after approximately 350 000 years. Unfortunately, increasing the observation frequency to intraday data does not help at all. This relates to using a step size $\frac{\Delta t}{k}$ and a sample size nk for some $k\in\mathbb{N}$. Under the above model assumptions, it holds

$$\mathrm{Var}\big[m_n\big]=\frac{\sigma^2}{n\,\Delta t},\qquad \mathrm{Var}\big[{s_n}^2\big]=\frac{2\sigma^4}{(n-1)}.$$

Therefore, by including more and more intraday returns, the volatility can be estimated more and more accurately. However, the variance of the drift estimator remains totally unaffected. m_n involves a telescoping sum of log-returns. Hence, only the initial and the final price level actually matter.



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