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Office Note 462

The Derivation of the Sigma Pressure Hybrid Coordinate Semi-Lagrangian Model Equations for the GFS

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Hydrodynamics

1 Derivation of the Model's Equations

The derivation is based on ECMWF's publications. These detailed notes intend to provide a documentation of what was actually derived and coded into the existing GFS Eulerian over-structure.

The model equations presented in this document are the equations found in Office Note 461 but cast in a Semi-Lagrangian "ready" form. Namely:

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\,(u,\ v,\ t,\ \ln p,\ \text{moisture, tracers}) = (\text{RHSU},\ \text{RHSV},\ \text{RHST},\ \dots\,)$$

Specifically, the momentum equations are not scaled $(U,V) = \cos\theta\,(u,v)$, and the surface pressure equation is expressed in terms of $\frac{\mathrm{d}p_s}{\mathrm{d}t}$, not $\frac{\partial p_s}{\partial t}$. The thermodynamic equation is included for completeness, and the adiabatic moisture equation simply expresses the vanishing of the moisture's total time derivative. The contributions of the physical parameterization are added in a separate scan of the grid using the splitting method. At its origin, the spectral model was cast in terms of divergence and vorticity. The transition to a velocity formulation was made in anticipation of a Semi-Lagrange implementation. This formulation permits the evaluation of total time derivatives in terms of upstream values in space and time. The specific choice of a time differencing scheme can be deferred to the final stage of the derivation, namely, after the space discretizations are completed. This paper will trace the detailed steps required to express the total time derivatives in terms of spherical harmonics in the horizontal (see appendix) and finite differences using the hybrid vertical coordinate described in Office Note 461. The time integration of the final form of the model equations will be performed in a semi-implicit fashion; the equations will be cast in a general way such that the actual time discretization could be either two or three time levels.

Some of the equations contain partial vertical sums which may be delicate to code. In these cases a detailed exposition is provided for a four level model so that all terms, especially near boundaries, are accounted for.

Indexing in the defining equations is from the top of the atmosphere to bottom. Indexing in the spectral model has always been from bottom to top, and the reader is cautioned to exercise care when switching between notes and codes. At times, upper case multilettered variables will appear in this document. These are presented in order to facilitate identification of variables in the code.

We begin with the definition of a hybrid coordinate.

Let $\quad \eta(0, p_s) = 0. \quad \eta(p_s, p_s) = 1. \quad \eta_{k+\frac{1}{2}} = \frac{A_{k+\frac{1}{2}}}{p_0} + B_{k+\frac{1}{2}}\, p_s$

$A_{k+\frac{1}{2}} \quad B_{k+\frac{1}{2}}$ are scaled to model units, $k = 0$ at top of atmosphere, and $p_0 = 101.325$ cb. It is noted that the actual values of η are only required for semi-lagrange interpolations, not for the derivation of the model's equations. Since the hybrid coordinate vanishes at the top, we set

$$A_{\frac{1}{2}} = B_{\frac{1}{2}} = 0.$$

1.1 The Momentum Equation

The momentum equations without diabatic forcing terms are:

$$\frac{\mathrm{d}\, u}{\mathrm{d}\, t} - fv + \frac{1}{a\cos\theta}\left(\frac{\partial\phi}{\partial\lambda} + R_d T_v \frac{\partial}{\partial\lambda}\ln p\right) = 0 \tag{1.1.1}$$

$$\frac{\mathrm{d}\, v}{\mathrm{d}\, t} + fu + \frac{1}{a}\left(\frac{\partial\phi}{\partial\theta} + R_d T_v \frac{\partial}{\partial\theta}\ln p\right) = 0 \tag{1.1.2}$$

For maintenance of conservation properties (Simmons and Burridge, 1981), we discretize $R_d T_v \nabla \ln p$ as follows:

$$R_d\left(T_v \nabla \ln p\right)_k = \frac{R_d (T_v)_k}{\Delta p_k}\left[\ln\left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right)\nabla p_{k-\frac{1}{2}} + \alpha_{k\nabla\Delta p_k}\right] \tag{1.1.3}$$

$$\alpha_k = 1 - \frac{p_{k-\frac{1}{2}}}{\Delta p_k} \ln \left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}} \right) \tag{1.1.4}$$

for $\quad k \geq 1$, $\quad$ and $\quad \alpha_1 = \ln 2$. $\quad$ (Another choice could be $\alpha = 1$)

The above choice of α_k will reduce eq. (1.1.3) to $R_d \left(T_v\right)_k \nabla \ln p$ when $A_k = 0$. In this case:

$\frac{R_d T}{p} \nabla p_k = R_d T_k \left\{ \frac{1}{\Delta p_k} \left(p_{k+\frac{1}{2}} \ln p_{k+\frac{1}{2}} - p_{k-\frac{1}{2}} \ln p_{k-\frac{1}{2}} \right) \right\}$

which is a discrete analog of $\quad R_d T \nabla \frac{\partial}{\partial p} \left(p \ln p \right) = R_d T \nabla \ln p$

We define coding variables as follows. (These terms will appear on the RHS, hence the negative sign.)

$$-\nabla \phi_k = \begin{cases} -\frac{1}{a \cos\theta} \left(\frac{\partial \phi}{\partial \lambda} \right)_k & = \mathrm{uphi} \\ -\frac{1}{a} \left(\frac{\partial \phi}{\partial \theta} \right)_k & = \mathrm{vphi} \end{cases}$$

$$R_d T_v \nabla \ln p = \begin{cases} \frac{R_d}{a \cos\theta} \left(T_v \frac{\partial}{\partial \lambda} \ln p \right) & = \mathrm{UPRS} \\ \frac{R_d}{a} T_v \frac{\partial \ln p}{\partial \theta} & = \mathrm{VPRS} \end{cases}$$

The geopotential ϕ is diagnosed from the hydrostatic equation.

$\phi_{k+\frac{1}{2}} - \phi_{k-\frac{1}{2}} = -R_d \left(T_v\right)_k \ln \left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}} \right)$

after some manipulation:

$\phi_{k+\frac{1}{2}} = \phi_s + \sum_{j=k+1}^{levs} R_d \left(T_v\right)_j \ln \left(\frac{p_{j+\frac{1}{2}}}{p_{j-\frac{1}{2}}} \right)$

Since we need ϕ_k in layers, we define α_k as in Simmons and Burridge, (1981)

$$\phi_k = \phi_{k+\frac{1}{2}} + \alpha_k R_d \left(T_v\right)_k \tag{1.1.5}$$

Calculation of (UPRS, VPRS)

For maintenance of conservation properties, the thermodynamic equation requires that we use eq. (1.1.3) :

$$R_d\left(T_v \nabla \ln p\right)_k = \frac{R_d (T_v)_k}{\Delta p_k}\left[\ln\left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right)\nabla p_{k-\frac{1}{2}} + \alpha_k \nabla \Delta p_k\right]$$

Since:

$$\nabla \Delta p_k = \nabla\left(A_{k+\frac{1}{2}} + B_{k+\frac{1}{2}}p_s - \left(A_{k-\frac{1}{2}} + B_{k-\frac{1}{2}}p_s\right)\right) = \nabla \Delta B_k p_s = \Delta B_k \nabla p_s$$

We may write

(UPRS, VPRS) =

$$R_d T_v \bigtriangledown \ln p = \frac{R_d (T_v)_k}{\Delta p_k}\left[\ln\left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right) B_{k-\frac{1}{2}}\nabla p_s + \alpha_k \Delta B_k \nabla p_s\right]$$

$$= \frac{R_d (T_v)_k}{\Delta p_k}\left[B_{k-\frac{1}{2}}\ln\left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right) + \alpha_k \Delta B_k\right] p_s \nabla \ln p_s$$

in the code $\quad \nabla \ln p_s = (\mathrm{DPDLAM}, \mathrm{DPDPHI})$

let $\quad \mathrm{cof}\, b = \frac{1}{\Delta p_k}\left[B_{k-\frac{1}{2}}\ln\left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right) + \alpha_k \Delta B_k\right]$

Then $\quad (\mathrm{UPRS}, \mathrm{VPRS}) = \mathrm{cof}\, b\, R_d \left(T_v\right)_k p_s\ (\mathrm{DPDLAM}, \mathrm{DPDPHI})$

for $\quad k = 1 \quad \ln\left(\frac{p_{\frac{3}{2}}}{p_{\frac{1}{2}}}\right)\nabla p_{\frac{1}{2}}$, $\quad$ at the top $\quad p_{\frac{1}{2}} = A_{\frac{1}{2}} = B_{\frac{1}{2}} = 0$

$$\lim_{p_{\frac{1}{2}}\to 0}\left(\ln\left(\frac{p_{\frac{3}{2}}}{p_{\frac{1}{2}}}\right)\nabla p_{\frac{1}{2}}\right) = 0$$

therefore $\quad (\mathrm{UPRS}, \mathrm{VPRS})_{k=1} = R_d \frac{(T_k)_1}{\Delta p_1}\ (\mathrm{DPDLAM}, \mathrm{DPDPHI}) + \alpha_1 \Delta B_1$

Calculation of (uphi, vphi)

$(\text{uphi, vphi}) = -\nabla\phi_k$

$\phi_k = \phi_{k+\frac{1}{2}} + \alpha_k R_d \left(T_v\right)_k \ , \quad \phi_{k+\frac{1}{2}} = \phi_s + \sum_{j=k+1}^{Levs} R_d \left(T_v\right)_j \ln\left(\frac{p_{j+\frac{1}{2}}}{p_{j-\frac{1}{2}}}\right)$

$\alpha_1 = \ln 2 \quad \text{or} \quad (=1) \ , \quad \alpha_k = 1 - \frac{p_{k+\frac{1}{2}}}{\Delta p_k} \ln\left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right) \quad k > 1 \ .$

then: $\quad -\nabla\phi_k = -\nabla\left[\phi_s + \sum_{j=k+1}^{Levs} R_d \left(T_v\right)_j \ln\left(\frac{p_{j+\frac{1}{2}}}{p_{j-\frac{1}{2}}}\right) + \alpha_k R_d \left(T_v\right)_k\right]$

$= -\nabla\phi_s - R_d\left\{\sum_{j=k+1}^{Levs}\left[\left(T_v\right)_j \nabla\pi_j\pi_i + \pi_j\nabla\left(T_v\right)_j\right] + \alpha_k\nabla\left(T_v\right)_k + \left(T_v\right)_k\nabla\alpha_k\right\}$

where $\quad \pi_j = \ln\frac{p_{j+\frac{1}{2}}}{p_{j-\frac{1}{2}}}$

let $\quad (\text{uphi, vphi}) = \overrightarrow{Px_1} + \overrightarrow{Px_2} + \overrightarrow{Px_3} + \overrightarrow{Px_4} + \overrightarrow{Px_5}$ (for coding purposes)

$$\overrightarrow{Px_1} = -\nabla\phi_s \ , \quad \overrightarrow{Px_2} = -R_d \sum_{j=k+1}^{Levs} \left(T_v\right)_j \nabla\pi_j$$

$$\overrightarrow{Px_3} = -R_d \sum_{j=k+1}^{Levs} \pi_j \nabla\left(T_v\right)_j$$

$$\overrightarrow{Px_4} = -R_d \alpha_k \nabla\left(T_v\right)_k \ , \quad \overrightarrow{Px_5} = -R_d \left(T_v\right)_k \nabla\alpha_k$$

Evaluate $\quad \overrightarrow{Px_2}$

$$\overrightarrow{Px_2} = -R_d \sum_{j=k+1}^{Levs} \left(T_v\right)_j \nabla\pi_j$$

$\nabla\pi_j = \nabla\ln\left(\frac{p_{j+\frac{1}{2}}}{p_{j-\frac{1}{2}}}\right) = \frac{\nabla p_{j+\frac{1}{2}}}{p_{j+\frac{1}{2}}} - \frac{\nabla p_{j-\frac{1}{2}}}{p_{j-\frac{1}{2}}}$

$$= \frac{B_{j+\frac{1}{2}}}{p_{j+\frac{1}{2}}} \nabla p_s - \frac{B_{j-\frac{1}{2}}}{p_{j-\frac{1}{2}}} \nabla p_s = \left(\frac{B_{j+\frac{1}{2}}}{p_{j+\frac{1}{2}}} - \frac{B_{j-\frac{1}{2}}}{p_{j-\frac{1}{2}}} \right) p_s \nabla \ln p_s \qquad (1.1.6)$$

$$\overrightarrow{Px_2} = -R_d \sum_{j=k+1}^{Levs} (T_v)_j \left(\frac{B_{j+\frac{1}{2}}}{p_{j+\frac{1}{2}}} - \frac{B_{j-\frac{1}{2}}}{p_{j-\frac{1}{2}}} \right) p_s \nabla \ln p_s$$

$$\overrightarrow{Px_2} = -R_d \sum_{j=k+1}^{Levs} (T_v)_j \left(\frac{B_{j+\frac{1}{2}}}{p_{j+\frac{1}{2}}} - \frac{B_{j-\frac{1}{2}}}{p_{j-\frac{1}{2}}} \right) \; p_s \underbrace{(\mathrm{DPDLAM}, \mathrm{DPDPHI})}_{\nabla \ln p_s}$$

for a σ coordinate $\quad A_{k+\frac{1}{2}} = 0 \quad \frac{B_{j+\frac{1}{2}}}{p_{j+\frac{1}{2}}} - \frac{B_{j-\frac{1}{2}}}{p_{j-\frac{1}{2}}} = \frac{1}{p_s} - \frac{1}{p_s} = 0$

$\overrightarrow{Px_2} = 0$ (useful for code validation)

Evaluate $\overrightarrow{Px_3}$

$$\overrightarrow{Px_3} = -R_d \sum_{j=k+1}^{Levs} \ln \frac{p_{j+\frac{1}{2}}}{p_{j-\frac{1}{2}}} \nabla (T_v)_j$$

$$\overrightarrow{Px_3} = -R_d \sum_{j=k+1}^{Levs} \ln \frac{p_{j+\frac{1}{2}}}{p_{j-\frac{1}{2}}} \; (\,\mathrm{dtdl(j)},\ \mathrm{dtdf(j)}\,)$$

$$\overrightarrow{Px_4} = -R_d \alpha_k \nabla (T_v)_k = -R_d \alpha_k \; (\,\mathrm{dtdl(k)},\ \mathrm{dtdf(k)}\,)$$

Evaluate $\overrightarrow{Px_5}$

$$\overrightarrow{Px_5} = -R_d (T_v)_k \nabla \alpha_k$$

now from eq. (1.1.4) :

$$\nabla \alpha_k = \nabla \left(1 - \frac{p_{k-\frac{1}{2}}}{\Delta p_k} \ln \left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}} \right) \right) = -\frac{p_{k-\frac{1}{2}}}{\Delta p_k} \left\{ \nabla \ln \left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}} \right) \right\} - \ln \left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}} \right) \left\{ \nabla \frac{p_{k-\frac{1}{2}}}{\Delta p_k} \right\}$$

from eq. (1.1.6) evaluate

$$\nabla \ln\left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right) = \frac{\nabla p_{k+\frac{1}{2}}}{p_{k+\frac{1}{2}}} - \frac{\nabla p_{k-\frac{1}{2}}}{p_{k-\frac{1}{2}}} = \left(\frac{B_{k+\frac{1}{2}}}{p_{k+\frac{1}{2}}} - \frac{B_{k-\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right) p_s \nabla \ln p_s$$

also:

$$\nabla \frac{p_{k-\frac{1}{2}}}{\Delta p_k} = \frac{B_{k-\frac{1}{2}}}{\Delta p_k} \nabla p_s - p_{k-\frac{1}{2}} \frac{\nabla \Delta p_k}{(\Delta p_k)^2}$$

$$= \frac{B_{k-\frac{1}{2}}}{\Delta p_k} \nabla p_s - \frac{p_{k-\frac{1}{2}}}{(\Delta p_k)^2} \left[\left(B_{k+\frac{1}{2}} - B_{k-\frac{1}{2}}\right) \nabla p_s\right] = \left(\frac{B_{k-\frac{1}{2}}}{\Delta p_k} - \frac{p_{k-\frac{1}{2}}}{(\Delta p_k)^2} \Delta B_k\right) p_s \nabla \ln p_s$$

then

$$\nabla \alpha_k = -\frac{p_{k-\frac{1}{2}}}{\Delta p_k} \left(\frac{B_{k+\frac{1}{2}}}{p_{k+\frac{1}{2}}} - \frac{B_{k-\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right) p_s \nabla \ln p_s - \ln\left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right) \left(\frac{B_{k-\frac{1}{2}}}{\Delta p_k} - \frac{p_{k-\frac{1}{2}}}{(\Delta p_k)^2} \Delta B_k\right) p_s \nabla \ln p_s$$

$$\nabla \alpha_k = -\frac{1}{\Delta p_k} \left\{\frac{p_{k-\frac{1}{2}}}{p_{k+\frac{1}{2}}} B_{k+\frac{1}{2}} - B_{k-\frac{1}{2}} + \ln\left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right) \left(B_{k-\frac{1}{2}} - \frac{p_{k-\frac{1}{2}}}{\Delta p_k} \Delta B_k\right)\right\} p_s \nabla \ln p_s$$

let

$$\text{cofa(k)} = -\frac{1}{\Delta p_k} \left\{\frac{p_{k-\frac{1}{2}}}{p_{k+\frac{1}{2}}} B_{k+\frac{1}{2}} - B_{k-\frac{1}{2}} + \ln\left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right) \left(B_{k-\frac{1}{2}} - \frac{p_{k-\frac{1}{2}}}{\Delta p_k} \Delta B_k\right)\right\} \quad \text{in code}$$

then:

$$\nabla \alpha_k = \text{ cofa(k)}\, p_s \nabla \ln p_s$$

and finally:

$$\overrightarrow{Px_5} = -R_d \left(T_v\right)_k \nabla \alpha_k = -R_d \left(T_v\right)_k \text{ cofa(k)}\, p_s \nabla \ln p_s$$

$$\overrightarrow{Px_5} = -R_d \, (T_v)_k \ \mathrm{cofa(k)} \, p_s \ (\,\mathrm{DPDLAM,\ DPDPHI}\,) \quad \text{in code}$$

$$\begin{cases} \text{for coordinate sigma} \quad A_{k+\frac{1}{2}} = 0 \\ \mathrm{cofa(k)} = -\frac{1}{\Delta p_k} \left\{ \frac{B_{k-\frac{1}{2}} p_s}{B_{k+\frac{1}{2}} p_s} B_{k+\frac{1}{2}} - B_{k-\frac{1}{2}} + \ln \left(\frac{B_{k+\frac{1}{2}} p_s}{B_{k-\frac{1}{2}} p_s} \right) \left(B_{k-\frac{1}{2}} - \frac{\Delta B_k B_{k-\frac{1}{2}} p_s}{\Delta B_k p_s} \right) \right\} p_s \nabla \ln p_s \\ \mathrm{cofa(k)} = 0 \quad \overrightarrow{Px_5} = 0 \quad \text{(useful for code validation)} \end{cases}$$

1.2 The surface pressure equation

Unlike the Eulerian framework, the surface pressure requires an expression for $\frac{\mathrm{d}p_s}{\mathrm{d}t}$, $\left(\text{not } \frac{\partial p_s}{\partial t}\right)$.

We start with the continuity equation:

$$\frac{\partial}{\partial t} \frac{\partial p}{\partial \eta} + \nabla_3 \cdot \frac{\partial p}{\partial \eta} \overrightarrow{\mathrm{V}}_3 = 0 \qquad \text{expand and regroup} \tag{1.2.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial p}{\partial \eta} = -\frac{\partial p}{\partial \eta} \left(D + \frac{\partial \dot{\eta}}{\partial \eta} \right) , \qquad D = \nabla \cdot \overrightarrow{\mathrm{v}}_H \tag{1.2.2}$$

$$\text{now express} \quad \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial p}{\partial \eta} \quad \text{using} \quad p = A\,(\eta) + B\,(\eta)\, p_s \tag{1.2.3}$$

(In principle, we could continue with $\frac{\partial p}{\partial \eta}$)

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial p}{\partial \eta} = \frac{\partial}{\partial t} \frac{\partial}{\partial \eta} (A + B p_s) + \overrightarrow{\mathrm{V}} \cdot \nabla \frac{\partial}{\partial \eta} (A + B p_s) + \dot{\eta} \frac{\partial}{\partial \eta} \frac{\partial p}{\partial \eta} \tag{1.2.4}$$

$$= \frac{\partial B}{\partial \eta}\frac{\partial p_s}{\partial t} + \overrightarrow{\mathrm{V}} \cdot \nabla \frac{\partial B}{\partial \eta} p_s + \dot{\eta}\frac{\partial}{\partial \eta}\frac{\partial p}{\partial \eta} \tag{1.2.5}$$

$$= \frac{\partial B}{\partial \eta}\left(\frac{\partial p_s}{\partial t} + \overrightarrow{\mathrm{V}} \cdot \nabla p_s + \dot{\eta}\frac{\partial p_s}{\partial \eta}\right) + \dot{\eta}\frac{\partial}{\partial \eta}\frac{\partial p}{\partial \eta} \tag{1.2.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial p}{\partial \eta} = \frac{\partial B}{\partial \eta}\frac{\mathrm{d}p_s}{\mathrm{d}t} + \dot{\eta}\frac{\partial}{\partial \eta}\frac{\partial p}{\partial \eta} \tag{1.2.7}$$

From eqs. (1.2.2) and (1.2.7),

$$\text{equate the two expressions for} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial p}{\partial \eta} \tag{1.2.8}$$

$$\frac{\partial B}{\partial \eta}\frac{\mathrm{d}p_s}{\mathrm{d}t} + \dot{\eta}\frac{\partial}{\partial \eta}\frac{\partial p}{\partial \eta} = -\frac{\partial p}{\partial \eta}\left(D + \frac{\partial \dot{\eta}}{\partial \eta}\right) \tag{1.2.9}$$

$$\frac{\partial B}{\partial \eta}\frac{\mathrm{d}p_s}{\mathrm{d}t} + \frac{\partial p}{\partial \eta}D + \frac{\partial}{\partial \eta}\left(\dot{\eta}\frac{\partial p}{\partial \eta}\right) = 0 \tag{1.2.10}$$

Vertical discretization yields:

$$\Delta B_k \frac{\mathrm{d}\,p_s}{\mathrm{d}\,t} + \Delta p_k D_k + \left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{k+\frac{1}{2}} - \left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{k-\frac{1}{2}} = 0 \tag{1.2.11}$$

for $\quad k = 1, \ldots, K \;, \qquad K \;=\;$ number of levels.

Recall that $\quad \displaystyle\sum_{k=1}^{K} \Delta B_k = 1$

Divide eq. (1.2.11) by $\quad p_s \quad$ to get $\quad \frac{\mathrm{d}}{\mathrm{d}\,t} \ln p_s$

$$\Delta B_k \frac{\mathrm{d}\,\ln p_s}{\mathrm{d}\,t} + \frac{1}{p_s}\left[\Delta p_k D_k + \left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{k+\frac{1}{2}} - \left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{k-\frac{1}{2}}\right] = 0 \qquad (1.2.12)$$

The details of the final form of eq. (1.2.12) will be discussed when the time descretization is presented.

1.3 The Thermodynamic Equation

Conservation of energy and hydrostatic conditions imply:

$\mathrm{d}Q = c_p \mathrm{d}T - \alpha \mathrm{d}p \;, \quad \alpha = \frac{1}{\rho} = \frac{R_d T_v}{p}$

If $\quad \mathrm{d}Q = 0$

$c_p \frac{\mathrm{d}T}{\mathrm{d}t} = \alpha \frac{\mathrm{d}p}{\mathrm{d}t}$

$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\alpha}{c_p}\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\alpha\omega}{c_p} = \frac{R_d T_v \omega}{c_p p} \;, \quad \alpha = \frac{R_d T_v}{p}$

We also have

$c_p = C_{pd} + (C_{pv} - C_{pd})\,q = C_{pd}\left[1 + \left(\frac{C_{pv}}{C_{pd}} - 1\right) q\right]$

and $\quad \kappa = \frac{R_d}{C_{pd}}$. We may then write:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{R_d T_v \omega}{C_{pd}\left[1 + \left(\frac{C_{pv}}{C_{pd}} - 1\right) q\right] p} = \frac{\kappa T_v \omega}{(1 + (\delta - 1)\,q)\,p} \qquad (1.3.1)$$

where $\quad \delta = \frac{C_{pv}}{C_{pd}} = \frac{1846}{1004.6} = 1.837$

$\delta - 1 = 0.8375$

For T_v as history variable: let $\quad \epsilon = \frac{R_v}{R_d} - 1 = 0.6077$

$T_v = (1 + \epsilon q)\, T \;, \quad \frac{\mathrm{d}T_v}{\mathrm{d}t} = (1 + \epsilon q)\, \frac{\mathrm{d}T}{\mathrm{d}t} + \epsilon T \frac{\mathrm{d}q}{\mathrm{d}t}$

$\frac{\mathrm{d}T_v}{\mathrm{d}t} = (1 + \epsilon q)\, \frac{\kappa T_v \omega}{(1+(\delta-1)q)p} + \epsilon T \frac{\mathrm{d}q}{\mathrm{d}t}$

The thermodynamic equation can therefore be written as:

$$\frac{\mathrm{d}T_v}{\mathrm{d}t} = \frac{\kappa T_v \omega}{p} \underbrace{\frac{(1 + \epsilon q)}{(1 + (\delta - 1)\, q)}}_{\Uparrow \text{ calculated in dynamics}} + \underbrace{\epsilon T \frac{\mathrm{d}q}{\mathrm{d}t}}_{\Uparrow \text{ calculated in physics}} \qquad (1.3.2)$$

The "correction" term $\quad \frac{1+\epsilon q}{1+(\delta-1)q} = \frac{1+0.6077q}{1+0.8375q}$

for large $\quad q = \frac{50}{1000}$

$\frac{1+\epsilon q}{1+(\delta-1)q} = 0.99$

The energy conversion term.

let $C_q = \frac{1+\epsilon q}{1+(\delta-1)q}$, the ECMWF energy conv. term (Ritchie, et al, 95 eq. 2.25) is:

$$\left(\kappa \frac{T_v \omega}{p} \frac{1+\epsilon q}{1+(\delta-1)q}\right)_k = \kappa \left(T_v\right)_k \frac{1+\epsilon q_k}{1+(\delta-1)q_k} \Bigg\langle$$

$$-\frac{1}{\Delta p_k} \left\{ \ln\left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right) \sum_{j=1}^{k-1} \left[D_j \Delta p_j + p_s \left(\vec{v}_j \cdot \nabla \ln p_s\right) \Delta B_j\right] \right.$$

$$\left. + \ \alpha_k \left[D_k \Delta p_k + p_s \left(\vec{v}_k \cdot \nabla \ln p_s\right) \Delta B_k\right] \right\}$$

$$+\frac{p_s}{\Delta p_k} \left[\Delta B_k + \frac{C_k}{\Delta p_k} \ln\left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right)\right] \left(\vec{v}_k \cdot \nabla \ln p_s\right) \Bigg\rangle$$

where

$$C_k = A_{k+\frac{1}{2}} B_{k-\frac{1}{2}} - A_{k-\frac{1}{2}} B_{k+\frac{1}{2}}$$

We now define variables for coding purposes:

$$\text{worka}_k = \frac{\kappa \left(T_v\right)_k}{\Delta p_k} \cdot \frac{1+\epsilon q_k}{1+(\delta-1)q_k}$$

$$\text{workb}_k = \ln\left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right) \sum_{j=1}^{k-1} \left[D_j \Delta p_j + p_s \left(\vec{v}_j \cdot \nabla \ln p_s\right) \Delta B_j\right]$$

$$+ \alpha_k \left[D_k \Delta p_k + p_s \left(\vec{v}_k \cdot \nabla \ln p_s\right) \Delta B_k\right]$$

$$\text{workc}_k = p_s \left[\Delta B_k + \frac{C_k}{\Delta p_k} \ln\left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right)\right] \vec{v}_k \cdot \nabla \ln p_s$$

Noting that $\frac{1}{\Delta p_k}$ was absorbed in worka_k the adiabatic form of eq. (1.3.2) becomes:

$$\left(\frac{\mathrm{d}T_v}{\mathrm{d}t}\right)_k = \left(\frac{\kappa T_v \omega}{p} \frac{1+\epsilon q}{1+(\delta-1)q}\right)_k = \text{worka}_k \left(-\text{workb}_k + \text{workc}_k\right)$$

1.4 Moisture Equation

The moisture specific humidity equation is:

$$\frac{\mathrm{d}\,q}{\mathrm{d}\,t} = S_q \tag{1.4.1}$$

S_q is the sum of sources and sinks.

1.5 Vertical Velocity Equation

ω is diagnosed as in the Eulerian frame.

$$\omega = -\int_0^\eta \nabla \cdot \left(v_H \frac{\partial p}{\partial \eta} \right) \mathrm{d}\eta + \vec{v}_H \cdot \nabla p \tag{1.5.1}$$

By definition: $\omega = \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\partial p}{\partial t} + \vec{v}_H \cdot \nabla p + \dot{\eta} \frac{\partial p}{\partial \eta}$

substitute $\frac{\partial p}{\partial t}$ in the continuity equation (1.2.1)

$\frac{\partial}{\partial \eta} \left(\omega - \vec{v}_H \cdot \nabla p - \dot{\eta} \frac{\partial p}{\partial \eta} \right) + \nabla \cdot v_H \frac{\partial p}{\partial \eta} + \frac{\partial}{\partial \eta} \left(\dot{\eta} \frac{\partial p}{\partial \eta} \right) = 0$

integrate 0 to η $(\omega - v_H \cdot \nabla p) \mid_0^\eta + \int_0^\eta \nabla \cdot \vec{v}_H \frac{\partial p}{\partial \eta} \mathrm{d}\eta = 0$

if $\omega_{\eta=0} = 0$ and $\left(\vec{v}_H \cdot \nabla p \right)_{\eta=0} = 0$ eq. (1.5.1) follows.

1.6 Time Discretization

The momentum equation can be manipulated into the following form:

$$u^+ = u^A_{t+\Delta t} = \tilde{U} - \frac{\Delta t}{2} UL^A_{t+\Delta t} \tag{1.6.1}$$

$$v^+ = v^A_{t+\Delta t} = \tilde{V} - \frac{\Delta t}{2} VL^A_{t+\Delta t}$$

where superscript "A" denotes arrival point, and $(\ )^+$ denotes values out $t + \Delta t$ at arrival point.

$$\text{Here} \quad \overrightarrow{VL} = (UL, VL) = \nabla \left(AT + R_d T_r \ln p_S \right)$$

is the linear part of the momentum eqs., to be treated in a semi-implicit fashion. A is a matrix, and T_r is a reference temperature.

Also let

$$q = \ln p_s$$

Then, taking the divergence of eq (1.6.1) we have:

$$D^+ = X - \frac{\beta \, \Delta t}{2} \, \nabla^2 \left(AT^+ + R_d T_r q^+ \right) \tag{1.6.2}$$

$$\text{where} \qquad X = \nabla \cdot (\tilde{u}, \tilde{v})$$

The corresponding vorticity equation is integrated explicitly.

The thermodynamic and surface pressure equations are:

$$T^{+} = T^{A}_{t+\Delta t} = Y - \frac{\beta\,\Delta t}{2}\,\tau\,D^{A}_{t+\Delta t} \tag{1.6.3}$$

$$q^{+} = q^{A}_{t+\Delta t} = Z - \Delta t\,S \cdot D^{A}_{t+\Delta t} \tag{1.6.4}$$

Equations (1.6.2), (1.6.3), (1.6.4) comprise a coupled system and are solved for all spectral components.

It is noted that the coupled system of divergence thermodynamic and surface pressure equations contains arrival point variables at $t+\Delta t$, and the terms $\tilde{U}$, $\tilde{V}$, Y and Z . These last terms contain the sum total of all the required operations in time and in space, namely calculations of departure points, interpolation of neighboring values to the departure points, as well as, rotation of the velocity components to account for the sperical coordinate system. Viewed in this light, it becomes clear that the center of gravity of the implementation of the Semi-Lagrange method is in the above details. For this reason, a special effort has to be made in order to create a useful and focused documentation which will be presented in a subsequent Office Note.

In the following, all history variables, as well as X, Y, and Z, are the spectral coefficients in the appropriate expansions.

let $\quad F^{+} = F^{A}_{t+\Delta t}\ , \qquad b = \dfrac{\beta\,\Delta t}{2}\ , \qquad r = R_d T_r$

where T_r is a reference temperature elaborated on in the detailed documentation of the Semi-Lagrange implementation.

Then:

$$D^{+} = X - b\,\nabla^2\left[A\left(Y - b\,\tau\,D^{+}\right) + r\left(Z - \Delta t\,S \cdot D^{+}\right)\right]$$

where $\quad S = \dfrac{1}{2\,p^r_s}\,(\Delta p^r_1\ \cdots\ \Delta p^r_K)$

regrouping:

$$\left[I + b\,\nabla^2\left(-bA\tau - r\,\Delta t\,S\right)\right] D^+ = X - b\,\nabla^2\left(AY + rZ\right)$$

expressing the Laplacian explicitly:

$$\left[I + \frac{n\,(n+1)}{a^2}\,b\,(bA\tau + r\,\Delta t\,S)\right] D^+ = X + \frac{n\,(n+1)}{a^2}\,b\,\left(AY + rz\right) \qquad (1.6.5)$$

let $$\mathrm{D\_m} = I + \frac{n\,(n+1)}{a^2}\left[\left(\frac{\beta\Delta t}{2}\right)^2 A\tau + \left(\frac{\beta\Delta t}{2}\right)^2 \frac{R_d}{P^r} T_r \cdot \Delta p^r\right]$$

$$\mathrm{D\_m} = I + \frac{n\,(n+1)}{a^2}\,\frac{\beta^2\,(\Delta t)^2}{4}\left[A\tau + \frac{R_d}{P^r} T_r \cdot \Delta p^r\right]$$

and $T_r \cdot \Delta P^r$ is the LEVS x LEVS matrix:

$$T_r \cdot \Delta P^r = \begin{pmatrix} T_1^r \\ \vdots \\ T_{Levs}^r \end{pmatrix} \left(\Delta p_1^r \;\cdots\; \Delta p_{Levs}^r\right)$$

Equation (1.6.5) can now be solved for D^+ .

$$D^+ = D_m^{-1}\left[x + \frac{n\,(n+1)}{a^2}\,\frac{\beta\Delta t}{2}\left(AY + R_d T_r Z\right)\right]$$

Finally T^+ and q^+ can also be isolated.

Table 1: Vertical Structure Indexing top to bottom

k	$k+\frac{1}{2}$	ak5	bk5	pk5	$\eta\frac{\partial p}{\partial \eta}$
0	$\frac{1}{2}$	ak5(1)	bk5(1)	pk5(1)	$\left(\eta\frac{\partial p}{\partial \eta}\right)_{\frac{1}{2}}$=Dot(1)
1	$1\frac{1}{2}$	ak5(2)	bk5(2)	pk5(2)	$\left(\eta\frac{\partial p}{\partial \eta}\right)_{1\frac{1}{2}}$=Dot(2)
2	$2\frac{1}{2}$	ak5(3)	bk5(3)	pk5(3)	$\left(\eta\frac{\partial p}{\partial \eta}\right)_{2\frac{1}{2}}$=Dot(3)
3	$3\frac{1}{2}$	ak5(4)	bk5(4)	pk5(4)	$\left(\eta\frac{\partial p}{\partial \eta}\right)_{3\frac{1}{2}}$=Dot(4)

1.7 Indexing and Looping

consider $\overrightarrow{Px_2}$:

$$\overrightarrow{Px_2}\,(k) = -R_d \left[\sum_{j=k+1}^{Levs} (T_v)_j \left(\frac{B_{j+\frac{1}{2}}}{p_{j+\frac{1}{2}}} - \frac{B_{j-\frac{1}{2}}}{p_{j-\frac{1}{2}}} \right) \right] p_s \;\; (\,\mathrm{DPDLAM},\ \mathrm{DPDPHI}\,) = (\,\mathrm{px2u},\ \mathrm{px2v}\,)$$

Using the notation in Table 1

$$\text{let} \quad f_{j+\frac{1}{2}} = \frac{B_{j+\frac{1}{2}}}{p_{j+\frac{1}{2}}} \,, \quad \mathrm{px(k)} = \sum_{j=k+1}^{Levs} (T_v)_j \left(f_{j+\frac{1}{2}} - f_{j-\frac{1}{2}} \right) \quad k = 1,\, \ldots,\, Levs$$

Let Levs=4 k=1, 2, 3, 4 $Px = \mathrm{px2\ scalar}\,,\ \mathrm{in\ code}$

$$\overrightarrow{Px_2} = (\,\mathrm{px2u},\ \mathrm{px2v}\,)$$

k=Levs 4	$\text{px}\,(4) = \sum_{j=4+1}^{4} \equiv 0$
k=Levs-1 3	$\text{px}\,(3) = \sum_{j=3+1}^{4} = (T_v)_4 \left(f_{4+\frac{1}{2}} - f_{4-\frac{1}{2}}\right)$
k=Levs-2 2	$\text{px}\,(2) = \sum_{j=2+1}^{4} = \sum_{j=3}^{4} = (T_v)_3 \left(f_{3+\frac{1}{2}} - f_{3-\frac{1}{2}}\right) + (T_v)_4 \left(f_{4+\frac{1}{2}} - f_{4-\frac{1}{2}}\right)$ $= (T_v)_3 \left(f_{3+\frac{1}{2}} - f_{3-\frac{1}{2}}\right) + \text{px}\,(3)$
k=Levs-3 1	$\text{px}\,(1) = \sum_{j=1+1}^{4} = \sum_{j=2}^{4} = (T_v)_2 \left(f_{2+\frac{1}{2}} - f_{2-\frac{1}{2}}\right) + \text{px}\,(2)$

$$
\begin{array}{ll}
 & \left\{ \begin{array}{l}
\text{DO}\ \ \text{k} = 2,\, 4-1 \quad (\text{k} = 2,\, 3\,) \\[6pt]
\quad \text{px}\,(2) = \text{px}\,(3) - R_d \left(\frac{\text{B}(4)}{\text{p}(4)} - \frac{\text{B}(3)}{\text{p}(3)} \qquad \right) \ \text{Tg}\,(3) \\[6pt]
\quad \text{px}\,(1) = \text{px}\,(2) - R_d \left(\frac{\text{B}(3)}{\text{p}(3)} - \frac{\text{B}(2)}{\text{p}(2)} \qquad \right) \ \text{Tg}\,(2)
\end{array} \right.
\end{array}
$$

k = 2

k = 3

$\omega\,(\eta)$ in the physics.

The vertical velocity ω is calculated as in the energy conversion term and is passed on to the physical parameterization routines.

Indexing is assumed top to bottom

$$P_{1/2} = PK5(1) \text{ ———— } 1/2$$

$$P_{1\,1/2} = PK5(2) \text{ ———— } 1\,1/2$$

$$P_{2\,1/2} = PK5(3) \text{ ———— } 2\,1/2$$

$$P_{3\,1/2} = PK5(4) \text{ ———— } 3\,1/2$$

$$P_{4\,1/2} = PK5(5) \text{ ———— } 4\,1/2$$

1.8 Schematic Vertical Grid

LEVS	= number of model layers
$p_{K+\frac{1}{2}}$	= pressure at interfaces
p_k	= pressure in model layers

$$
\begin{array}{rcl}
p_{\frac{1}{2}} = 0 & \underline{\hspace{3cm}} & \\
 & - - - - - - - - & p_1 \\
p_{\frac{3}{2}} & \underline{\hspace{3cm}} & \\
 & - - - - - - - - & p_2 \\
p_{\frac{5}{2}} & \underline{\hspace{3cm}} & \\
 & \vdots & \\
 & \underline{\hspace{3cm}} & \\
 & - - - - - - - - & p_{LEVS-1} \\
p_{LEVS-\frac{1}{2}} & \underline{\hspace{3cm}} & \\
 & - - - - - - - - & p_{LEVS} \\
p_{LEVS+\frac{1}{2}} & \underline{\hspace{3cm}} &
\end{array}
$$

$a_{k+\frac{1}{2}}$, $b_{k+\frac{1}{2}}$ are defined at interfaces

1.9 Notation

λ	longitude
θ	latitude
η	hybrid vertical coordinate
A_k B_k	level dependent constants defining η
t	time
a	radius of the earth
f	coriolis parameter
(u, v)	horizontal velocity components
p	pressure
p_s	surface pressure
p_r	reference pressure used in linearization for semi-implicit time integration
$\omega = \frac{\mathrm{d}p}{\mathrm{d}t}$	
T	temperature
T_r	reference temperature
T_v	virtual temperature
ϕ	geopotential
ρ	density
$\alpha = \frac{1}{\rho}$	specific volume
q	specific humidity
Q	energy per unit mass
R_d	dry air gas constant
c_p	dry air specific heat at constant pressure
c_v	dry air specific heat at constant volume
κ	$= R_d/c_p$
ϵ	$= R_v/R_d + 1$
δ	C_{pv}/C_{pd}

Y	hydrostatic matrix
$X^A_{t+\Delta t}$	X at arrival point and at time $t + \Delta t$
X^D_t	X at departure point at time t
$X^M_{t+\frac{\Delta t}{2}}$	X at mid-point of trajectory and at mid-time

1.10 References

Simmons and Burridge, MWR, vol 109, An Energy and Angular-Momentum Conserving Vertical Finite-Difference Scheme and Hybrid Vertical Coordinates.

Ritchie, et al, MWR, vol 123. Implementation of the Semi-Lagrangian Method in a High-Resolution Version of the ECMWF Forecast Model.

IFS Documentation CY28r1, 2004. Dynamics and Numerical Procedures.

2 Appendix: Overview of the Spectral Technique

2.1 Definition of Expansion in Spherical Harmonics

In the following, all prognostic variables as well as the tendencies will be assumed to possess a spherical harmonic representation of the form

$$F(\varphi, \lambda) = \sum_{l=-J}^{J} \sum_{n=|l|}^{N(l)} F_n^l Y_n^l \tag{2.1.1}$$

where the spherical harmonics Y_n^l are given by

$$Y_n^l(\varphi, \lambda) = p_n^l(\sin\varphi)\, e^{il\lambda} \tag{2.1.2}$$

(p_n^l computed in subroutine PLN.)

The upper limit of the second sum is left as a general function of l. The model can be integrated with any resolution, and the specific resolution is determined by a control array, set at the beginning of the integration.

The expansion coefficients F_n^l are functions of time and sigma and the p_n^l are Legendre polynomials given by

$$p_n^l(x) = \frac{1}{2^n n!}\left[\frac{(2n+1)(n-l)}{2(n+l)!}\right]^{\frac{1}{2}} \left(1-x^2\right)^{\frac{l}{2}} \frac{\mathrm{d}^{l+n}\left(1-x^2\right)^n}{\mathrm{d}x^{l+n}} \tag{2.1.3}$$

where $x = \sin\varphi$ and the normalization of the p_n^l is such that

$$\int_{-1}^{1} p_n^l p_m^l \mathrm{d}x = \delta_{mn} \tag{2.1.4}$$

The p_n^l can be generated from the recursion relation

$$\epsilon_{n+1}^l p_{n+1}^l(x) = x p_n^l(x) - \epsilon_n^l p_{n-1}^l(x) \tag{2.1.5}$$

where

$$\epsilon_n^l = \left(\frac{n^2 - l^2}{4n^2 - 1}\right)^{\frac{1}{2}} \tag{2.1.6}$$

When horizontal gradients require north-south derivatives, they can be obtained from

$$\left(1-x^2\right)\frac{\mathrm{d}p_n^l(x)}{\mathrm{d}x} = (n+1)\,\epsilon_n^l p_{n-1}^l(x) - n\epsilon_{n+1}^l p_{n+1}^l(x) = \cos\varphi \frac{\mathrm{d}p_n^l}{\mathrm{d}\varphi} \tag{2.1.7}$$

2.2 Grid to Spectral Transform

Assume that the grid values of a given field are known, and that we require the field's expansion coefficients. In practice, the field could represent a history variable or a tendency.

Let the field be represented by Eq. (2.1.1). Multiplication of this equation by $\left(Y_n^l\right)^*$ and a surface integration over the sphere yields:

$$F_n^l = \frac{1}{2\pi}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F\left(\varphi,\lambda\right) p_n^l\left(\sin\varphi\right) e^{-il\lambda}\cos\varphi \mathrm{d}\varphi \mathrm{d}\lambda \tag{2.2.1}$$

The numerical evaluation of the integrals in Eq. (2.2.1) is carried out in two steps:

Step 1. We define the field's Fourier coefficients at a given latitude as

$$F^l\left(\varphi\right) = \frac{1}{2\pi}\int_0^{2\pi} F\left(\varphi,\lambda\right) e^{-il\lambda}\mathrm{d}\lambda \tag{2.2.2}$$

This integral can be evaluated using a discrete Fourier transform, provided F is a trigonometric polynomial.

In general, if $f(x)$ is a trigonometric polynomial of degree not exceeding $M-1$,

$$\int_0^{2\pi} f\left(x\right)\mathrm{d}x = \frac{2\pi}{M}\sum_{j=0}^{M-1} f\left(\frac{2\pi j}{M}\right) \tag{2.2.3}$$

is an exact evaluation of the integral (Krylov, 1962).

Since the model's variables are assumed to have a spherical harmonic expansion, they are represented in the zonal direction by a trigonometric polynomial of degree J. Quadratic terms will contain powers of at most $2J$, and the integrand in Eq. (2.2.2) will be of degree not exceeding $3J$. For exact integration in the zonal direction, we therefore require at least $3J+1$ points around a latitude circle.

Step 2. The integration in the meridional direction is performed by Gaussian Quadrature (Krylov, 1962, p. 108). The Gaussian weights are

$$W_k = 2\left(1-x^2\right)^{-1}\left[\left(\frac{\mathrm{d}p_N^0}{\mathrm{d}x}\right)_{x=x_k}\right]^{-\frac{1}{2}} \tag{2.2.4}$$

where the x_k are the zeros of p_N^0

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} y\left(\varphi\right)\cos\varphi \mathrm{d}\varphi = \int_{-1}^{1} y\left(x\right)\mathrm{d}x = \sum_{k=1}^{K} W_k y\left(x_k\right) \tag{2.2.5}$$

is an exact integration of the function y, provided $y\left(x\right)$ is a polynomial of degree not exceeding $2N-1$, and the function y is known at the gaussian points x_k (see Fig. 2.2 and Table 2.1).

Using this method

$$F_n^l = \int_{-1}^{1} F^l(x)\, p_n^l(x)\, \mathrm{d}x = \sum_{k=1}^{N} W_k F^l(x_k)\, p_n^l(x_k) \tag{2.2.6}$$

For rhomboidal truncation, we must have $N \geq \frac{5J+1}{2}$ while for triangular truncation, $N \geq \frac{3J+1}{2}$. For an arbitrary resolution, the maximum power of $F_n^l(x)\, p_n^l(x)$ must be evaluated, and the condition imposed by the Gaussian quadrature requirement must be applied.

2.3 Spectral to Grid Transform

The computation of grid values from the spectral expansion given by Eq. (2.1.1) is also carried out in two steps. First, for real F, we take advantage of the relation

$$F_n^{-l} = (-1)^l \left(F_n^l\right)^* \tag{2.3.1}$$

to derive

$$F(\varphi, \lambda) = \sum_{n=0}^{J} F_n^l p_n^0 + 2\,\mathrm{Re} \sum_{l=1}^{J} \sum_{n=l}^{N(l)} F_n^l Y_n^l \tag{2.3.2}$$

Here, the limit $N(l)$ is equal to J for triangular truncation, and to $J+l$ for rhomboidal truncation. It is assumed that grid values are required at a given latitude φ_k (corresponding to x_k). The Legendre polynomials can be computed at this latitude, and the partial sums

$$F^l(x_k) = \sum_{n=l}^{N(l)} F_n^l p_n^l(x_k) \tag{2.3.3}$$

are evaluated for $l = 0, \ldots, J$. Eq. (2.3.2) can, therefore, be written as

$$F(\varphi_k, \lambda) = \sum_{l=0}^{J} F^l(x_k)\, e^{il\lambda} \tag{2.3.4}$$

If equally spaced values of F are required at the latitude φ_k , a discrete Fourier Transform can be applied to Eq. (2.3.4):

$$F(\varphi_k, \lambda_j) = \sum_{l=0}^{M-1} F^l(\varphi_k)\, e^{\frac{2\pi ijl}{M}} \tag{2.3.5}$$

with inverse

$$F^l(\varphi_k) = \sum_{j=0}^{M-1} F(\varphi_k, \lambda_j)\, e^{-\frac{2\pi ijl}{M}} \tag{2.3.6}$$

2.4 Velocity and Divergence-Vorticity Relations

While the kinematic history variables in the model are divergence and vorticity, the wind velocity is also required. In order to compute the spectral coefficients of the scaled velocity, we decompose the velocity vector field into its rotational and divergent components. We introduce the stream function ψ and velocity potential χ and write

$$\vec{v} = \nabla\chi + \vec{k} \times \nabla\psi \tag{2.4.1}$$

$$u = \frac{1}{a\cos\varphi}\left(\frac{\partial\chi}{\partial\lambda} - \cos\varphi\frac{\partial\psi}{\partial\varphi}\right) \tag{2.4.2}$$

$$v = \frac{1}{a\cos\varphi}\left(\frac{\partial\psi}{\partial\lambda} + \cos\varphi\frac{\partial\chi}{\partial\varphi}\right) \tag{2.4.3}$$

Clearly, all the results in terms of ψ and χ can also be expressed in terms of ζ and D , since

$$\zeta = \nabla^2\psi\ , \qquad D = \nabla^2\chi \tag{2.4.4}$$

and

$$\nabla^2 F_n^l = -\frac{n\,(n+1)}{a^2} F_n^l \tag{2.4.5}$$

Assume that ψ and χ are expressed by

$$\psi = \sum_{l=-J}^{J} \sum_{n=|l|}^{|l|+J} \psi_n^l Y_n^l$$

$$\chi = \sum_{l=-J}^{J} \sum_{n=|l|}^{|l|+J} \chi_n^l Y_n^l \tag{2.4.6}$$

Substitution into Eqs. (2.4.2) and (2.3.4) yields

$$aU = \sum_{l=-J}^{J} \sum_{n=|l|}^{|l|+J} \chi_n^l\, il\, Y_n^l - \cos\varphi \sum_{l=-J}^{J} \sum_{n=|l|}^{|l|+J} \psi_n^l\, e^{il\lambda}\, \frac{\mathrm{d}p_n^l}{\mathrm{d}\varphi} \tag{2.4.7}$$

Now, since $\cos\varphi\frac{\mathrm{d}p_n^l}{\mathrm{d}\varphi}$ can be expressed as a linear combination of p_{n-1}^l and p_{n+1}^l (Eq. 2.1.7), U can also be expressed as a spherical harmonic series. We find, after some algebra,

$$aU_n^l = (n-1)\,\epsilon_n^l \psi_{n-1}^l + il\,\chi_n^l - (n+2)\,\epsilon_{n+1}^l \psi_{n+1}^l \tag{2.4.8}$$

Similarly,

$$aV_n^l = (1-n)\,\epsilon_n^l \chi_{n-1}^l - il\,\psi_n^l + (n+2)\,\epsilon_n^l \chi_{n+1}^l \tag{2.4.9}$$

It should be noted that for each zonal wave number, the meridional derivative introduces an additional term. Thus, the series of U and V are given by

$$U = \sum_{l=-J}^{J} \sum_{n=|l|}^{|l|+J+1} U_n^l Y_n^l$$

$$V = \sum_{l=-J}^{J} \sum_{n=|l|}^{|l|+J+1} V_n^l Y_n^l \tag{2.4.10}$$

We should point out that the above method is not the only way to compute the velocity from the divergence and vorticity. It is possible, at each Gaussian latitude, to use the Fourier coefficients of ψ and χ (or ζ and D) and compute the Fourier coefficients of U and V directly from Eq. (2.4.7). If we let

$$\chi^l = \sum_{n=|l|}^{|l|+J} \chi_n^l p_n^l$$

$$\psi^l = \sum_{n=|l|}^{|l|+J} \psi_n^l S_n^l \tag{2.4.11}$$

where

$$S_n^l = \cos\varphi \frac{\mathrm{d}p_n^l}{\mathrm{d}\varphi} \tag{2.4.12}$$

then

$$aU = \sum_{l=-J}^{J} il\,\chi^l e^{il\lambda} - \sum_{l=-J}^{J} \psi^l e^{il\lambda} \tag{2.4.13}$$

Thus,

$$U^l = \frac{1}{a}\left(il\,\chi^l - \psi^l\right) \tag{2.4.14}$$

are the Fourier coefficients of U at the given latitude. A similar expression can be found for the V component. This method is useful when computer memory is limited and the usual trade off between memory and computations applies. In this case, the computation of the Fourier coefficients of U or V requires two summations per latitude, while the method using the spherical harmonic expansion of U or V requires only one summation.

2.5 Spectral Form of the Gradient

At this point, it is convenient to discuss the computation of the gradient of a function F:

$$\nabla F = \frac{1}{a\cos\varphi}\left(\frac{\partial F}{\partial\lambda}\,,\ \cos\varphi\frac{\partial F}{\partial\varphi}\right) \tag{2.5.1}$$

Since S_n^l is easily computed (Eq. 2.4.12), it is simple to compute

$$\cos\varphi\nabla F = \frac{1}{a}\left(\frac{\partial F}{\partial\lambda}\,,\ \cos\varphi\frac{\partial F}{\partial\varphi}\right) \tag{2.5.2}$$

Introducing a spherical harmonic series for F, we find

$$\begin{aligned}\cos\varphi\nabla F = \frac{1}{a}\Bigg(&\sum_{l=-J}^{J}\sum_{n=|l|}^{|l|+J} il\,F_n^l p_n^l e^{il\lambda}\ ,\\ &\sum_{l=-J}^{J}\sum_{n=|l|}^{|l|+J} F_n^l S_n^l e^{il\lambda}\Bigg)\end{aligned} \tag{2.5.3}$$

where, once again, the meridional derivative introduces for each l an extra term in the expansion.

and the full thermodynamic equation can now be presented as used by ECMWF:

$$\frac{T^A_{t+\Delta t} - T^D_t}{\Delta t} = \left(\frac{\kappa T_v \omega (1+\epsilon q)}{1+(\delta-1)q}\right)^M_{t+\frac{\Delta t}{2}} - \frac{\beta}{2}\Delta_{tt}\left(\tau D\right)_k$$

The matrix τ is defined by:

$$\left(\tau D\right)_k = \kappa T_r \left(\frac{1}{\Delta p^r_k}\ln\left(\frac{p^r_{k+\frac{1}{2}}}{p^r_{k-\frac{1}{2}}}\right)\sum_{j=1}^{k-1} D_j \Delta p^r_j + \alpha^r_k D_k\right)$$

Δ_{tt} has the semi-lagrangian appropriate definition, presented in the derivation of the details concerning the geometric and kinematic details. $\tilde{D}$ is the transpose of the divergence column.

$$\tilde{D} = \left(D_1 \ \ldots \ D_{Levs}\right)$$