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Sketching functions as a digital task with automated feedback

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Abstract: This paper describes and illustrates a digital task implemented in STACK in which students are asked to sketch the graph of a function. Students can draw a sketch of the function's graph which is evaluated automatically afterwards. Besides the front-end task design, some details of the underlying feedback tree and didactical aspects are discussed in relation to the task. Finally, potentials and limitations of the task will be discussed.

Keywords: STACK, sketching and drawing functions' graphs, qualitative automated feedback

1 Introduction

Self-assessment is an important part of doing mathematics. Good students check their result of a calculation, or, at least, assess it in terms of plausibility. Self-assessment is more difficult for students when they need to assess their performance on a topic that they have not yet mastered. In a specific content domain, a lack of (not yet obtained) knowledge and skills can be an obstacle for successful self-assessment. Research indicates that the quality of self-assessment is influenced by many factors, e.g. age, its implementation in a learning context, or specific instructions and training for self-assessment undergone beforehand with an instructor ([Ro06]). To support the objectivity of feedback, automated feedback for self-assessment opportunities can be helpful.

Mathematics allows for automatization of feedback in self-assessments, because of its formal nature. When a single numerical value is the answer to a math problem, then automatic assessment appears to be very easy. The correct evaluation of terms and formulae is not as simple. The STACK software package for computer aided assessment offers the means to handle the input of mathematical formulae as well as the opportunity to create individual feedback to a given student's answer to a problem (c.f. [Sa13]). This approach focuses on an algebraic representation of mathematics and is very powerful.

However, mathematics consists of more than algebraic representations such as formulae

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and terms. According to [HSV09] (p. 536) “it is widely claimed that the facility to use multiple representations and to flexibly switch between a range of representations (including graphical, tabular, algebraic and verbal ones) is a critical component of the skill of solving mathematical problems”. Undeniably, a wide variation of possibilities to represent mathematics exists. Hence, we developed a solution to work with another form of mathematical representations in STACK. In this paper we illustrate and discuss a task which allows students to sketch graphs of functions freehand. Following the idea of automated feedback, detailed individual feedback on the sketched graph is given.

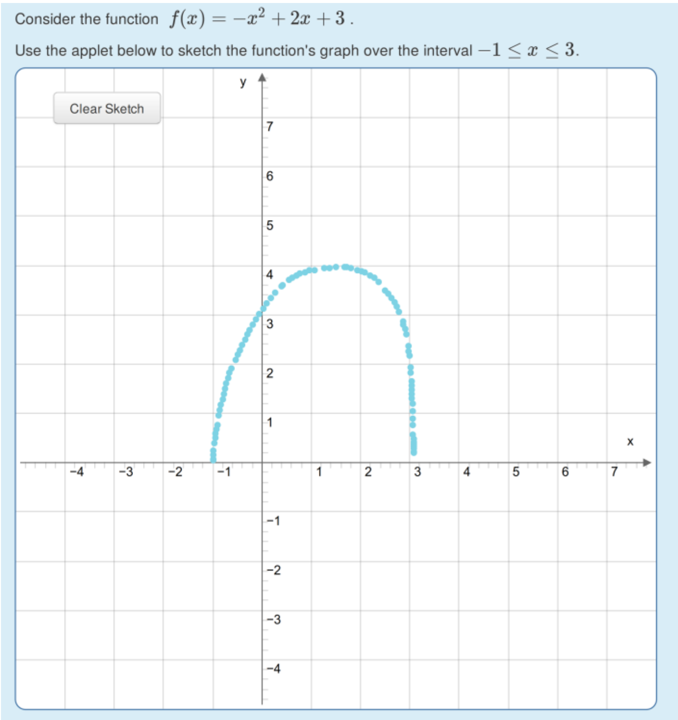

Fig. 1: Screenshot of the task with an exemplary graph sketched to show a possible input.

2 Design, implementation, and illustration of the digital task

The starting point of the task at hand was the idea of connecting dynamic geometry software (DGS) with STACK as a means of input, as suggested in [Se16] and [Va18]. So far, they have used an advanced drag and drop alike mechanism via DGS, either to move points in a coordinate system or to mimic complex inputs like graphs of linear functions (via moving the two points necessary to uniquely define it). Advancing from this groundwork we developed a tool (JSXGraph applet) to let students sketch freehand. By design, the applet can be integrated into a STACK task. To exemplify the new possibilities of our development, we invented the task shown in figure 1: a task to sketch a specific function freehand within a given interval.

The applet reacts to mouse input, pens, or touch events, which can all be used to draw the demanded sketch of the function's graph. In case of a mistake the button "Clear Sketch" in the top left corner erases the board and one can start anew. As presented in figure 1, the applet shows a trace of points rather than a continuous line. This is a design decision with the intention of making it more transparent how the system processes the answer afterwards. With enough points, the sketch appears and is perceived as approximately continuous. All specifics of the task – the chosen function, the interval limits and the displayed section of the coordinate system – are designed to be variable in future versions of the task. Thus, these parameters can be changed or randomized later. In the given example from figure 1, it is a coincidence that the interval's limits are the same as the zeroes of the function in the given example.

For the implementation of the illustrated prototype version from figure 1 JSXGraph was used. JSXGraph is a DGS system that helps to realize the input method for the task. Furthermore, JSXGraph is used to display the drawn sketch and the ideal graph in the feedback afterwards. Some additional lines of JavaScript source code link the STACK answer fields and JSXGraph[3].

As indicated in figure 2, a student's answer is evaluated in detail and leads to more than a simple right or wrong feedback. For providing an adequate feedback, several properties of a student's answer are considered. Internally, the evaluation uses a list of all points drawn. First, we look at all these points as well as points from selected intervals. This is to make sure that the demanded interval was respected and that all relevant intervals on the horizontal axis were covered by the sketch. Second, we calculate the minimal distance of every logged point to the graph of the ideal function and compute the arithmetic mean to get the average distance. Currently for the given setting an average distance below 0.21 is considered tolerable and an average below 0.16 is accepted as an accurate sketch. A sketch which meets these initial requirements should be roughly accurate at least. Furthermore, the feedback tree is designed to investigate in how far the zeroes and the maximum of the function have been sketched precisely. If in doubt the answer contains a specific hint regarding the inaccuracy. A reasonable margin of error is still tolerated for each of these special points of the graph.

During the development and testing of the prototype task we experimented with different thresholds. Our current choices seem to be adequate for the given example. However, these thresholds should also be considered as variable to support different intentions of the task (see section 4 for more details). We also noticed that using a pen for the input is more precise and feels more natural than using a mouse. Using a touch device to draw with a finger allows precise movements, but the size of the finger and the fact that it

[3] The prototype was developed with STACK 4.1. From version 4.2 onwards, STACK supports using JSXGraph, and consequently the next version of the task will utilize this feature.

covers parts of the display while drawing leads to a reduced accuracy. Our thresholds are intended to compromise between these input methods.

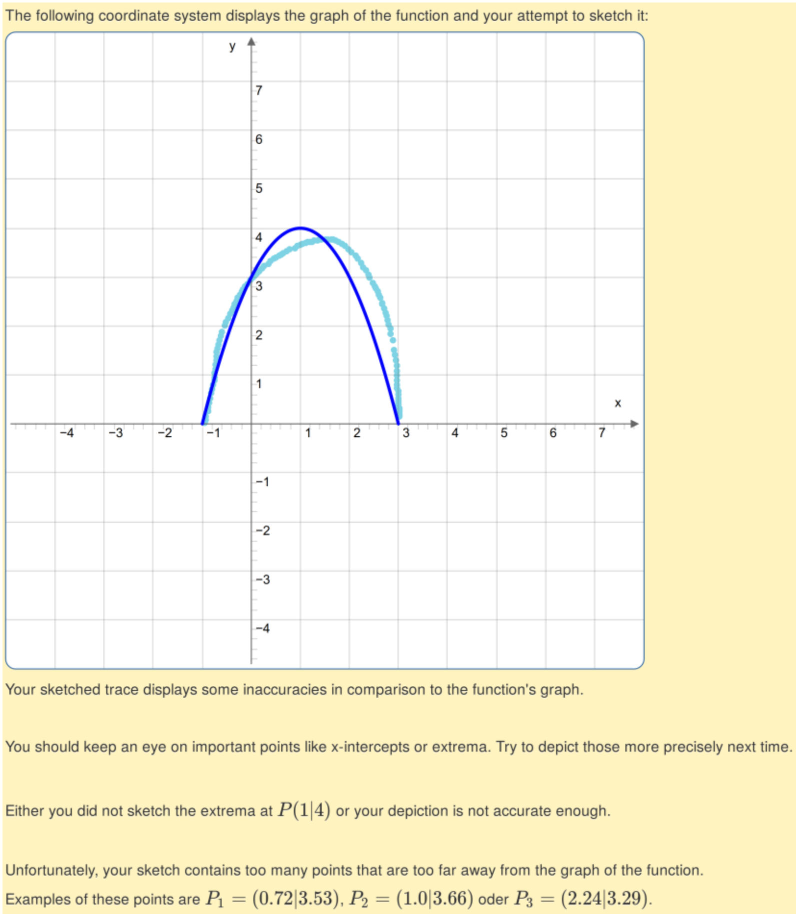

Fig. 2: Screenshot of a possible feedback to an answer.

3 A didactical perspective on the task and functions

Depending on the context, different properties of a function can become relevant. For example, the illustrated task above (see figure 1) does not refer explicitly to any codomain and the domain is only given implicitly with the demanded interval scope of the graph. [LP05] suggest different forms of representation as important for functions and teaching the concept of a function. According to [LP05] it is vital to know about these forms of representation and to be able to deliberately transit from one to another.

Four representational forms of a function can be distinguished:

1. Situation or context
2. Table
3. Graph
4. Term

Students should be able to switch between these representations at will and decide, which is the most suitable representation in order to find the answer to the question they are currently working on. Of course there are mathematical problems which are used for teaching for every possible transition from one these four representations to another. However, automated and detailed feedback on a digital function graph drawn freehand has, to our knowledge, not been implemented so far. Now it is possible to design digital tasks which aim at having students make a transition from any form of representation to a graph and to provide automated feedback on their answers.

Before employing such a task, the knowledge to be taught (cf. [CB14]) should be considered in detail. Afterwards, the task's potentials and limits towards this knowledge can be considered. The outcome of these considerations will depend on the specific utilization of the task. A task like our prototype could support learners to develop an idea on what a graph looks like. It could support the development of basic strategies like calculating enough function values to sketch the graph, or it could even deliver a reason to think about properties of a class of functions (in our case zeroes and the maximum). Depending on its domain, a function could have neither zeroes nor a maximum, or end at the zeroes as in the example from figure 1. Even at school level, where the notion of domain and codomain might be less relevant, every graph displayed is restricted to a chosen window of the coordinate system. These parameters can heavily influence the difficulty of a task. Characteristics of the task's parameters can also either lead to misconceptions or help to avoid their manifestation. The next paragraph elaborates on what kinds of misconceptions might occur.

[Ev98] conducted a qualitative study with prospective secondary mathematics teachers, which discusses different approaches (global vs. local) to functions and their implications. Concerning quadratic functions specifically, she found that some students had trouble linking a function's symbolic representation as a term with the appearance of its graph. Also, the impact of minor manipulations of graphs (translations, stretching or compressing) or minor alterations in the function's term and their impact on the graph seemed to create difficulties for the students. [LZS90] (p. 30) distinguished different types of difficulties and misconceptions: "(a) what is and is not a function; (b) correspondence; (c) linearity; (d) continuous versus discrete graphs; (e) representations of functions; (f) relative reading and interpretation; (g) concept of variable; and (h) notation". Teaching of and learning about functions, misconceptions and the role of technology are still a concern in research (e.g., [Ku16]). So far, there is no evidence to which misconceptions the prototype task could lead, and which misconceptions it could help to overcome, yet.

4 Potential and perspectives

Our prototype task can evaluate a given sketch of a function by interpreting the sketch as a set of points and give feedback on how accurate the sketch was and whether there are some details (within the sketch) that could be improved (run-away points, zeroes, maximum). Also, a distinction between very accurate and less accurate but tolerable sketches is possible. In the following we discuss aspects for further elaboration and development of the task.

Future versions of our task are intended to support randomization. The support of randomization in connection with the detailed feedback possibilities is one strength of STACK and can be utilized here, too. Of course, teachers should be given the option to randomize the parameters of the quadratic functions. Different kinds of functions, e.g. the exponential function, should also be handled correctly by the task. This would make other kinds of tasks possible. For example, “Give the term *f(x)* of a function that satisfies […] and sketch it afterwards.” It would be interesting to explore students’ choices of function terms and whether the sketching part of the task influences their choices.

Adapting the thresholds in the evaluation process will make a wider range of tasks possible. For example, the threshold for the average distance to the ideal graph could be high, but the threshold on the zeroes relatively low. Such a setting could support tasks to sketch polynomial functions of a higher degree given in a factorial representation.

We see further room for improvement for the user experience in the interface offered by the applet. We would like to allow students to change the window of the coordinate system by shifting it or zooming in and out, at least in case the teacher decides to enable such an option. With this feature, finding the right window for the sketch could be part of the task. Further, an *undo* button and a *redo* button should be added next to the *clear all* button in the applet together with an *erase* tool to selectively remove points.

Qualitative feedback whether a sketch is similar to the graph of any parabola, or whether a sketched parabola opens in the right direction at least, is not supported so far. Unfortunately, this also applies when the form of the graph is correct, but it is located elsewhere in the coordinate system. Partial credit can only be given to students’ answers which are somewhat close the ideal solution, although educated guesses might be possible for some kinds of functions.

5 Conclusion and outlook

We presented a task to sketch the graph of a given quadratic function and gave an idea how it can be implemented using STACK and JSXGraph[4] to give automated feedback. The brief didactical discussion of functions already showed that functions are a complex subject to teach because four different forms of representation must be considered. Moreover, one can rather focus on a function as a whole or concentrate on specific values and their mappings. Our task can support learning scenarios which use self-assessment by making the transition from any representation of a function to a graph available and by offering automated feedback. Hence, an important aspect of working with functions can now be covered in terms of digital tasks.

After developing the task, a logical next step is to discover a scenario to deploy it and analyze its effects on the learners. Exploring the task in a real learning situation would also reveal potential for further improvements from a practical point of view. A comparison of students learning to sketch graphs with and without this tool in a controlled setting might reveal interesting insights.

Thinking even further ahead, we consider working out the task design in a fashion that makes it feasible as a new built-in question type for STACK. In this case every STACK user could easily create similar tasks in the same way other STACK questions are created. Before we will suggest this, some more work on the task must be done.

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