# PRISMS-PF
# Mechanics (Infinitesimal Strain)

Consider a strain energy expression of the form:

$$\Pi(\varepsilon) = \int_{\Omega} \frac{1}{2}\varepsilon : C : \varepsilon \; dV \tag{1}$$

where $\varepsilon$ is the infinitesimal strain tensor, $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ is the fourth order elasticity tensor and ($\lambda$, $mu$) are the Lame parameters.

## 1 Governing equation

Considering variations on the displacement $u$ of the from $u + \alpha w$, we have

$$\delta\Pi = \frac{d}{d\alpha}\int_{\Omega} \frac{1}{2}\epsilon(u+\alpha w) : C : \epsilon(u+\alpha w) \; dV \bigg|_{\alpha=0} \tag{2}$$

$$= -\int_{\Omega} \nabla w : C : \epsilon \; dV + \int_{\partial\Omega} w \cdot [C : \epsilon \cdot n] \; dS \tag{3}$$

$$= -\int_{\Omega} \nabla w : \sigma \; dV + \int_{\partial\Omega} w \cdot (\sigma \cdot n) \; dS \tag{4}$$

$$= -\int_{\Omega} \nabla w : \sigma \; dV + \int_{\partial\Omega} w \cdot t \; dS \tag{5}$$

where $\sigma = C : \varepsilon$ is the stress tensor and $t = \sigma \cdot n$ is the surface traction.

The minimization of the variation, $\delta\Pi = 0$, gives the weak formulation of the governing equation of mechanics:

$$\int_{\Omega} \nabla w : \sigma \; dV - \int_{\partial\Omega} w \cdot t \; dS = 0 \tag{6}$$

If surface tractions are zero:

$$R = \int_{\Omega} \nabla w : \sigma \; dV = 0 \tag{7}$$

## 2 Residual expressions

In PRISMS-PF, two sets of residuals are required for elliptic PDEs (such as this one), one for the left-hand side of the equation (LHS) and one for the right-hand side of the equation (RHS). We solve $R = 0$ by casting this in a form that can be solved as a matrix inversion problem. This will involve a brief detour into the discretized form of the equation. First we derive an expression for the solution, given an initial guess, $u_0$:

$$0 = R(u) = R(u_0 + \Delta u) \tag{8}$$

where $\Delta u = u - u_0$. Then, applying the discretization that $u = \sum_i w^i U^i$, we can write the following linearization:

$$\frac{\delta R(u)}{\delta u} \Delta U = -R(u_0) \tag{9}$$

The discretized form of this equation can be written as a matrix inversion problem. However, in PRISMS-PF, we only care about the product $\frac{\delta R(u)}{\delta u}\Delta U$. Taking the variational derivative of $R(u)$ yields:

$$\frac{\delta R(u)}{\delta u} = \frac{d}{d\alpha} \int_\Omega \nabla w : C : \epsilon(u + \alpha w)\ dV \bigg|_{\alpha=0} \tag{10}$$

$$= \int_\Omega \nabla w : C : \frac{1}{2} \frac{d}{d\alpha} \left[ \nabla(u + \alpha w) + \nabla(u + \alpha w)^T \right]\ dV \bigg|_{\alpha=0} \tag{11}$$

$$= \int_\Omega \nabla w : C : \frac{d}{d\alpha} \nabla(u + \alpha w)\ dV \bigg|_{\alpha=0} \quad (\textit{due to the symmetry of } C) \tag{12}$$

$$= \int_\Omega \nabla w : C : \nabla w\ dV \tag{13}$$

In its discretized form $\frac{\delta R(u)}{\delta u}\Delta U$ is:

$$\frac{\delta R(u)}{\delta u} \Delta U = \sum_i \sum_j \int_\Omega \nabla N^i : C : \nabla N^j dV\ \Delta U^j \tag{14}$$

Moving back to the non-discretized form yields:

$$\frac{\delta R(u)}{\delta u} \Delta U = \int_\Omega \nabla w : C : \nabla(\Delta u) dV \tag{15}$$

Thus, the full equation relating $u_0$ and $\Delta u$ is:

$$\int_\Omega \nabla w : \underbrace{C : \nabla(\Delta u)}_{r_{ux}^{LHS}} dV = - \int_\Omega \boldsymbol{\nabla} w : \underbrace{\sigma}_{r_{ux}}\ dV \tag{16}$$

The above values of $r_{ux}^{LHS}$ and $r_{ux}$ are used to define the residuals in the following input file:
*applications/mechanics/equations.h*