The Best Matrix Conjecture

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Abstract

This note describes the *best matrix conjecture* from combinatorial design theory and the latest results that are known on the conjecture. In particular, examples of best matrices in orders $r^2 + r + 1$ for all r up to and including 6 are given.

1 Introduction

Best matrices were introduced by Georgiou, Koukouvinos, and Seberry [2001] and further studied by Koukouvinos and Stylianou [2008] and Đoković [2009]. A quadruple of matrices A, B, C, D are known as *best matrices* if they are square matrices of order n with ± 1 entries and satisfy the following axioms:

- (1) A I, B I, C I are skew matrices and D is a symmetric matrix.
- (2) A, B, C, D commute pairwise.
- (3) $AA^T + BB^T + CC^T + DD^T$ is the scalar matrix 4nI.

Note that a matrix X is *symmetric* if $X = X^T$, a matrix X is *skew* if $X = -X^T$, and two matrices X, Y *commute* if XY = YX. Best matrices can be used to generate skew Hadamard matrices via a construction introduced by Goethals and Seidel [1970]. In particular, if A, B, C, D are best matrices then the Goethals–Seidel array

$$\begin{pmatrix} A & BR & CR & DR \\ -BR & A & -D^{T}R & C^{T}R \\ -CR & D^{T}R & A & -B^{T}R \\ -DR & -C^{T}R & B^{T}R & A \end{pmatrix}$$

gives a skew Hadamard matrix of order 4n where R is the exchange matrix (anti-diagonal identity matrix) of order n.

Furthermore, X is *circulant* if its (i, j) entry is the same as its (i + 1, j + 1) entry for all indices i and j (reducing mod n if necessary). For the purposes of this note we will only consider circulant best matrices. In this case condition (2) is always satisfied.

Georgiou, Koukouvinos, and Seberry [2001] show that if circulant best matrices exist in odd order n then n must be of the form $(m^2+3)/4$ for odd m. In other words, letting m=2r+1 we have that $n=r^2+r+1$ and the possible values for n are

$$\{1,3,7,13,21,31,43,57,73,91,111,\ldots\}.$$

Georgiou, Koukouvinos, and Seberry [2001] found that best matrices exist for all $r \le 5$ and for many years no additional best matrices were found. Recently the situation changed as Đoković and Kotsireas [2018] found that best matrices also exist for r = 6, i.e., in order n = 43.

We call the *best matrix conjecture* the conjecture that best matrices exist in all orders of the form $r^2 + r + 1$. The conjecture is currently open for each $r \ge 7$.

2 Examples

We now explicitly give examples of best matrices for r = 1, 2, ..., 6. The first five examples were found by Georgiou, Koukouvinos, and Seberry [2001] and the sixth was found by Đoković and Kotsireas [2018]. In each example the four matrices A, B, C, D are drawn using a different colour. The coloured squares represent 1 and the grey squares represent -1.

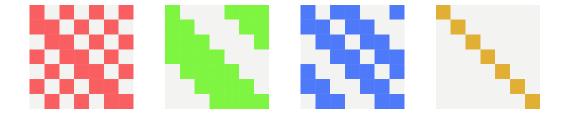




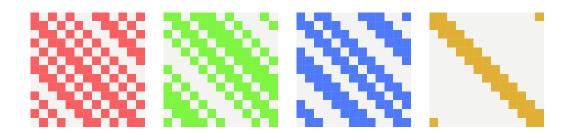




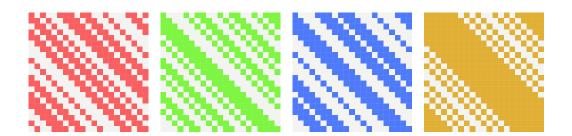
Case r = 1: Best matrices of order 3.



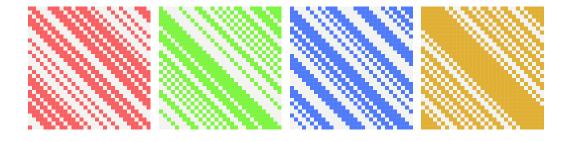
Case r = 2: Best matrices of order 7.



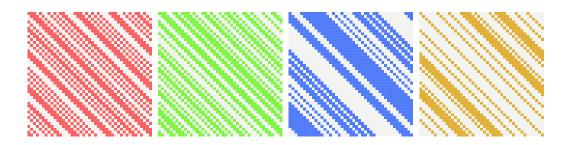
Case r = 3: Best matrices of order 13.



Case r = 4: Best matrices of order 21.



Case r = 5: Best matrices of order 31.



Case r = 6: Best matrices of order 43.

References

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