Patterns in Pythagorean Triples Using Single and Double Variable Procedures

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Abstract

*The Pythagoras theorem is very famous in the literature of mathematics. The aim of this work is to extend in a symmetrical way the some **Pythagorean triples** resulting in **patterns**. These symmetric extensions are in such a way that we reach to good patterns. In some cases, the final sums also give a good pattern. In some cases examples are with interesting **pandigital palindromic-type patterns**. The patterns are obtained based on five procedures for single variable functions, and four for double variable functions. This work is a combination of authors previous two works [8, 9].*

Contents

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1 Introduction

By Pythagoras theorem it is understood that

$$a^2 + b^2 = c^2, \ \forall a, \ b, \ c \in N_+$$

For simplicity, let's write it as (a, b, c) . For example, Pythagorean triple (3,4,5) means $3^2 + 4^2 = 5^2$.

The symmetric extensions of above triple we call **patterns in Pythagorean triples**. There two obvious ways of getting **patterns in Pythagorean triples**.

1.1 Simple Patterns

- **Multiplication by 10, 100, 1000, ... :**

See two examples

$$\begin{aligned} 3^2 + 4^2 &= 5^2 := 25 \\ 30^2 + 40^2 &= 50^2 := 2500 \\ 300^2 + 400^2 &= 500^2 := 250000 \end{aligned} \tag{1.1}$$

$$\begin{aligned} 9^2 + 40^2 &= 41^2 := 1681 \\ 90^2 + 400^2 &= 410^2 := 168100 \\ 900^2 + 4000^2 &= 4100^2 := 16810000 \end{aligned} \tag{1.2}$$

The difference in above two examples is that the first one is with single digit in each case, i.e., 3, 4 and 5, and in second example (1.2) not all the same digits are with same length, i.e., the first one is of length 1 and other two are with length 2. This process is generally true for all types of **Pythagorean triples**.

- **Repetition of Digits:**

$$\begin{aligned} 3^2+4^2 &= 5^2 := 25\\ 33^2+44^2 &= 55^2 := 3025\\ 333^2+444^2 &= 555^2 := 308035 \end{aligned} \tag{1.3}$$

$$\begin{aligned} 12^2+35^2 &= 37^2 := 1369\\ 1212^2+3535^2 &= 3737^2 := 13965169\\ 121212^2+353535^2 &= 373737^2 := 139679345169 \end{aligned} \tag{1.4}$$

$$\begin{aligned} 119^2+120^2 &= 169^2 := 28561\\ 119119^2+120120^2 &= 169169^2 := 28618150561\\ 119119119^2+120120120^2 &= 169169169^2 := 28618207740150561 \end{aligned} \tag{1.5}$$

In the examples (1.3), (1.4) and (1.5), we observe that the repetition of digits give us patterned results when we work with same length triples. Here we have (3,4,5), (12, 35, 27) and (119, 120, 169). In the example (3, 4, 5) each digit is of length 1. In the example (12, 35, 27) each digit is of length 2, and the triple (119, 120, 169) each digit is of length 3. This theory doesn't work when the triples are with different lengths, for example, (5, 12, 13). This gives

$$\begin{aligned} 5^2+12^2 = 13^2 &\Rightarrow 169 = 169\\ 55^2+1212^2 \neq 1313^2 &\Rightarrow 1471969 \neq 1723969 \end{aligned} \tag{1.6}$$

In this case the pattern (1.6) is not extendable as in case examples (1.3), (1.4) and (1.5).

Remark 1. *Analysing the final sums, we observe that the examples (1.1) and (1.2) are good for **patterns with final sums**. The example (1.3) is not so good but acceptable. In case of examples (1.4) and (1.5), the patterns with final sums are not good. In this work, we shall write patterns with final sums only if they are good.*

The aim of this work is to give different procedures to find Pythagorean triples, and then give examples giving patterns based on functional representations. This work is limited to single variable cases. Two variable cases are done in another part.

2 Single Variable Procedures

2.1 Procedure 1

Let's consider the following three functions:

$$\begin{aligned} f_1(n) &:= 2n(n+1)\\ g_1(n) &:= 2n+1\\ h_1(n) &:= 2n^2+2n+1 \end{aligned} \tag{2.1}$$

We can easily check that

$$
\begin{aligned}
f_1(n)^2 + g_1(n)^2 &= 4n^2(n+1)^2 + (2n+1)^2 \\
&= 4n^4 + 8n^3 + 4n^2 + +4n^2 + 4n + 1 \\
&= 4n^4 + 8n^3 + 8n^2 + 4n + 1 \\
&= (2n^2 + 2n + 1)^2 \\
&= h_1(n)^2
\end{aligned}
$$

This proves that the triple (f_1, g_1, h_1) is a **Pythagorean triple**. Let's see some particular values. For $n = 1, 2, ..., 9, 10$ in (2.1), we get

$$
\begin{aligned}
(f_1(1), g_1(1), h_1(1)) &= (4, 3, 5) \\
(f_1(2), g_1(2), h_1(2)) &= (12, 5, 13) \\
(f_1(3), g_1(3), h_1(3)) &= (24, 7, 25) \\
(f_1(4), g_1(4), h_1(4)) &= (40, 9, 41) \\
(f_1(5), g_1(5), h_1(5)) &= (60, 11, 61) \\
(f_1(6), g_1(6), h_1(6)) &= (85, 13, 85) \\
(f_1(7), g_1(7), h_1(7)) &= (112, 15, 113) \\
(f_1(8), g_1(8), h_1(8)) &= (114, 17, 145) \\
(f_1(9), g_1(9), h_1(9)) &= (180, 19, 181) \\
(f_1(10), g_1(10), h_1(10)) &= (220, 21, 221)
\end{aligned}
$$

We observe that the **Pythagorean triples** are primitive and out of three two values are consecutive. This procedure is very well known in the literature [1].

2.1.1 Examples

Below are some Pythagorean triples based on functional values given in (2.1).

- **For $n = 10, 100, 1000, ...$ in (2.1):**

$$
\begin{aligned}
220^2 + 21^2 &= 221^2 &&:= 48841 \\
20200^2 + 201^2 &= 20201^2 &&:= 408080401 \\
2002000^2 + 2001^2 &= 2002001^2 &&:= 4008008004001 \\
200020000^2 + 20001^2 &= 200020001^2 &&:= 40008000800040001
\end{aligned} \tag{2.2}
$$

- **For $n = 20, 200, 2000, ...$ in (2.1):**

$$
\begin{aligned}
840^2 + 41^2 &= 841^2 && \\
80400^2 + 401^2 &= 80401^2 &&:= 6464320801 \\
8004000^2 + 4001^2 &= 8004001^2 &&:= 64064032008001 \\
800040000^2 + 40001^2 &= 800040001^2 &&:= 640064003200080001
\end{aligned} \tag{2.3}
$$

- **For $n = 30, 300, 3000, ...$ in (2.1):**

$$\begin{aligned}
1860^2 + 61^2 &= 1861^2 \\
180600^2 + 601^2 &= 180601^2 \\
18006000^2 + 6001^2 &= 18006100^2 := 324216072012001 \\
1800060000^2 + 60001^2 &= 1800060001^2 := 3240216007200120001 \\
180000600000^2 + 600001^2 &= 180000600001^2 := 32400216000720001200001
\end{aligned} \tag{2.4}$$

- **For $n = 40, 400, 4000, ...$ in (2.1):**

$$\begin{aligned}
3280^2 + 81^2 &= 3281^2 \\
320800^2 + 801^2 &= 320801^2 \\
32008000^2 + 8001^2 &= 32008001^2 := 1024512128016001 \\
3200080000^2 + 80001^2 &= 3200080001^2 := 10240512012800160001 \\
320000800000^2 + 800001^2 &= 320000800001^2 := 102400512001280001600001
\end{aligned} \tag{2.5}$$

- **For $n = 50, 500, 5000, ...$ in (2.1):**

$$\begin{aligned}
5100^2 + 101^2 &= 5101^2 \\
501000^2 + 1001^2 &= 501001^2 := 251002002001 \\
50010000^2 + 10001^2 &= 50010001^2 := 2501000200020001 \\
5000100000^2 + 100001^2 &= 5000100001^2 := 25001000020000200001
\end{aligned} \tag{2.6}$$

- **For $n = 600, 6000, 60000, ...$ in (2.1):**

$$\begin{aligned}
721200^2 + 1201^2 &= 721201^2 \\
72012000^2 + 12001^2 &= 72012001^2 \\
7200120000^2 + 120001^2 &= 7200120001^2 := 51841728028800240001 \\
720001200000^2 + 1200001^2 &= 720001200001^2 := 518401728002880002400001 \\
72000012000000^2 + 12000001^2 &= 72000012000001^2 := 5184001728000288000024000001
\end{aligned} \tag{2.7}$$

The first triple $(7320, 121, 7321)$ for $n = 60$ is not written above as it doesn't give good pattern.

- **For $n = 700, 7000, 70000, ...$ in (2.1):**

$$\begin{aligned}
981400^2 + 1401^2 &= 981401^2 \\
98014000^2 + 14001^2 &= 98014001^2 \\
9800140000^2 + 140001^2 &= 9800140001^2 := 96042744039200280001 \\
980001400000^2 + 1400001^2 &= 980001400001^2 := 960402744003920002800001 \\
98000014000000^2 + 14000001^2 &= 98000014000001^2 := 9604002744000392000028000001
\end{aligned} \tag{2.8}$$

The first triple $(9940, 141, 9941)$ for $m = 70$ is not written above as it doesn't give good pattern.

- **For $n = 800, 8000, 80000, \ldots$ in (2.1):**

$$\begin{aligned}
1281600^2 + 1601^2 &= 1281601^2 \\
128016000^2 + 16001^2 &= 128016001^2 \\
12800160000^2 + 160001^2 &= 12800160001^2 := 163844096051200320001 \\
1280001600000^2 + 1600001^2 &= 1280001600001^2 := 1638404096005120003200001 \\
128000016000000^2 + 16000001^2 &= 128000016000001^2 := 16384004096000512000032000001
\end{aligned} \tag{2.9}$$

The first triple (12960, 161, 12961) for $m = 80$ is not written above as it doesn't give good pattern.

- **For $n = 900, 9000, 90000, \ldots$ in (2.1):**

$$\begin{aligned}
1621800^2 + 1801^2 &= 1621801^2 \\
162018000^2 + 18001^2 &= 162018001^2 \\
16200180000^2 + 180001^2 &= 16200180001^2 := 262445832064800360001 \\
1620001800000^2 + 1800001^2 &= 1620001800001^2 := 2624405832006480003600001 \\
162000018000000^2 + 18000001^2 &= 162000018000001^2 := 26244005832000648000036000001
\end{aligned} \tag{2.10}$$

The first triple (16380, 181, 16381) for $n = 90$ is not written above as it doesn't give good pattern.

- **For $n = 11, 101, 1001, 10001, \ldots$ in (2.1):**

$$\begin{aligned}
264^2 + 23^2 &= 265^2 \\
20604^2 + 203^2 &= 20605^2 := 424566025 \\
2006004^2 + 2003^2 &= 2006005^2 := 4024056060025 \\
200060004^2 + 20003^2 &= 200060005^2 := 40024005600600025
\end{aligned} \tag{2.11}$$

- **For $n = 201, 2001, 20001, \ldots$ in (2.1):**

$$\begin{aligned}
81204^2 + 403^2 &= 81205^2 \\
8012004^2 + 4003^2 &= 8012005^2 := 64192224120025 \\
800120004^2 + 40003^2 &= 800120005^2 := 640192022401200025 \\
80001200004^2 + 400003^2 &= 80001200005^2 := 6400192002240012000025
\end{aligned} \tag{2.12}$$

The first triple (924, 43, 925) for $n = 21$ is not written above as it doesn't give good pattern.

- **For $n = 202, 2002, 20002, \ldots$ in (2.1):**

$$\begin{aligned}
82012^2 + 405^2 &= 82013^2 \\
8020012^2 + 4005^2 &= 8020013^2 := 64320608520169 \\
800200012^2 + 40005^2 &= 800200013^2 := 640320060805200169 \\
80002000012^2 + 400005^2 &= 80002000013^2 := 6400320006080052000169
\end{aligned} \tag{2.13}$$

The first triple (1012, 45, 1013) for $n = 22$ is not written above as it doesn't give good pattern.

- **For $n = 301, 3001, 30001, \ldots$ in (2.1):**

$$\begin{aligned}
181804^2 + 603^2 &= 181805^2 \\
18018004^2 + 6003^2 &= 18018005^2 &&:= 324648504180025 \\
1800180004^2 + 60003^2 &= 1800180005^2 &&:= 3240648050401800025 \\
180001800004^2 + 600003^2 &= 180001800005^2 &&:= 32400648005040018000025
\end{aligned} \tag{2.14}$$

The first triple $(1984, 63, 1985)$ for $n = 31$ is not written above as it doesn't give good pattern.

- **For $n = 302, 3002, 30002, \ldots$ in (2.1):**

$$\begin{aligned}
183012^2 + 605^2 &= 183013^2 \\
18030012^2 + 6005^2 &= 18030013^2 \\
1800300012^2 + 60005^2 &= 1800300013^2 &&:= 3241080136807800169 \\
180003000012^2 + 600005^2 &= 180003000013^2 &&:= 32401080013680078000169 \\
18000030000012^2 + 6000005^2 &= 18000030000013^2 &&:= 324001080001368000780000169
\end{aligned} \tag{2.15}$$

The first triple $(2112, 65, 2113)$ for $n = 32$ is not written above as it doesn't give good pattern.

- **For $n = 303, 3003, 30003, \ldots$ in (2.1):**

$$\begin{aligned}
184224^2 + 607^2 &= 184225^2 \\
18042024^2 + 6007^2 &= 18042025^2 \\
1800420024^2 + 60007^2 &= 1800420025^2 \\
180004200024^2 + 600007^2 &= 180004200025^2
\end{aligned} \tag{2.16}$$

The first triple $(2244, 67, 2245)$ for $n = 33$ is not written above as it doesn't give good pattern.

- **For $n = 13, 133, 1333, 13333, \ldots$ in (2.1):**

$$\begin{aligned}
364^2 + 27^2 &= 365^2 \\
35644^2 + 267^2 &= 35645^2 \\
3556444^2 + 2667^2 &= 3556445^2 \\
355564444^2 + 26667^2 &= 355564445^2
\end{aligned} \tag{2.17}$$

- **For $n = 1, 16, 166, 1666, \ldots$ in (2.1):**

$$\begin{aligned}
4^2 + 3^2 &= 5^2 \\
544^2 + 33^2 &= 545^2 \\
55444^2 + 333^2 &= 55445^2 \\
5554444^2 + 3333^2 &= 5554445^2 \\
555544444^2 + 33333^2 &= 555544445^2
\end{aligned} \tag{2.18}$$

- **For $n = 199, 1999, 19999, ...$ in (2.1):**

$$\begin{aligned} 79600^2 + 399^2 &= 79601^2 &&:= 6336319201 \\ 7996000^2 + 3999^2 &= 7996001^2 &&:= 63936031992001 \\ 799960000^2 + 39999^2 &= 799960001^2 &&:= 639936003199920001 \\ 79999600000^2 + 399999^2 &= 79999600001^2 &&:= 6399936000319999200001 \end{aligned} \tag{2.19}$$

The first triple $(760, 39, 761)$ for $n = 19$ is not written above as it doesn't give good pattern.

- **For $n = 29, 299, 2999, 29999, ...$ in (2.1):**

$$\begin{aligned} 1740^2 + 59^2 &= 1741^2 \\ 179400^2 + 599^2 &= 179401^2 &&:= 32184718801 \\ 17994000^2 + 5999^2 &= 17994001^2 &&:= 323784071988001 \\ 1799940000^2 + 59999^2 &= 1799940001^2 &&:= 3239784007199880001 \\ 179999400000^2 + 599999^2 &= 179999400001^2 &&:= 32399784000719998800001 \end{aligned} \tag{2.20}$$

- **For $n = 4, 49, 499, 4999, ...$ in (2.1):**

$$\begin{aligned} 40^2 + 9^2 &= 41^2 \\ 4900^2 + 99^2 &= 4901^2 &&:= 24019801 \\ 499000^2 + 999^2 &= 499001^2 &&:= 249001998001 \\ 49990000^2 + 9999^2 &= 49990001^2 &&:= 2499000199980001 \\ 4999900000^2 + 99999^2 &= 4999900001^2 &&:= 24999000019999800001 \end{aligned} \tag{2.21}$$

- **For $n = 1, 31, 331, 3331, 33331, ...$ in (2.1):**

$$\begin{aligned} 4^2 + 3^2 &= 5^2 \\ 1984^2 + 63^2 &= 1985^2 \\ 219784^2 + 663^2 &= 219785^2 \\ 22197784^2 + 6663^2 &= 22197785^2 \\ 2221977784^2 + 66663^2 &= 2221977785^2 \end{aligned} \tag{2.22}$$

- **For $n = 2, 32, 332, 3332, ...$ in (2.1):**

$$\begin{aligned} 12^2 + 5^2 &= 13^2 \\ 2112^2 + 65^2 &= 2113^2 \\ 221112^2 + 665^2 &= 221113^2 \\ 22211112^2 + 6665^2 &= 22211113^2 \\ 2222111112^2 + 66665^2 &= 2222111113^2 \end{aligned} \tag{2.23}$$

- **For $n = 3, 33, 333, 3333, ...$ in (2.1):**

$$\begin{aligned}
24^2 + 7^2 &= 25^2\\
2244^2 + 67^2 &= 2245^2\\
222444^2 + 667^2 &= 222445^2\\
22224444^2 + 6667^2 &= 22224445^2\\
2222244444^2 + 66667^2 &= 2222244445^2
\end{aligned} \tag{2.24}$$

- **For $n = 3, 36, 336, 3336, 33336, ...$ in (2.1):**

$$\begin{aligned}
24^2 + 7^2 &= 25^2\\
2664^2 + 73^2 &= 2665^2\\
226464^2 + 673^2 &= 226465^2\\
22264464^2 + 6673^2 &= 22264465^2\\
2222644464^2 + 66673^2 &= 2222644465^2
\end{aligned} \tag{2.25}$$

- **For $n = 339, 3339, 33339, ...$ in (2.1):**

$$\begin{aligned}
230520^2 + 679^2 &= 230521^2\\
22304520^2 + 6679^2 &= 22304521^2\\
2223044520^2 + 66679^2 &= 2223044521^2
\end{aligned} \tag{2.26}$$

The first two triples $(24, 7, 25)$ and $(3120, 79, 3121)$ for $n = 3$ and 39 are not written above as they don't give good pattern.

- **For $n = 1, 61, 661, 6661, 66661, ...$ in (2.1):**

$$\begin{aligned}
4^2 + 3^2 &= 5^2\\
7564^2 + 123^2 &= 7565^2\\
875164^2 + 1323^2 &= 875165^2\\
88751164^2 + 13323^2 &= 88751165^2\\
8887511164^2 + 133323^2 &= 8887511165^2
\end{aligned} \tag{2.27}$$

- **For $n = 2, 62, 662, 6662, 66662, ...$ in (2.1):**

$$\begin{aligned}
12^2 + 5^2 &= 13^2\\
812^2 + 125^2 &= 7813^2\\
877812^2 + 1325^2 &= 877813^2\\
88777812^2 + 13325^2 &= 88777813^2\\
8887777812^2 + 133325^2 &= 8887777813^2
\end{aligned} \tag{2.28}$$

- **For $n = 63, 663, 6663, 66663, \ldots$ in (2.1):**

$$\begin{aligned} 8064^2 + 127^2 &= 8065^2 \\ 880464^2 + 1327^2 &= 880465^2 \\ 88804464^2 + 13327^2 &= 88804465^2 \\ 8888044464^2 + 133327^2 &= 8888044465^2 \end{aligned} \tag{2.29}$$

The first triple (24, 7, 25) for $n = 3$ is not written above as it doesn't give good pattern.

- **For $n = 6, 66, 666, 6666, 66666, \ldots$ in (2.1):**

$$\begin{aligned} 84^2 + 13^2 &= 85^2 \\ 8844^2 + 133^2 &= 8845^2 \\ 888444^2 + 1333^2 &= 888445^2 \\ 88884444^2 + 13333^2 &= 88884445^2 \\ 8888844444^2 + 133333^2 &= 8888844445^2 \end{aligned} \tag{2.30}$$

- **For $n = 69, 669, 6669, 66669, \ldots$ in (2.1):**

$$\begin{aligned} 9660^2 + 139^2 &= 9661^2 \\ 896460^2 + 1339^2 &= 896461^2 \\ 88964460^2 + 13339^2 &= 88964461^2 \\ 8889644460^2 + 133339^2 &= 8889644461^2 \end{aligned} \tag{2.31}$$

The first triple (84, 13, 85) for $n = 6$ is not written above as it doesn't give good pattern.

- **For $n = 699, 6999, 69999, 699999, \ldots$ in (2.1):**

$$\begin{aligned} 978600^2 + 1399^2 &= 978601 \\ 97986000^2 + 13999^2 &= 97986001 \\ 9799860000^2 + 139999^2 &= 9799860001 \\ 979998600000^2 + 1399999^2 &= 979998600001 \end{aligned} \tag{2.32}$$

The first two triples (84, 13, 85) and (9660, 139, 9661) for $n = 6$ and 69 are not written above as they don't give good pattern.

- **For $n = 991, 9991, 99991, \ldots$ in (2.1):**

$$\begin{aligned} 1966144^2 + 1983^2 &= 1966145^2 \\ 199660144^2 + 19983^2 &= 199660145^2 \\ 19996600144^2 + 199983^2 &= 19996600145^2 \end{aligned} \tag{2.33}$$

The first two triples (4, 3, 5) and (16744, 183, 16745) for $n = 1$ and 91 are not written above as they don't give good pattern.

- **For $n = 2, 92, 992, 9992, 99992, \ldots$ in (2.1):**

$$\begin{aligned} 12^2 + 5^2 &= 13^2 \\ 17112^2 + 185^2 &= 17113^2 \\ 1970112^2 + 1985^2 &= 1970113^2 \\ 199700112^2 + 19985^2 &= 199700113^2 \\ 19997000112^2 + 199985^2 &= 19997000113^2 \end{aligned} \tag{2.34}$$

- **For $n = 93, 993, 9993, 99993, \ldots$ in (2.1):**

$$\begin{aligned} 17484^2 + 187^2 &= 17485^2 \\ 1974084^2 + 1987^2 &= 1974085^2 \\ 199740084^2 + 19987^2 &= 199740085^2 \\ 19997400084^2 + 199987^2 &= 19997400085^2 \end{aligned} \tag{2.35}$$

The first triple (24, 7, 25) for $n = 3$ is not written above as it doesn't give good pattern.

- **For $n = 96, 996, 9996, 99996, \ldots$ in (2.1):**

$$\begin{aligned} 18624^2 + 193^2 &= 18625^2 \\ 1986024^2 + 1993^2 &= 1986025^2 := 3944295300625 \\ 199860024^2 + 19993^2 &= 199860025^2 := 39944029593000625 \\ 19998600024^2 + 199993^2 &= 19998600025^2 := 399944002959930000625 \end{aligned} \tag{2.36}$$

The first triple (180, 19, 181) for $n = 9$ is not written above as it doesn't give good pattern.

- **For $n = 9, 99, 999, 9999, \ldots$ in (2.1):**

$$\begin{aligned} 180^2 + 19^2 &= 181^2 := 32761 \\ 19800^2 + 199^2 &= 19801^2 := 392079601 \\ 1998000^2 + 1999^2 &= 1998001^2 := 3992007996001 \\ 199980000^2 + 19999^2 &= 199980001^2 := 39992000799960001 \\ 19999800000^2 + 199999^2 &= 19999800001^2 := 399992000079999600001 \end{aligned} \tag{2.37}$$

Remark 2. *Since out of three values in a triple, two are consecutive, obviously the patterns are with primitive values.*

2.2 Procedure 2

Let's consider the following three functions:

$$\begin{aligned} f_2(m) &:= m(m+2) \\ g_2(m) &:= 2(m+1) \\ h_2(m) &:= m^2+2m+2 \end{aligned} \tag{2.38}$$

We can easily check that

$$\begin{aligned} f_2(m)^2+g_2(m)^2 &= (m(m+2))^2+(2(m+1))^2 \\ &= m^4+4m^3+4m^2+4m^2+8m+4 \\ &= m^4+4m^3+8m^2+8m+4 \\ &= (m^2+2m+2)^2 \\ &= h_2(m)^2 \end{aligned}$$

This proves that the triple (f_2, g_2, h_2) is a **Pythagorean triple**. Let's see some particular values. For $m = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ in (2.38), we get

$$\begin{aligned} (f_2(1), g_2(1), h_2(1)) &= (3, 4, 5) \\ (f_2(2), g_2(2), h_2(2)) &= (8, 6, 10) \\ (f_2(3), g_2(3), h_2(3)) &= (15, 8, 17) \\ (f_2(4), g_2(4), h_2(4)) &= (24, 10, 26) \\ (f_2(5), g_2(5), h_2(5)) &= (35, 12, 37) \\ (f_2(6), g_2(6), h_2(6)) &= (48, 14, 50) \\ (f_2(7), g_2(7), h_2(7)) &= (63, 16, 65) \\ (f_2(8), g_2(8), h_2(8)) &= (80, 18, 82) \\ (f_2(9), g_2(9), h_2(9)) &= (99, 20, 101) \\ (f_2(10), g_3(10), h_5(10)) &= (120, 22, 122) \end{aligned}$$

We observe that not all **Pythagorean triples** are primitive. While, in case of Procedure 1 given in (2.1), all the triples are with primitive values.

2.2.1 Examples

See below examples of patterns based on Procedure 2 given in subsection 2.2 and equation (2.38).

- **For $m = 10, 100, 1000, 10000, ...$ in (2.38):**

$$\begin{aligned} 120^2+22^2 &= 122^2 &&:= 14884 \\ 10200^2+202^2 &= 10202^2 &&:= 104080804 \\ 1002000^2+2002^2 &= 1002002^2 &&:= 1004008008004 \\ 100020000^2+20002^2 &= 100020002^2 &&:= 10004000800080004 \\ 10000200000^2+200002^2 &= 10000200002^2 &&:= 100004000080000800004 \\ 1000002000000^2+2000002^2 &= 1000002000002^2 &&:= 1000004000008000008000004 \end{aligned} \tag{2.39}$$

- **Division by 2**

$$\begin{aligned}
5100^2+101^2 &= 5101^2 \\
501000^2+1001^2 &= 501001^2 := 251002002001 \\
50010000^2+10001^2 &= 50010001^2 := 2501000200020001 \\
5000100000^2+100001^2 &= 5000100001^2 := 25001000020000200001
\end{aligned} \tag{2.40}$$

The first triple (60, 11, 61) is not written above as it doesn't give good pattern.

- **For $m = 20, 200, 2000, 20000, ...$ in (2.38):**

$$\begin{aligned}
440^2+42^2 &= 442^2 \\
40400^2+402^2 &= 40402^2 := 1632321604 \\
4004000^2+4002^2 &= 4004002^2 := 16032032016004 \\
400040000^2+40002^2 &= 400040002^2 := 160032003200160004 \\
40000400000^2+400002^2 &= 40000400002^2 := 1600032000320001600004 \\
4000004000000^2+4000002^2 &= 4000004000002^2 := 16000032000032000016000004
\end{aligned} \tag{2.41}$$

  - **Division by 2**

$$\begin{aligned}
220^2+21^2 &= 221^2 := 48841 \\
20200^2+201^2 &= 20201^2 := 408080401 \\
2002000^2+2001^2 &= 2002001^2 := 4008008004001 \\
200020000^2+20001^2 &= 200020001^2 := 40008000800040001 \\
20000200000^2+200001^2 &= 20000200001^2 := 400008000080000400001
\end{aligned} \tag{2.42}$$

- **For $m = 30, 300, 3000, 30000, ...$ in (2.38):**

$$\begin{aligned}
960^2+62^2 &= 962^2 \\
90600^2+602^2 &= 90602^2 \\
9006000^2+6002^2 &= 9006002^2 := 81108072024004 \\
900060000^2+60002^2 &= 900060002^2 := 810108007200240004 \\
90000600000^2+600002^2 &= 90000600002^2 := 8100108000720002400004 \\
9000006000000^2+6000002^2 &= 9000006000002^2 := 81000108000072000024000004
\end{aligned} \tag{2.43}$$

  - **Division by 2**

$$\begin{aligned}
45300^2+301^2 &= 45301^2 \\
4503000^2+3001^2 &= 4503001^2 \\
450030000^2+30001^2 &= 450030001^2 := 202527001800060001 \\
45000300000^2+300001^2 &= 45000300001^2 := 2025027000180000600001
\end{aligned} \tag{2.44}$$

The first triple (480, 31, 481) is not written above as it doesn't give good pattern.

- **For $m = 40, 400, 4000, 40000, \ldots$ in (2.38):**

$$\begin{aligned}
1680^2 + 82^2 &= 1682^2 \\
160800^2 + 802^2 &= 160802^2 \\
16008000^2 + 8002^2 &= 16008002^2 &&:= 256256128032004 \\
1600080000^2 + 80002^2 &= 1600080002^2 &&:= 2560256012800320004 \\
160000800000^2 + 800002^2 &= 160000800002^2 &&:= 25600256001280003200004 \\
16000008000000^2 + 8000002^2 &= 16000008000002^2 &&:= 256000256000128000032000004
\end{aligned} \tag{2.45}$$

▶ **Division by 2**

$$\begin{aligned}
840^2 + 41^2 &= 841^2 \\
80400^2 + 401^2 &= 80401^2 &&:= 6464320801 \\
8004000^2 + 4001^2 &= 8004001^2 &&:= 64064032008001 \\
800040000^2 + 40001^2 &= 800040001^2 &&:= 640064003200080001 \\
80000400000^2 + 400001^2 &= 80000400001^2 &&:= 6400064000320000800001
\end{aligned} \tag{2.46}$$

- **For $m = 500, 5000, 50000, \ldots$ in (2.38):**

$$\begin{aligned}
251000^2 + 1002^2 &= 251002^2 \\
25010000^2 + 10002^2 &= 25010002^2 &&:= 625500200040004 \\
2500100000^2 + 100002^2 &= 2500100002^2 &&:= 6250500020000400004 \\
250001000000^2 + 1000002^2 &= 250001000002^2 &&:= 62500500002000004000004 \\
25000010000000^2 + 10000002^2 &= 25000010000002^2 &&:= 625000500000200000040000004
\end{aligned} \tag{2.47}$$

The first triple (2600, 102, 2602) for $m = 50$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
125500^2 + 501^2 &= 125501^2 \\
12505000^2 + 5001^2 &= 12505001^2 \\
1250050000^2 + 50001^2 &= 1250050001^2 \\
125000500000^2 + 500001^2 &= 125000500001^2
\end{aligned} \tag{2.48}$$

The first triple (1300, 51, 1301) is not written above as it doesn't give good pattern.

- **For $m = 600, 6000, 60000, \ldots$ in (2.38):**

$$\begin{aligned}
361200^2 + 1202^2 &= 361202^2 \\
36012000^2 + 12002^2 &= 36012002^2 &&:= 1296864288048004 \\
3600120000^2 + 120002^2 &= 3600120002^2 &&:= 12960864028800480004 \\
360001200000^2 + 1200002^2 &= 360001200002^2 &&:= 129600864002880004800004 \qquad (2.49)
\end{aligned}$$

The first triple $(3720, 122, 3722)$ for $m = 60$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
1860^2 + 61^2 &= 1861^2 \\
180600^2 + 601^2 &= 180601^2 \\
18006000^2 + 6001^2 &= 18006001^2 &&:= 324216072012001 \\
1800060000^2 + 60001^2 &= 1800060001^2 &&:= 3240216007200120001 \\
180000600000^2 + 600001^2 &= 180000600001^2 &&:= 32400216000720001200001 \qquad (2.50)
\end{aligned}$$

- **For $m = 1, 11, 111, 1111, 11111, \ldots$ in (2.38):**

$$\begin{aligned}
3^2 + 4^2 &= 5^2 \\
143^2 + 24^2 &= 145^2 \\
12543^2 + 224^2 &= 12545^2 \\
1236543^2 + 2224^2 &= 1236545^2 \\
123476543^2 + 22224^2 &= 123476545^2 \\
12345876543^2 + 222224^2 &= 12345876545^2 \qquad (2.51)
\end{aligned}$$

- **For $m = 3, 33, 333, 3333, 3333, \ldots$ in (2.38):**

$$\begin{aligned}
15^2 + 8^2 &= 17^2 \\
1155^2 + 68^2 &= 1157^2 \\
111555^2 + 668^2 &= 111557^2 \\
11115555^2 + 6668^2 &= 11115557^2 \\
1111155555^2 + 66668^2 &= 1111155557^2 \\
111111555555^2 + 666668^2 &= 111111555557^2 \qquad (2.52)
\end{aligned}$$

- **For $m = 66, 666, 6666, 66666, ...$ in (2.38):**

$$\begin{aligned} 4488^2 + 134^2 &= 4490^2 \\ 444888^2 + 1334^2 &= 444890^2 \\ 44448888^2 + 13334^2 &= 44448890^2 \\ 4444488888^2 + 133334^2 &= 4444488890^2 \\ 444444888888^2 + 1333334^2 &= 444444888890^2 \end{aligned} \quad (2.53)$$

The first triple (48, 14, 50) for $m = 6$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned} 24^2 + 7^2 &= 25^2 \\ 2244^2 + 67^2 &= 2245^2 \\ 222444^2 + 667^2 &= 222445^2 \\ 22224444^2 + 6667^2 &= 22224445^2 \\ 2222244444^2 + 66667^2 &= 2222244445^2 \\ 222222444444^2 + 666667^2 &= 222222444445^2 \\ 22222224444444^2 + 6666667^2 &= 22222224444445^2 \end{aligned} \quad (2.54)$$

- **For $m = 9, 99, 999, 9999, 99999, ...$ in (2.38):**

$$\begin{aligned} 99^2 + 20^2 &= 101^2 &&:= 10201 \\ 9999^2 + 200^2 &= 10001^2 &&:= 100020001 \\ 999999^2 + 2000^2 &= 1000001^2 &&:= 1000002000001 \\ 99999999^2 + 20000^2 &= 100000001^2 &&:= 10000000200000001 \\ 9999999999^2 + 200000^2 &= 10000000001^2 &&:= 100000000020000000001 \\ 999999999999^2 + 2000000^2 &= 1000000000001^2 &&:= 1000000000002000000000001 \end{aligned} \quad (2.55)$$

- **For $m = 1, 11, 101, 1001, 10001, ...$ in (2.38):**

$$\begin{aligned} 3^2 + 4^2 &= 5^2 \\ 143^2 + 24^2 &= 145^2 \\ 10403^2 + 204^2 &= 10405^2 &&:= 108264025 \\ 1004003^2 + 2004^2 &= 1004005^2 &&:= 1008026040025 \\ 100040003^2 + 20004^2 &= 100040005^2 &&:= 10008002600400025 \\ 10000400003^2 + 200004^2 &= 10000400005^2 &&:= 100008000260004000025 \end{aligned} \quad (2.56)$$

- **For $m = 202, 2002, 20002, ...$ in (2.38):**

$$\begin{aligned}
41208^2 + 406^2 &= 41210^2 \\
4012008^2 + 4006^2 &= 4012010^2 &&:= 16096224240100 \\
400120008^2 + 40006^2 &= 400120010^2 &&:= 160096022402400100 \\
40001200008^2 + 400006^2 &= 40001200010^2 &&:= 1600096002240024000100
\end{aligned} \tag{2.57}$$

The first two triples $(8,6,10)$ and $(528,46,530)$ for $m = 2$ and 22 are not written above as they don't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
4^2 + 3^2 &= 5^2 \\
264^2 + 23^2 &= 265^2 \\
20604^2 + 203^2 &= 20605^2 &&:= 424566025 \\
2006004^2 + 2003^2 &= 2006005^2 &&:= 4024056060025 \\
200060004^2 + 20003^2 &= 200060005^2 &&:= 40024005600600025 \\
20000600004^2 + 200003^2 &= 20000600005^2 &&:= 400024000560006000025
\end{aligned} \tag{2.58}$$

- **For $m = 303, 3003, 30003, ...$ in (2.38):**

$$\begin{aligned}
92415^2 + 608^2 &= 92417^2 \\
9024015^2 + 6008^2 &= 9024017^2 \\
900240015^2 + 60008^2 &= 900240017^2 \\
90002400015^2 + 600008^2 &= 90002400017^2
\end{aligned} \tag{2.59}$$

The first two triples $(15,8,17)$ and $(1155,68,1157)$ for $m = 3$ and 33 are not written above as they don't give good pattern.

- **For $m = 606, 6006, 60006, ...$ in (2.38):**

$$\begin{aligned}
368448^2 + 1214^2 &= 368450^2 \\
36084048^2 + 12014^2 &= 36084050^2 \\
3600840048^2 + 120014^2 &= 3600840050^2 \\
360008400048^2 + 1200014^2 &= 360008400050^2
\end{aligned} \tag{2.60}$$

The first two triples $(48,14,50)$ and $(3720,122,3722)$ for $m = 6$ and 66 are not written above as they don't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
184224^2 + 607^2 &= 184225^2 \\
18042024^2 + 6007^2 &= 18042025^2 \\
1800420024^2 + 60007^2 &= 1800420025^2 \\
180004200024^2 + 600007^2 &= 180004200025^2
\end{aligned} \tag{2.61}$$

- **For $m = 909, 9009, 90009, ...$ in (2.38):**

$$\begin{aligned}
828099^2 + 1820^2 &= 828101^2 \\
81180099^2 + 18020^2 &= 81180101^2 \\
8101800099^2 + 180020^2 &= 8101800101^2 \\
810018000099^2 + 1800020^2 &= 810018000101^2 \\
81000180000099^2 + 18000020^2 &= 81000180000101^2
\end{aligned} \tag{2.62}$$

The first two triples $(99, 20, 101)$ and $(9999, 200, 10001)$ for $m = 9$ and 99 are not written above as they don't give good pattern.

2.3 Procedure 3

Let's consider the following three functions:

$$\begin{aligned}
f_3(m) &:= m^2 - 1 \\
g_3(m) &:= 2m \\
h_3(m) &:= m^2 + 1
\end{aligned} \tag{2.63}$$

Then we can easily check that

$$\begin{aligned}
f_3(m)^2 + g_3(m)^2 &= \left(m^2 - 1\right)^2 + (2m)^2 \\
&= m^4 - 2m^2 + 4m^2 + 1 \\
&= m^4 + 2m^2 + 1 \\
&= \left(m^2 + 1\right)^2 = h_3(m)^2
\end{aligned}$$

This proves that the triple (f_3, g_3, h_3) is a **Pythagorean triple**. Let's consider $m = 2, 3, 4, 5, 6, 7, 8, 9, 10$ in (6.11). This gives following triples:

$$\begin{aligned}
(f_3(2), g_3(2), h_3(2)) &= (3, 4, 5) \\
(f_3(3), g_3(3), h_3(3)) &= (8, 6, 10) \\
(f_3(4), g_3(4), h_3(4)) &= (15, 8, 17) \\
(f_3(5), g_3(5), h_3(5)) &= (24, 10, 26) \\
(f_3(6), g_3(6), h_3(6)) &= (35, 12, 37) \\
(f_3(7), g_3(7), h_3(7)) &= (48, 14, 50) \\
(f_3(8), g_3(8), h_3(8)) &= (63, 16, 65) \\
(f_3(9), g_3(9), h_3(9)) &= (80, 18, 82) \\
(f_3(10), g_3(10), h_3(10)) &= (99, 20, 101)
\end{aligned}$$

Remark 3. *Analysing the Procedures 2 and 3 given in (2.38) and (6.11) respectively, we observe that both the procedures give same results with difference of only one numbers, i.e.,*

$$(f_2(n), g_2(n), h_2(n)) = (g_3(n+1), f_3(n+1), h_3(n+1)).$$

In case of two variables both the procedures works separately (see [5]). Procedure 3 is very much known in the literature [1], but Procedure 2 seems to be new.

2.3.1 Examples

Below are examples of patterns in Pythagorean triples based on Procedures 3 given in 6.2.3 and equation 6.11

- **For $m = 10, 100, 1000, 10000, ...$ in (6.11):**

$$\begin{aligned} 99^2 + 20^2 &= 101^2 := 10201 \\ 9999^2 + 200^2 &= 10001^2 := 100020001 \\ 999999^2 + 2000^2 &= 1000001^2 := 1000002000001 \\ 99999999^2 + 20000^2 &= 100000001^2 := 10000002000001 \end{aligned} \tag{2.64}$$

- **For $m = 20, 200, 2000, 20000, ...$ in (6.11):**

$$\begin{aligned} 399^2 + 40^2 &= 401^2 := 160801 \\ 39999^2 + 400^2 &= 40001^2 := 1600080001 \\ 3999999^2 + 4000^2 &= 4000001^2 := 16000008000001 \\ 399999999^2 + 40000^2 &= 400000001^2 := 160000000800000001 \end{aligned} \tag{2.65}$$

The first triple $(3, 4, 5)$ for $m = 2$ is not written above as it doesn't give good pattern.

- **For $m = 30, 300, 3000, 30000, ...$ in (6.11):**

$$\begin{aligned} 899^2 + 60^2 &= 901^2 := 811801 \\ 89999^2 + 600^2 &= 90001^2 := 8100180001 \\ 8999999^2 + 6000^2 &= 9000001^2 := 81000018000001 \\ 899999999^2 + 60000^2 &= 900000001^2 := 810000001800000001 \end{aligned} \tag{2.66}$$

The first triple $(8, 6, 10)$ for $m = 3$ is not written above as it doesn't give good pattern.

- **For $m = 40, 400, 4000, 40000, ...$ in (6.11):**

$$\begin{aligned} 1599^2 + 80^2 &= 1601^2 := 2563201 \\ 159999^2 + 800^2 &= 160001^2 := 25600320001 \\ 15999999^2 + 8000^2 &= 16000001^2 := 256000032000001 \\ 1599999999^2 + 80000^2 &= 1600000001^2 := 2560000003200000001 \end{aligned} \tag{2.67}$$

The first triple $(15, 8, 17)$ for $m = 4$ is not written above as it doesn't give good pattern.

- **For $m = 50, 500, 5000, 50000, ...$ in (6.11):**

$$\begin{aligned} 2499^2 + 100^2 &= 2501^2 := 6255001 \\ 249999^2 + 1000^2 &= 250001^2 := 62500500001 \\ 24999999^2 + 10000^2 &= 25000001^2 := 625000050000001 \\ 2499999999^2 + 100000^2 &= 2500000001^2 := 6250000005000000001 \end{aligned} \tag{2.68}$$

The first triple $(24, 10, 26)$ for $m = 5$ is not written above as it doesn't give good pattern.

- **For $m = 60, 600, 6000, 60000, \ldots$ in (6.11):**

$$\begin{aligned}
3599^2 + 120^2 &= 3601^2 &&:= 12967201\\
359999^2 + 1200^2 &= 360001^2 &&:= 129600720001\\
35999999^2 + 12000^2 &= 36000001^2 &&:= 1296000072000001\\
3599999999^2 + 120000^2 &= 3600000001^2 &&:= 12960000007200000001
\end{aligned} \tag{2.69}$$

The first triple $(35, 12, 37)$ for $m = 6$ is not written above as it doesn't give good pattern.

- **For $m = 70, 700, 7000, 70000, \ldots$ in (6.11):**

$$\begin{aligned}
4899^2 + 140^2 &= 4901^2 &&:= 24019801\\
489999^2 + 1400^2 &= 490001^2 &&:= 240100980001\\
48999999^2 + 14000^2 &= 49000001^2 &&:= 2401000098000001\\
4899999999^2 + 140000^2 &= 4900000001^2 &&:= 24010000009800000001
\end{aligned} \tag{2.70}$$

The first triple $(48, 14, 50)$ for $m = 7$ is not written above as it doesn't give good pattern.

- **For $m = 80, 800, 8000, 80000, \ldots$ in (6.11):**

$$\begin{aligned}
6399^2 + 160^2 &= 6401^2 &&\\
639999^2 + 1600^2 &= 640001^2 &&:= 409601280001\\
63999999^2 + 16000^2 &= 64000001^2 &&:= 4096000128000001\\
6399999999^2 + 160000^2 &= 6400000001^2 &&:= 40960000012800000001
\end{aligned} \tag{2.71}$$

The first triple $(63, 16, 65)$ for $m = 8$ is not written above as it doesn't give good pattern.

- **For $m = 90, 900, 9000, 90000, \ldots$ in (6.11):**

$$\begin{aligned}
8099^2 + 180^2 &= 8101^2 &&\\
809999^2 + 1800^2 &= 810001^2 &&:= 656101620001\\
80999999^2 + 18000^2 &= 81000001^2 &&:= 6561000162000001\\
8099999999^2 + 180000^2 &= 8100000001^2 &&:= 65610000016200000001
\end{aligned} \tag{2.72}$$

The first triple $(80, 18, 82)$ for $m = 9$ is not written above as it doesn't give good pattern.

- **For $m = 11, 111, 1111, 11111, \ldots$ in (6.11):**

$$\begin{aligned}
\mathbf{1}\,20^2 + 22^2 &= \mathbf{1}\,22^2\\
\mathbf{123}\,20^2 + 222^2 &= \mathbf{123}\,22^2\\
\mathbf{12343}\,20^2 + 2222^2 &= \mathbf{12343}\,22^2\\
\mathbf{1234543}\,20^2 + 22222^2 &= \mathbf{1234543}\,22^2\\
\mathbf{123456543}\,20^2 + 222222^2 &= \mathbf{123456543}\,22^2\\
\mathbf{12345676543}\,20^2 + 2222222^2 &= \mathbf{12345676543}\,22^2\\
\mathbf{1234567876543}\,20^2 + 22222222^2 &= \mathbf{1234567876543}\,22^2\\
\mathbf{123456789876543}\,20^2 + 222222222^2 &= \mathbf{123456789876543}\,22^2\\
\mathbf{12345679009876543}\,20^2 + 2222222222^2 &= \mathbf{12345679009876543}\,22^2
\end{aligned} \tag{2.73}$$

▶ **Division by 2**

$$\begin{aligned}
60^2 + 11^2 &= 61^2\\
6160^2 + 111^2 &= 6161^2\\
617160^2 + 1111^2 &= 617161^2\\
61727160^2 + 11111^2 &= 61727161^2\\
6172827160^2 + 111111^2 &= 6172827161^2
\end{aligned} \tag{2.74}$$

Here the division by 2 doesn't give good pattern as original one. For systematic example of this type see subsection ??.

- **For $m = 3, 33, 333, 3333, 33333, ...$ in (6.11):**

$$\begin{aligned}
8^2 + 6^2 &= 10^2\\
1088^2 + 66^2 &= 1090^2\\
110888^2 + 666^2 &= 110890^2\\
11108888^2 + 6666^2 &= 11108890^2\\
1111088888^2 + 66666^2 &= 1111088890^2\\
111110888888^2 + 666666^2 &= 111110888890^2
\end{aligned} \tag{2.75}$$

▶ **Division by 2**

$$\begin{aligned}
5^2 + 3^2 &= 5^2\\
544^2 + 33^2 &= 545^2\\
55444^2 + 333^2 &= 55445^2\\
5554444^2 + 3333^2 &= 5554445^2\\
555544444^2 + 33333^2 &= 555544445^2\\
55555444444^2 + 333333^2 &= 55555444445^2
\end{aligned} \tag{2.76}$$

- **For $m = 6, 66, 666, 6666, 66666, ...$ in (6.11):**

$$\begin{aligned}
35^2 + 12^2 &= 37^2 \\
4355^2 + 132^2 &= 4357^2 \\
443555^2 + 1332^2 &= 443557^2 \\
44435555^2 + 13332^2 &= 44435557^2 \\
4444355555^2 + 133332^2 &= 4444355557^2
\end{aligned} \tag{2.77}$$

- **For $m = 9, 99, 999, 9999, 99999, ...$ in (6.11):**

$$\begin{aligned}
80^2 + 18^2 &= 82^2 \\
9800^2 + 198^2 &= 9802^2 &&:= 96079204 \\
998000^2 + 1998^2 &= 998002^2 &&:= 996007992004 \\
99980000^2 + 19998^2 &= 99980002^2 &&:= 9996000799920004 \\
9999800000^2 + 199998^2 &= 9999800002^2 &&:= 99996000079999200004 \\
999998000000^2 + 1999998^2 &= 999998000002^2 &&:= 999996000007999992000004
\end{aligned} \tag{2.78}$$

► **Division by 2**

$$\begin{aligned}
40^2 + 9^2 &= 41^2 \\
4900^2 + 99^2 &= 4901^2 &&:= 24019801 \\
499000^2 + 999^2 &= 499001^2 &&:= 249001998001 \\
49990000^2 + 9999^2 &= 49990001^2 &&:= 2499000199980001 \\
4999900000^2 + 99999^2 &= 4999900001^2 &&:= 24999000019999800001 \\
499999000000^2 + 999999^2 &= 499999000001^2 &&:= 249999000001999998000001
\end{aligned} \tag{2.79}$$

- **For $m = 11, 101, 1001, 10001, ...$ in (6.11):**

$$\begin{aligned}
120^2 + 22^2 &= 122^2 &&:= 14884 \\
10200^2 + 202^2 &= 10202^2 &&:= 104080804 \\
1002000^2 + 2002^2 &= 1002002^2 &&:= 1004008008004 \\
100020000^2 + 20002^2 &= 100020002^2 &&:= 10004000800080004 \\
10000200000^2 + 200002^2 &= 10000200002^2 &&:= 100004000080000800004 \\
1000002000000^2 + 2000002^2 &= 1000002000002^2 &&:= 1000004000008000008000004
\end{aligned} \tag{2.80}$$

► **Division by 2**

$$
\begin{aligned}
5100^2 + 101^2 &= 5101^2 \\
501000^2 + 1001^2 &= 501001^2 && := 251002002001 \\
50010000^2 + 10001^2 &= 50010001^2 && := 2501000200020001 \\
5000100000^2 + 100001^2 &= 5000100001^2 && := 25001000020000200001 \\
500001000000^2 + 1000001^2 &= 500001000001^2 && := 250001000002000002000001
\end{aligned} \tag{2.81}
$$

The first triple (60, 11, 61) for $m = 11$ is not written above as it doesn't give good pattern.

- **For $m = 22, 202, 2002, 20002, \ldots$ in (6.11):**

$$
\begin{aligned}
483^2 + 44^2 &= 485^2 \\
40803^2 + 404^2 &= 40805^2 \\
4008003^2 + 4004^2 &= 4008005^2 && := 16064104080025 \\
400080003^2 + 40004^2 &= 400080005^2 && := 160064010400800025 \\
40000800003^2 + 400004^2 &= 40000800005^2 && := 1600064001040008000025
\end{aligned} \tag{2.82}
$$

- **For $m = 3, 33, 303, 3003, 30003, 300003, \ldots$ in (6.11):**

$$
\begin{aligned}
8^2 + 6^2 &= 10^2 \\
1088^2 + 66^2 &= 1090^2 \\
91808^2 + 606^2 &= 91810^2 \\
9018008^2 + 6006^2 &= 9018010^2 && := 81324504360100 \\
900180008^2 + 60006^2 &= 900180010^2 && := 810324050403600100 \\
90001800008^2 + 600006^2 &= 90001800010^2 && := 8100324005040036000100 \\
9000018000008^2 + 6000006^2 &= 9000018000010^2 && := 81000324000504000360000100
\end{aligned} \tag{2.83}
$$

▶ **Division by 2**

$$
\begin{aligned}
4^2 + 3^2 &= 5^2 \\
544^2 + 33^2 &= 545^2 \\
45904^2 + 303^2 &= 45905^2 \\
4509004^2 + 3003^2 &= 4509005^2 \\
450090004^2 + 30003^2 &= 450090005^2 && := 202581012600900025 \\
45000900004^2 + 300003^2 &= 45000900005^2 && := 2025081001260009000025 \\
4500009000004^2 + 3000003^2 &= 4500009000005^2 && := 20250081000126000090000025
\end{aligned} \tag{2.84}
$$

- **For $m = 404, 4004, 40004, 400004, \ldots$ in (6.11):**

$$\begin{aligned} 163215^2 + 808^2 &= 163217^2 \\ 16032015^2 + 8008^2 &= 16032017^2 \\ 1600320015^2 + 80008^2 &= 1600320017^2 := 2561024156810880289 \\ 160003200015^2 + 800008^2 &= 160003200017^2 := 25601024015680108800289 \end{aligned} \tag{2.85}$$

The first triple $(1935, 88, 1937)$ for $m = 44$ is not written above as it doesn't give good pattern.

- **For $m = 505, 5005, 50005, 500005, \ldots$ in (6.11):**

$$\begin{aligned} 255024^2 + 1010^2 &= 255026^2 \\ 25050024^2 + 10010^2 &= 25050026^2 \\ 2500500024^2 + 100010^2 &= 2500500026^2 := 6252500380026000676 \\ 250005000024^2 + 1000010^2 &= 250005000026^2 := 62502500038000260000676 \\ 25000050000024^2 + 10000010^2 &= 25000050000026^2 := 625002500003800002600000676 \end{aligned} \tag{2.86}$$

The first triple $(3024, 110, 3026)$ for $m = 55$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned} 12525012^2 + 5005^2 &= 12525013^2 \\ 1250250012^2 + 50005^2 &= 1250250013^2 \\ 125002500012^2 + 500005^2 &= 125002500013^2 := 15625625009500065000169 \\ 12500025000012^2 + 5000005^2 &= 12500025000013^2 := 156250625000950000650000169 \end{aligned} \tag{2.87}$$

The first triple $(127512, 505, 127513)$ is not written above as it doesn't give good pattern.

- **For $m = 606, 6006, 60006, 600006, \ldots$ in (6.11):**

$$\begin{aligned} 367235^2 + 1212^2 &= 367237^2 \\ 36072035^2 + 12012^2 &= 36072037^2 \\ 3600720035^2 + 120012^2 &= 3600720037^2 := 12965184784853281369 \\ 360007200035^2 + 1200012^2 &= 360007200037^2 := 129605184078480532801369 \end{aligned} \tag{2.88}$$

The first triple $(4355, 132, 4357)$ for $m = 66$ is not written above as it doesn't give good pattern.

- **For $m = 707, 7007, 70007, 700007, \ldots$ in (6.11):**

$$\begin{aligned} 499848^2 + 1414^2 &= 499850^2 \\ 49098048^2 + 14014^2 &= 49098050^2 \\ 4900980048^2 + 140014^2 &= 4900980050^2 \\ 490009800048^2 + 1400014^2 &= 490009800050^2 \\ 49000098000048^2 + 14000014^2 &= 49000098000050^2 \end{aligned} \tag{2.89}$$

The first triple $(5928, 154, 5930)$ for $m = 77$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned} 24549024^2 + 7007^2 &= 24549025^2 \\ 2450490024^2 + 70007^2 &= 2450490025^2 \\ 245004900024^2 + 700007^2 &= 245004900025^2 \\ 24500049000024^2 + 7000007^2 &= 24500049000025^2 \end{aligned} \tag{2.90}$$

The first triple (249924, 707, 249925) for $m = 707$ is not written above as it doesn't give good pattern.

- **For $m = 8008, 80008, 800008, \ldots$ in (6.11):**

$$\begin{aligned} 64128063^2 + 16016^2 &= 64128065^2 \\ 6401280063^2 + 160016^2 &= 6401280065^2 \\ 640012800063^2 + 1600016^2 &= 640012800065^2 := 409616384247041664004225 \\ 64000128000063^2 + 16000016^2 &= 64000128000065^2 := 4096016384024704016640004225 \end{aligned} \tag{2.91}$$

The first two triples (7743, 176, 7745) and (652863, 1616, 652865) for $m = 88$ and 808 are not written above as they don't give good pattern.

- **For $m = 9009, 90009, 900009, \ldots$ in (6.11):**

$$\begin{aligned} 81162080^2 + 18018^2 &= 81162082^2 \\ 8101620080^2 + 180018^2 &= 8101620082^2 \\ 810016200080^2 + 1800018^2 &= 810016200082^2 := 656126244395282656806724 \\ 81000162000080^2 + 18000018^2 &= 81000162000082^2 := 6561026244039528026568006724 \end{aligned} \tag{2.92}$$

The first two triples (9800, 198, 9802) and (826280, 1818, 826282) for $m = 99$ and 909 are not written above as they don't give good pattern.

▶ **Division by 2**

$$\begin{aligned} 40581040^2 + 9009^2 &= 40581041^2 \\ 4050810040^2 + 90009^2 &= 4050810041^2 \\ 405008100040^2 + 900009^2 &= 405008100041^2 \\ 40500081000040^2 + 9000009^2 &= 40500081000041^2 \end{aligned} \tag{2.93}$$

- **For $m = 10, 110, 1110, 11110, \ldots$ in (6.11):**

$$\begin{aligned} 99^2 + 20^2 &= 101^2 \\ 12099^2 + 220^2 &= 12101^2 \\ 1232099^2 + 2220^2 &= 1232101^2 \\ 123432099^2 + 22220^2 &= 123432101^2 \end{aligned} \tag{2.94}$$

- **For $m = 3, 13, 133, 1333, 13333, 133333, ...$ in (6.11):**

$$\begin{aligned}
8^2 + 6^2 &= 10^2 \\
168^2 + 26^2 &= 170^2 \\
17688^2 + 266^2 &= 17690^2 \\
1776888^2 + 2666^2 &= 1776890^2 \\
177768888^2 + 26666^2 &= 177768890^2 \\
17777688888^2 + 266666^2 &= 17777688890^2
\end{aligned} \tag{2.95}$$

  - ▶ **Division by 2**

$$\begin{aligned}
4^2 + 3^2 &= 5^2 \\
84^2 + 13^2 &= 85^2 \\
8844^2 + 133^2 &= 8845^2 \\
888444^2 + 1333^2 &= 888445^2 \\
88884444^2 + 13333^2 &= 88884445^2 \\
8888844444^2 + 133333^2 &= 8888844445^2 \\
888888444444^2 + 1333333^2 &= 888888444445^2
\end{aligned} \tag{2.96}$$

- **For $m = 233, 2333, 23333, 233333, ...$ in (6.11):**

$$\begin{aligned}
54288^2 + 466^2 &= 54290^2 \\
5442888^2 + 4666^2 &= 5442890^2 \\
544428888^2 + 46666^2 &= 544428890^2 \\
54444288888^2 + 466666^2 &= 54444288890^2
\end{aligned} \tag{2.97}$$

The first triple $(528, 46, 530)$ for $m = 23$ is not written above as it doesn't give good pattern.

  - ▶ **Division by 2**

$$\begin{aligned}
27144^2 + 233^2 &= 27145^2 \\
2721444^2 + 2333^2 &= 2721445^2 \\
272214444^2 + 23333^2 &= 272214445^2 \\
27222144444^2 + 233333^2 &= 27222144445^2 \\
2722221444444^2 + 2333333^2 &= 2722221444445^2
\end{aligned} \tag{2.98}$$

- **For $m = 16, 166, 1666, 16666, 166666, ...$ in (6.11):**

$$\begin{aligned}
255^2 + 32^2 &= 257^2 \\
27555^2 + 332^2 &= 27557^2 \\
2775555^2 + 3332^2 &= 2775557^2 \\
277755555^2 + 33332^2 &= 277755557^2
\end{aligned} \tag{2.99}$$

- **For $m = 266, 2666, 26666, 266666, \ldots$ in (6.11):**

$$\begin{aligned} 70755^2 + 532^2 &= 70757^2 \\ 7107555^2 + 5332^2 &= 7107557^2 \\ 711075555^2 + 53332^2 &= 711075557^2 \\ 71110755555^2 + 533332^2 &= 71110755557^2 \end{aligned} \tag{2.100}$$

The first triple (675, 52, 677) for $m = 26$ is not written above as it doesn't give good pattern.

- **For $m = 19, 199, 1999, 19999, 199999, \ldots$ in (6.11):**

$$\begin{aligned} 360^2 + 38^2 &= 362 \\ 39600^2 + 398^2 &= 39602^2 && := 1568318404 \\ 3996000^2 + 3998^2 &= 3996002^2 && := 15968031984004 \\ 399960000^2 + 39998^2 &= 399960002^2 && := 159968003199840004 \\ 39999600000^2 + 399998^2 &= 39999600002^2 && := 1599968000319998400004 \end{aligned} \tag{2.101}$$

▶ **Division by 2**

$$\begin{aligned} 180^2 + 19^2 &= 181^2 \\ 19800^2 + 199^2 &= 19801^2 && := 392079601 \\ 1998000^2 + 1999^2 &= 1998001^2 && := 3992007996001 \\ 199980000^2 + 19999^2 &= 199980001^2 && := 39992000799960001 \\ 19999800000^2 + 199999^2 &= 19999800001^2 && := 399992000079999600001 \end{aligned} \tag{2.102}$$

- **For $m = 29, 299, 2999, 29999, 299999, \ldots$ in (6.11):**

$$\begin{aligned} 840^2 + 58^2 &= 842^2 \\ 89400^2 + 598^2 &= 89402^2 \\ 8994000^2 + 5998^2 &= 8994002^2 && := 80892071976004 \\ 899940000^2 + 59998^2 &= 899940002^2 && := 809892007199760004 \\ 89999400000^2 + 599998^2 &= 89999400002^2 && := 8099892000719997600004 \end{aligned} \tag{2.103}$$

▶ **Division by 2**

$$\begin{aligned} 44700^2 + 299^2 &= 44701^2 \\ 4497000^2 + 2999^2 &= 4497001^2 \\ 449970000^2 + 29999^2 &= 449970001^2 && := 202473001799940001 \\ 44999700000^2 + 299999^2 &= 44999700001^2 && := 2024973000179999400001 \end{aligned} \tag{2.104}$$

The first triple (420, 29, 421) for $m = 29$ is not written above as it doesn't give good pattern.

- **For $m = 39, 399, 3999, 39999, 399999, ...$ in (6.11):**

$$\begin{aligned}
1520^2 + 78^2 &= 1522^2 \\
159200^2 + 798^2 &= 159202^2 \\
15992000^2 + 7998^2 &= 15992002^2 := 255744127968004 \\
1599920000^2 + 79998^2 &= 1599920002^2 := 2559744012799680004 \\
159999200000^2 + 799998^2 &= 159999200002^2 := 25599744001279996800004
\end{aligned} \tag{2.105}$$

▶ Division by 2

$$\begin{aligned}
760^2 + 39^2 &= 761^2 \\
79600^2 + 399^2 &= 79601^2 := 6336319201 \\
7996000^2 + 3999^2 &= 7996001^2 := 63936031992001 \\
799960000^2 + 39999^2 &= 799960001^2 := 639936003199920001 \\
79999600000^2 + 399999^2 &= 79999600001^2 := 6399936000319999200001 \\
7999996000000^2 + 3999999^2 &= 7999996000001^2 := 63999936000031999992000001
\end{aligned} \tag{2.106}$$

2.4 Procedure 4

Let's consider the following three functions:

$$\begin{aligned}
f_4(m) &:= 4m^2 - 1 \\
g_4(m) &:= 4m \\
h_4(m) &:= 4m^2 + 1
\end{aligned} \tag{2.107}$$

Then we can easily check that

$$\begin{aligned}
f_4(m)^2 + g_4(m)^2 &= \left(4m^2 - 1\right)^2) + (4m)^2 \\
&= 16m^4 - 8m^2 + 1 + 16m^2 \\
&= 16m^4 + 8m^2 + 1 \\
&= \left(4m^2 + 1\right)^2 = h_4(m)^2
\end{aligned}$$

This proves that the triple (f_4, g_4, h_4) is a **Pythagorean triple**. Let's consider $m = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ in (6.13). This gives following triples:

$$\begin{aligned}
(f_4(1), g_4(1), h_4(1)) &= (3, 4, 5) \\
(f_4(2), g_4(2), h_4(2)) &= (15, 8, 17) \\
(f_4(3), g_4(3), h_4(3)) &= (35, 12, 37) \\
(f_4(4), g_4(4), h_4(4)) &= (63, 16, 65) \\
(f_4(5), g_4(5), h_4(5)) &= (99, 20, 101) \\
(f_4(6), g_4(6), h_4(6)) &= (143, 24, 145) \\
(f_4(7), g_4(7), h_4(7)) &= (195, 28, 197) \\
(f_4(8), g_4(8), h_4(8)) &= (255, 32, 257) \\
(f_4(9), g_4(9), h_4(9)) &= (323, 36, 325) \\
(f_4(10), g_4(10), h_4(10)) &= (399, 40, 401)
\end{aligned}$$

The following property among Procedures 3 and 4 hold:

$$(f_4(m), g_4(m), h_4(m)) = (f_3(2m), g_3(2m), h_3(2m))$$

In particular, we have

$$\begin{aligned}
(f_4(1), g_4(1), h_4(1)) &= (f_3(2), g_3(2), h_3(2)) &&= (3, 4, 5)\\
(f_4(2), g_4(2), h_4(2)) &= (f_3(4), g_3(4), h_3(4)) &&= (15, 8, 17)\\
(f_4(3), g_4(3), h_4(3)) &= (f_3(6), g_3(6), h_3(6)) &&= (35, 12, 37)\\
(f_4(4), g_4(4), h_4(4)) &= (f_3(8), g_3(8), h_3(8)) &&= (63, 16, 65)\\
(f_4(5), g_4(5), h_4(5)) &= (f_3(10), g_3(10), h_3(10)) &&= (99, 20, 101)
\end{aligned}$$

2.4.1 Examples

Below are some patterns obtained by use of formulas (6.13).

- **For $m = 1, 16, 166, 1666, 16666, ...$ in (6.13):**

$$\begin{aligned}
3^2 + 4^2 &= 5^2\\
1023^2 + 64^2 &= 1025^2\\
110223^2 + 664^2 &= 110225^2\\
11102223^2 + 6664^2 &= 11102225^2\\
1111022223^2 + 66664^2 &= 1111022225^2
\end{aligned} \tag{2.108}$$

- **For $m = 366, 3666, 36666, ...$ in (6.13):**

$$\begin{aligned}
535823^2 + 1464^2 &= 535825^2\\
53758223^2 + 14664^2 &= 53758225^2\\
5377582223^2 + 146664^2 &= 5377582225^2\\
537775822223^2 + 1466664^2 &= 537775822225^2
\end{aligned} \tag{2.109}$$

The first two triples $(35, 12, 37)$ and $(5183, 144, 5185)$ for $m = 3$ and 36 are not written above as they don't give good pattern.

- **For $m = 633, 6333, 63333, 633333, ...$ in (6.13):**

$$\begin{aligned}
1602755^2 + 2532^2 &= 1602757^2\\
160427555^2 + 25332^2 &= 160427557^2\\
16044275555^2 + 253332^2 &= 16044275557^2\\
1604442755555^2 + 2533332^2 &= 1604442755557^2
\end{aligned} \tag{2.110}$$

The first two triples $(143, 24, 145)$ and $(15875, 252, 15877)$ for $m = 6$ and 63 are not written above as they don't give good pattern.

- **For $m = 6, 66, 666, 6666, \ldots$ in (6.13):**

$$\begin{aligned} 143^2 + 24^2 &= 145^2 \\ 17423^2 + 264^2 &= 17425^2 \\ 1774223^2 + 2664^2 &= 1774225^2 \\ 177742223^2 + 26664^2 &= 177742225^2 \\ 17777422223^2 + 266664^2 &= 17777422225^2 \end{aligned} \tag{2.111}$$

- **For $m = 699, 6999, 69999, 699999, \ldots$ in (6.13):**

$$\begin{aligned} 1954403^2 + 2796^2 &= 1954405^2 \\ 195944003^2 + 27996^2 &= 195944005^2 \\ 19599440003^2 + 279996^2 &= 19599440005^2 \\ 1959994400003^2 + 2799996^2 &= 1959994400005^2 \end{aligned} \tag{2.112}$$

The first two triples $(143, 24, 145)$ and $(19043, 276, 19045)$ for $m = 6$ and 69 are not written above as they don't give good pattern.

- **For $m = 39, 399, 3999, 39999, \ldots$ in (6.13):**

$$\begin{aligned} 6083^2 + 156^2 &= 6085^2 \\ 636803^2 + 1596^2 &= 636805^2 \\ 63968003^2 + 15996^2 &= 63968005^2 \\ 6399680003^2 + 159996^2 &= 6399680005^2 \end{aligned} \tag{2.113}$$

The first triple $(35, 12, 37)$ for $m = 3$ is not written above as it doesn't give good pattern.

- **For $m = 933, 9333, 93333, 933333, \ldots$ in (6.13):**

$$\begin{aligned} 3481955^2 + 3732^2 &= 3481957^2 \\ 348419555^2 + 37332^2 &= 348419557^2 \\ 34844195555^2 + 373332^2 &= 34844195557^2 \\ 3484441955555^2 + 3733332^2 &= 3484441955557^2 \end{aligned} \tag{2.114}$$

The first two triples $(323, 36, 325)$ and $(34595, 373, 34597)$ for $m = 9$ and 93 are not written above as they don't give good pattern.

- **For $m = 966, 9666, 96666, 966666, \ldots$ in (6.13):**

$$\begin{aligned} 3732623^2 + 3864^2 &= 3732625^2 \\ 373726223^2 + 38664^2 &= 373726225^2 \\ 37377262223^2 + 386664^2 &= 37377262225^2 \\ 3737772622223^2 + 3866664^2 &= 3737772622225^2 \end{aligned} \tag{2.115}$$

The first two triples $(323, 36, 325)$ and $(36863, 384, 36865)$ for $m = 9$ and 96 are not written above as they don't give good pattern.

- **For $m = 9, 99, 999, 9999, \ldots$ in (6.13):**

$$\begin{aligned}
323^2 + 36^2 &= 325^2 \\
39203^2 + 396^2 &= 39205^2 \\
3992003^2 + 3996^2 &= 3992005^2 &&:= 15936103920025 \\
399920003^2 + 39996^2 &= 399920005^2 &&:= 159936010399200025 \\
39999200003^2 + 399996^2 &= 39999200005^2 &&:= 1599936001039992000025
\end{aligned} \tag{2.116}$$

First two expressions don't give a good pattern with final sums.

2.5 Procedure 5

Let's consider the following three functions:

$$\begin{aligned}
f_5(m) &:= 16m^2 - 1 \\
g_5(m) &:= 8m \\
h_5(m) &:= 16m^2 + 1
\end{aligned} \tag{2.117}$$

Then we can easily check that

$$\begin{aligned}
f_5(m)^2 + g_5(m)^2 &= \left(16m^2 - 1\right)^2 + (8m)^2 \\
&= 256m^4 - 32m^2 + 1 + 64m^2 \\
&= 256m^4 + 32m^2 + 1 \\
&= \left(16m^2 + 1\right)^2 = h_5(m)^2.
\end{aligned}$$

This proves that the triple (f_5, g_5, h_5) is a **Pythagorean triple**. Let's consider $m = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ in (2.117). This gives following triples:

$$\begin{aligned}
(f_5(1), g_5(1), h_5(1)) &= (15, 8, 17) \\
(f_5(2), g_5(2), h_5(2)) &= (63, 16, 65) \\
(f_5(3), g_5(3), h_5(3)) &= (143, 24, 145) \\
(f_5(4), g_5(4), h_5(4)) &= (255, 32, 257) \\
(f_5(5), g_5(5), h_5(5)) &= (399, 40, 401) \\
(f_5(6), g_5(6), h_5(6)) &= (575, 48, 577) \\
(f_5(7), g_5(7), h_5(7)) &= (783, 56, 785) \\
(f_5(8), g_5(8), h_5(8)) &= (1023, 64, 1025) \\
(f_5(9), g_5(9), h_5(9)) &= (1295, 72, 1297) \\
(f_5(10), g_5(10), h_5(10)) &= (1599, 80, 1601)
\end{aligned}$$

The following property among Procedures 4 and 5 hold:

$$(f_5(m), g_5(m), h_3(m)) = (f_4(2m), g_4(2m), h_4(2m))$$

In particular, we have

$$\begin{aligned}
(f_5(1), g_5(1), h_5(1)) &= (f_4(2), g_4(2), h_4(2)) &&= (f_3(4), g_3(4), h_3(4)) &&= (15, 8, 17)\\
(f_5(2), g_5(2), h_5(2)) &= (f_4(4), g_4(4), h_4(4)) &&= (f_3(8), g_3(8), h_3(8)) &&= (63, 16, 65)\\
(f_5(3), g_5(3), h_5(3)) &= (f_4(6), g_4(6), h_4(6)) &&= (f_3(12), g_3(12), h_3(12)) &&= (143, 24, 145)\\
(f_5(4), g_5(4), h_5(4)) &= (f_4(8), g_4(8), h_4(8)) &&= (f_3(16), g_3(16), h_3(16)) &&= ((255, 32, 257)\\
(f_5(5), g_5(5), h_5(5)) &= (f_4(10), g_4(10), h_4(10)) &&= (f_3(12), g_3(20), h_3(20)) &&= (399, 40, 401)\,.
\end{aligned}$$

Remark 4. *Procedures 1 and 4 are with primitive values, while procedures 2 and 3 are primitive and non-primitive values. The Procedure 5 again give all the triples primitive.*

2.5.1 Examples

Below are examples of patterns in Pythagorean triples based on Procedures 5 given in 2.5 and equation 2.117.

- **For $m = 10, 100, 1000, 10000, 10000, ...$ in (2.117):**

$$\begin{aligned}
1599^2 + 80^2 &= 1601^2 &&:= 2563201\\
159999^2 + 800^2 &= 160001^2 &&:= 25600320001\\
15999999^2 + 8000^2 &= 16000001^2 &&:= 256000032000001\\
1599999999^2 + 80000^2 &= 1600000001^2 &&:= 2560000003200000001\\
159999999999^2 + 800000^2 &= 160000000001^2 &&:= 25600000000320000000001
\end{aligned} \tag{2.118}$$

- **For $m = 20, 200, 2000, 20000, 20000, ...$ in (2.117):**

$$\begin{aligned}
6399^2 + 160^2 &= 6401^2 &&\\
639999^2 + 1600^2 &= 640001^2 &&:= 409601280001\\
63999999^2 + 16000^2 &= 64000001^2 &&:= 4096000128000001\\
6399999999^2 + 160000^2 &= 6400000001^2 &&:= 40960000012800000001\\
639999999999^2 + 1600000^2 &= 640000000001^2 &&:= 409600000001280000000001
\end{aligned} \tag{2.119}$$

- **For $m = 30, 300, 3000, 30000, 30000, ...$ in (2.117):**

$$\begin{aligned}
14399^2 + 240^2 &= 14401^2 &&\\
1439999^2 + 2400^2 &= 1440001^2 &&:= 2073602880001\\
143999999^2 + 24000^2 &= 144000001^2 &&:= 20736000288000001\\
14399999999^2 + 240000^2 &= 14400000001^2 &&:= 207360000028800000001\\
1439999999999^2 + 2400000^2 &= 1440000000001^2 &&:= 2073600000002880000000001
\end{aligned} \tag{2.120}$$

- **For $m = 40, 400, 4000, 40000, 40000, ...$ in (2.117):**

$$\begin{aligned} 25599^2 + 320^2 &= 25601^2 \\ 2559999^2 + 3200^2 &= 2560001^2 := 6553605120001 \\ 255999999^2 + 32000^2 &= 256000001^2 := 65536000512000001 \\ 25599999999^2 + 320000^2 &= 25600000001^2 := 655360000051200000001 \\ 2559999999999^2 + 3200000^2 &= 2560000000001^2 := 6553600000005120000000001 \end{aligned} \tag{2.121}$$

- **For $m = 5, 50, 500, 5000, 50000, 50000, ...$ in (2.117):**

$$\begin{aligned} 399^2 + 40^2 &= 401^2 := 160801 \\ 39999^2 + 400^2 &= 40001^2 := 1600080001 \\ 3999999^2 + 4000^2 &= 4000001^2 := 16000008000001 \\ 399999999^2 + 40000^2 &= 400000001^2 := 160000000800000001 \\ 39999999999^2 + 400000^2 &= 40000000001^2 := 1600000000080000000001 \\ 3999999999999^2 + 4000000^2 &= 4000000000001^2 := 16000000000008000000000001 \end{aligned} \tag{2.122}$$

- **For $m = 60, 600, 6000, 60000, 60000, ...$ in (2.117):**

$$\begin{aligned} 57599^2 + 480^2 &= 57601^2 \\ 5759999^2 + 4800^2 &= 5760001^2 := 33177611520001 \\ 575999999^2 + 48000^2 &= 576000001^2 := 331776001152000001 \\ 57599999999^2 + 480000^2 &= 57600000001^2 := 3317760000115200000001 \\ 5759999999999^2 + 4800000^2 &= 5760000000001^2 := 33177600000011520000000001 \end{aligned} \tag{2.123}$$

- **For $m = 70, 700, 7000, 70000, 70000, ...$ in (2.117):**

$$\begin{aligned} 78399^2 + 560^2 &= 78401^2 \\ 7839999^2 + 5600^2 &= 7840001^2 := 61465615680001 \\ 783999999^2 + 56000^2 &= 784000001^2 := 614656001568000001 \\ 78399999999^2 + 560000^2 &= 78400000001^2 := 6146560000156800000001 \\ 7839999999999^2 + 5600000^2 &= 7840000000001^2 := 61465600000015680000000001 \end{aligned} \tag{2.124}$$

- **For $m = 80, 800, 8000, 80000, 80000, ...$ in (2.117):**

$$\begin{aligned} 102399^2 + 640^2 &= 102401^2 \\ 10239999^2 + 6400^2 &= 10240001^2 := 104857620480001 \\ 1023999999^2 + 64000^2 &= 1024000001^2 := 1048576002048000001 \\ 102399999999^2 + 640000^2 &= 102400000001^2 := 10485760000204800000001 \\ 10239999999999^2 + 6400000^2 &= 10240000000001^2 := 104857600000020480000000001 \end{aligned} \tag{2.125}$$

- **For $m = 90, 900, 9000, 90000, 90000, ...$ in (2.117):**

$$\begin{aligned}
129599^2 + 720^2 &= 129601^2 \\
12959999^2 + 7200^2 &= 12960001^2 &&:= 167961625920001 \\
1295999999^2 + 72000^2 &= 1296000001^2 &&:= 1679616002592000001 \\
129599999999^2 + 720000^2 &= 129600000001^2 &&:= 16796160000259200000001 \\
12959999999999^2 + 7200000^2 &= 12960000000001^2 &&:= 167961600000025920000000001
\end{aligned} \tag{2.126}$$

- **For $m = 101, 1001, 10001, 100001, ...$ in (2.117):**

$$\begin{aligned}
163215^2 + 808^2 &= 163217^2 \\
16032015^2 + 8008^2 &= 16032017^2 \\
1600320015^2 + 80008^2 &= 1600320017^2 &&:= 2561024156810880289 \\
160003200015^2 + 800008^2 &= 160003200017^2 &&:= 25601024015680108800289 \\
16000032000015^2 + 8000008^2 &= 16000032000017^2 &&:= 256001024001568001088000289
\end{aligned} \tag{2.127}$$

- **For $m = 202, 2002, 20002, 200002, ...$ in (2.117):**

$$\begin{aligned}
652863^2 + 1616^2 &= 652865^2 \\
64128063^2 + 16016^2 &= 64128065^2 \\
6401280063^2 + 160016^2 &= 6401280065^2 \\
640012800063^2 + 1600016^2 &= 640012800065^2 &&:= 409616384247041664004225 \\
64000128000063^2 + 16000016^2 &= 64000128000065^2 &&:= 4096016384024704016640004225
\end{aligned} \tag{2.128}$$

- **For $m = 303, 3003, 30003, 300003, ...$ in (2.117):**

$$\begin{aligned}
1468943^2 + 2424^2 &= 1468945^2 \\
144288143^2 + 24024^2 &= 144288145^2 \\
14402880143^2 + 240024^2 &= 14402880145^2 \\
1440028800143^2 + 2400024^2 &= 1440028800145^2 \\
144000288000143^2 + 24000024^2 &= 144000288000145^2
\end{aligned} \tag{2.129}$$

- **For $m = 5, 55, 505, 5005, 50005, 500005, ...$ in (2.117):**

$$
\begin{aligned}
399^2 + 40^2 &= 401^2 \\
48399^2 + 440^2 &= 48401^2 \\
4080399^2 + 4040^2 &= 4080401^2 \\
400800399^2 + 40040^2 &= 400800401^2 \\
40008000399^2 + 400040^2 &= 40008000401^2 := 1600640096086416160801 \\
4000080000399^2 + 4000040^2 &= 4000080000401^2 := 16000640009608064160160801 \\
400000800000399^2 + 40000040^2 &= 400000800000401^2 := 160000640000960800641600160801
\end{aligned} \tag{2.130}
$$

- **For $m = 808, 8008, 80008, 800008, \ldots$ in (2.117):**

$$
\begin{aligned}
10445823^2 + 6464^2 &= 10445825^2 \\
1026049023^2 + 64064^2 &= 1026049025^2 \\
102420481023^2 + 640064^2 &= 102420481025^2 \\
10240204801023^2 + 6400064^2 &= 10240204801025^2 \\
1024002048001023^2 + 64000064^2 &= 1024002048001025^2
\end{aligned} \tag{2.131}
$$

2.6 Pandigital Palindromic-Type Patterns

Below are some examples of **palindormic-type pandigial patterns**. These are of three types with different possibilities.

▶ First-Type: Part 1

- **For $m = 10, 110, 1110, 11110, \ldots$ in (6.11): or**
- **For $m = 5, 55, 555, 5555, 55555, \ldots$ in (6.13): or**
- **For $m = 5/2, 55/2, 555/2, 5555/2, 55555/2, \ldots$ in (2.117):**

$$
\begin{aligned}
099^2 + 20^2 &= 1\,01^2 \\
12\,099^2 + 220^2 &= 121\,01^2 \\
1232\,099^2 + 2220^2 &= 12321\,01^2 \\
123432\,099^2 + 22220^2 &= 1234321\,01^2 \\
12345432\,099^2 + 222220^2 &= 123454321\,01^2 \\
1234565432\,099^2 + 2222220^2 &= 12345654321\,01^2 \\
123456765432\,099^2 + 22222220^2 &= 1234567654321\,01^2 \\
12345678765432\,099^2 + 222222220^2 &= 123456787654321\,01^2 \\
1234567898765432\,099^2 + 2222222220^2 &= 12345678987654321\,01^2
\end{aligned} \tag{2.132}
$$

▶ First-Type: Part 2

- **For $m = 100, 1100, 11100, 111100, ...$ in (6.11): or**
- **For $m = 50, 550, 5550, 55550, 555550, ...$ in (6.13): or**
- **For $m = 50/2, 550/2, 5550/2, 55550/2, 555550/2, ...$ in (2.117):**

$$
\begin{aligned}
09999^2 + 200^2 &= 1\,0001^2 \\
12\,09999^2 + 2200^2 &= 121\,0001^2 \\
1232\,09999^2 + 22200^2 &= 12321\,0001^2 \\
123432\,09999^2 + 222200^2 &= 1234321\,0001^2 \\
12345432\,09999^2 + 2222200^2 &= 123454321\,0001^2 \\
1234565432\,09999^2 + 22222200^2 &= 12345654321\,0001^2 \\
123456765432\,09999^2 + 222222200^2 &= 1234567654321\,0001^2 \\
12345678765432\,09999^2 + 2222222200^2 &= 123456787654321\,0001^2 \\
1234567898765432\,09999^2 + 22222222200^2 &= 12345678987654321\,0001^2
\end{aligned}
\tag{2.133}
$$

▶ First-Type: Part 3

- **For $m = 1000, 11000, 111000, 1111000, ...$ in (6.11): or**
- **For $m = 1000/2, 11000/2, 111000/2, 1111000/2, ...$ in (6.13): or**
- **For $m = 1000/4, 11000/4, 111000/4, 1111000/4, ...$ in (2.117):**

$$
\begin{aligned}
0999999^2 + 2000^2 &= 1\,000001^2 \\
12\,0999999^2 + 22000^2 &= 121\,000001^2 \\
1232\,0999999^2 + 222000^2 &= 12321\,000001^2 \\
123432\,0999999^2 + 2222000^2 &= 1234321\,000001^2 \\
12345432\,0999999^2 + 22222000^2 &= 123454321\,000001^2 \\
1234565432\,0999999^2 + 222222000^2 &= 12345654321\,000001^2 \\
123456765432\,0999999^2 + 2222222000^2 &= 1234567654321\,000001^2 \\
12345678765432\,0999999^2 + 22222222000^2 &= 123456787654321\,000001^2 \\
1234567898765432\,0999999^2 + 222222222000^2 &= 12345678987654321\,000001^2
\end{aligned}
\tag{2.134}
$$

Remark 5. *Extending the above study for two variables, the first case lead us to 9 values, the second case lead us to 99 values and the third case lead us to 999 values. This has been done separately in Taneja [6].*

▶ Second-Type: Part 1

- **For $m = 10, 1010, 101010, 10101010, ...$ in (6.11): or**
- **For $m = 10/2, 1010/2, 101010/2, 10101010/2, ...$ in (6.13): or**
- **For $m = 10/4, 1010/4, 101010/4, 10101010/4, ...$ in (2.117):**

$$
\begin{aligned}
0099^2 + 20^2 &= 1\,01^2 \\
102\,0099^2 + 2020^2 &= 10201\,01^2 \\
1020302\,0099^2 + 202020^2 &= 102030201\,01^2 \\
10203040302\,0099^2 + 20202020^2 &= 1020304030201\,01^2 \\
102030405040302\,0099^2 + 2020202020^2 &= 10203040504030201\,01^2 \\
1020304050605040302\,0099^2 + 202020202020^2 &= 102030405060504030201\,01^2 \\
10203040506070605040302\,0099^2 + 20202020202020^2 &= 1020304050607060504030201\,01^2 \\
102030405060708070605040302\,0099^2 + 2020202020202020^2 &= 10203040506070807060504030201\,01^2 \\
1020304050607080908070605040302\,0099^2 + 202020202020202020^2 &= 102030405060708090807060504030201\,01^2
\end{aligned} \tag{2.135}
$$

▶ Second-Type: Part 2

- **For $m = 100, 10100, 1010100, 101010100, 10101010100, ...$ in (6.11): or**
- **For $m = 50, 5050, 505050, 50505050, 5050505050, ...$ in (6.13): or**
- **For $m = 25, 2525, 252525, 25252525, ...$ in (2.117):**

$$
\begin{aligned}
009999^2 + 200^2 &= 1\,0001^2 \\
102\,009999^2 + 20200^2 &= 10201\,0001^2 \\
1020302\,009999^2 + 2020200^2 &= 102030201\,0001^2 \\
10203040302\,009999^2 + 202020200^2 &= 1020304030201\,0001^2 \\
102030405040302\,009999^2 + 20202020200^2 &= 10203040504030201\,0001^2 \\
1020304050605040302\,009999^2 + 2020202020200^2 &= 102030405060504030201\,0001^2 \\
10203040506070605040302\,009999^2 + 202020202020200^2 &= 1020304050607060504030201\,0001^2 \\
102030405060708070605040302\,009999^2 + 20202020202020200^2 &= 10203040506070807060504030201\,0001^2 \\
1020304050607080908070605040302\,009999^2 + 2020202020202020200^2 &= 102030405060708090807060504030201\,0001^2
\end{aligned} \tag{2.136}
$$

▶ Second-Type: Part 3

- **For $m = 1000, 101000, 10101000, 1010101000, 101010101000, ...$ in (6.11): or**
- **For $m = 1000/2, 101000/2, 10101000/2, 1010101000/2, 101010101000/2, ...$ in (6.13): or**
- **For $m = 1000/4, 101000/4, 10101000/4, 1010101000/4, 101010101000/4, ...$ in (2.117):**

$$\begin{aligned}
00999999^2 + 2000^2 &= 1\,000001^2\\
\mathbf{102}\,00999999^2 + 202000^2 &= \mathbf{10201}\,000001^2\\
\mathbf{1020302}\,00999999^2 + 20202000^2 &= \mathbf{102030201}\,000001^2\\
\mathbf{10203040302}\,00999999^2 + 2020202000^2 &= \mathbf{1020304030201}\,000001^2\\
\mathbf{102030405040302}\,00999999^2 + 202020202000^2 &= \mathbf{10203040504030201}\,000001^2\\
\mathbf{1020304050605040302}\,00999999^2 + 20202020202000^2 &= \mathbf{102030405060504030201}\,000001^2\\
\mathbf{10203040506070605040302}\,00999999^2 + 2020202020202000^2 &= \mathbf{1020304050607060504030201}\,000001^2\\
\mathbf{102030405060708070605040302}\,00999999^2 + 202020202020202000^2 &= \mathbf{10203040506070807060504030201}\,000001^2\\
\mathbf{1020304050607080908070605040302}\,00999999^2 + 20202020202020202000^2 &= \mathbf{102030405060708090807060504030201}\,0001^2
\end{aligned} \tag{2.137}$$

Remark 6. *Extending the above study for two variables, the first case lead us to 9 values, the second case lead us to 99 values and the third case lead us to 999 values. This has been done separately in Taneja [7].*

▶ Third-Type: Part 1

- **For $m = 100, 100100, 100100100, 100100100100, \ldots$ in (6.11): or**
- **For $m = 100/2, 100100/2, 100100100/2, 100100100100/2, \ldots$ in (6.13): or**
- **For $m = 100/4, 100100/4, 100100100/4, 100100100100/4, \ldots$ in (2.117):**

$$\begin{aligned}
0009999^2 + 200^2 &= 1\,0001^2\\
\mathbf{1002}\,0009999^2 + 200200^2 &= \mathbf{1002001}\,0001^2\\
\mathbf{1002003002}\,0009999^2 + 200200200^2 &= \mathbf{1002003002001}\,0001^2\\
\mathbf{1002003004003002}\,0009999^2 + 200200200200^2 &= \mathbf{1002003004003002001}\,0001^2\\
\mathbf{1002003004005004003002}\,0009999^2 + 200200200200200^2 &= \mathbf{1002003004005004003002001}\,0001^2\\
\mathbf{1002003004005006005004003002}\,0009999^2 + 200200200200200200^2 &= \mathbf{1002003004005006005004003002001}\,0001^2\\
\mathbf{1002003004005006007006005004003002}\,0009999^2 + 200200200200200200200^2 &= \mathbf{1002003004005006007006005004003002001}\,0001^2\\
\mathbf{1002003004005006007008007006005004003002}\,0009999^2 + 200200200200200200200200^2 &= \mathbf{1002003004005006007008007006005004003002001}\,0001^2\\
\mathbf{1002003004005006007008009008007006005004003002}\,0009999^2 + 200200200200200200200200200^2 &= \mathbf{1002003004005006007008009008007006005004003002001}\,0001^2
\end{aligned} \tag{2.138}$$

▶ Third-Type: Part 2

- **For $m = 500, 500500, 500500500, 500500500500, \ldots$ in (6.13): or**
- **For $m = 250, 250250, 250250250, 250250250250, \ldots$ in (2.117):**

$$\begin{aligned}
000999999^2 + 2000^2 &= 1\,000001^2\\
\mathbf{1002}\,000999999^2 + 2002000^2 &= \mathbf{1002001}\,000001^2\\
\mathbf{1002003002}\,000999999^2 + 2002002000^2 &= \mathbf{1002003002001}\,000001^2\\
\mathbf{1002003004003002}\,000999999^2 + 2002002002000^2 &= \mathbf{1002003004003002001}\,000001^2\\
\mathbf{1002003004005004003002}\,000999999^2 + 2002002002002000^2 &= \mathbf{1002003004005004003002001}\,000001^2\\
\mathbf{1002003004005006005004003002}\,000999999^2 + 2002002002002002000^2 &= \mathbf{1002003004005006005004003002001}\,000001^2\\
\mathbf{1002003004005006007006005004003002}\,000999999^2 + 2002002002002002002000^2 &= \mathbf{1002003004005006007006005004003002001}\,000001^2\\
\mathbf{1002003004005006007008007006005004003002}\,000999999^2 + 2002002002002002002002000^2 &= \mathbf{1002003004005006007008007006005004003002001}\,000001^2\\
\mathbf{1002003004005006007008009008007006005004003002}\,000999999^2 + 2002002002002002002002002000^2 &= \mathbf{1002003004005006007008009008007006005004003002001}\,000001^2
\end{aligned} \tag{2.139}$$

2.7 Remarks

(i) This section brings five procedures to get **patterns in Pythagorean triples**. In some cases, they give same results depending on the variations of variables. From the section 2.6, we observe that the **pandigital palindromic-type patterns** can be obtained from the Procedures 3, 4 or 5 by considering different values of the variables. This study is extended for two variables in [6, 7], where the **pandigital palindromic-type patterns** are obtained with more possibilities. Moreover, we observe that the results appearing in section 2.6 are based on the following pattern:

$$\begin{aligned}
99^2 + 20^2 &= 101^2 &&:= 10201\\
9999^2 + 200^2 &= 10001^2 &&:= 100020001\\
999999^2 + 2000^2 &= 1000001^2 &&:= 1000002000001\\
99999999^2 + 20000^2 &= 100000001^2 &&:= 10000000200000001\\
9999999999^2 + 200000^2 &= 10000000001^2 &&:= 100000000020000000001\\
999999999999^2 + 2000000^2 &= 1000000000001^2 &&:= 1000000000002000000000001
\end{aligned}$$

This is the same pattern as given in (2.55). It is good with final sum too, but some of the results given in section 2.6 aren't good with final sums. Moreover, this is also good for magic squares with perfect square sum [3]

(ii) The Procedure 5 can be extended further to Procedure 6 given by

$$\begin{aligned}
f_6(m) &:= 64m^2 - 1\\
g_6(m) &:= 16m\\
h_6(m) &:= 64m^2 + 1
\end{aligned} \tag{2.140}$$

There is not very much advantage in extending it as the same results can be obtained from the procedures Procedures 3, 4 or 5.

3 Two Variable Procedures

3.1 Procedure 1

For all $m, n \in N_+$, let's consider the following three functions:

$$\begin{aligned}
F_1(m,n) &:= m(2n+m)\\
G_1(m,n) &:= 2n(n+m)\\
H_1(m,n) &:= m^2 + 2mn + 2n^2
\end{aligned} \tag{3.1}$$

Then we can easily check that

$$\begin{aligned}
F_1(m,n)^2 + G_1(m,n)^2 &= (m(2n+m))^2 + (2n(n+m))^2\\
&= m^4 + 4m^3n + 4m^2n^2 + 4m^2n^2 + 8mn^3 + 4n^4\\
&= m^4 + 4m^3n + 8m^2n^2 + 8mn^3 + 4n^4\\
&= \left(m^2 + 2mn + 2n^2\right)^2 = H_1(m,n)^2.
\end{aligned}$$

This proves that the triple (F_1, G_1, H_1) is a **Pythagorean triple** for all $m, n \in N_+$. In particular, we have

$$\begin{aligned}(F_1(1,n), G_1(1,n), H_1(1,n)) &= (f_1(n), g_1(n), h_1(n))\\ (F_1(m,1), G_1(m,1), H_1(m,1)) &= (f_2(m), g_2(m), h_2(m))\end{aligned}$$

where

$$\begin{aligned}f_1(n) &:= 2n+1\\ g_1(n) &:= 2n(n+1)\\ h_1(n) &:= 2n^2+2n+1\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}f_2(m) &:= m(m+2)\\ g_2(m) &:= 2(m+1)\\ h_2(m) &:= m^2+2m+2\end{aligned} \tag{3.3}$$

The Procedures (3.2) and (3.3) are two single variable procedures studied in [5].

Some particular cases Procedure (3.1 are as follows:

$$\begin{aligned}
(F_1(1,2), G_1(1,2), H_1(1,2)) &\Rightarrow (5,12,13)\\
(F_1(3,1), G_1(3,1), H_1(3,1)) &\Rightarrow (5,8,17)\\
(F_1(3,2), G_1(3,2), H_1(3,2)) &\Rightarrow (21,20,29)\\
(F_1(2,5), G_1(2,5), H_1(2,5)) &\Rightarrow (24,70,74)\\
(F_1(3,4), G_1(3,4), H_1(3,4)) &\Rightarrow (33,56,65)\\
(F_1(2,6), G_1(2,6), H_1(2,6)) &\Rightarrow (28,96,100)\\
(F_1(5,5), G_1(5,5), H_1(5,5)) &\Rightarrow (75,100,125)\\
(F_1(3,7), G_1(3,7), H_1(3,7)) &\Rightarrow (51,140,149)\\
(F_1(6,7), G_1(6,7), H_1(6,7)) &\Rightarrow (120,182,218)
\end{aligned}$$

Some patterned examples of this Procedure are given below. There the options of fixing m or n are considered.

3.1.1 Examples

This subsection brings patterns based on Procedure (3.1) given in subsection 2.1. Below are examples based on particular values of m and n. In this procedure we have facilities to fix one and vary other.

- **For $n = 1$; $m = 30, 300, 3000, 30000, ...$ in (3.1):**

$$\begin{aligned}
960^2 + 62^2 &= 962^2\\
90600^2 + 602^2 &= 90602^2 &&:= 8208722404\\
9006000^2 + 6002^2 &= 9006002^2 &&:= 81108072024004\\
900060000^2 + 60002^2 &= 900060002^2 &&:= 810108007200240004\\
90000600000^2 + 600002^2 &= 90000600002^2 &&:= 8100108000720002400004\\
9000006000000^2 + 6000002^2 &= 9000006000002^2 &&:= 81000108000072000024000004
\end{aligned} \tag{3.4}$$

The first triple (15, 8, 17) for $n = 1; m = 3$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
480^2 + 31^2 &= 481^2 \\
45300^2 + 301^2 &= 45301^2 \\
4503000^2 + 3001^2 &= 4503001^2 \\
450030000^2 + 30001^2 &= 450030001^2 && := 202527001800060001 \\
45000300000^2 + 300001^2 &= 45000300001^2 && := 2025027000180000600001 \\
4500003000000^2 + 3000001^2 &= 4500003000001^2 && := 20250027000018000006000001
\end{aligned} \tag{3.5}$$

- **For $n = 2;\ m = 500, 5000, 50000, ...$ in (3.1):**

$$\begin{aligned}
252000^2 + 2008^2 &= 252008^2 \\
25020000^2 + 20008^2 &= 25020008^2 \\
2500200000^2 + 200008^2 &= 2500200008^2 && := 6251000080003200064 \\
250002000000^2 + 2000008^2 &= 250002000008^2 && := 62501000008000032000064 \\
25000020000000^2 + 20000008^2 &= 25000020000008^2 && := 625001000000800000320000064
\end{aligned} \tag{3.6}$$

The first two triples (45, 28, 53) and (2700, 208, 2708) for $n = 2; m = 5$ and, 50 are not written above as they don't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
126000^2 + 1004^2 &= 126004^2 \\
12510000^2 + 10004^2 &= 12510004^2 \\
1250100000^2 + 100004^2 &= 1250100004^2 \\
125001000000^2 + 1000004^2 &= 125001000004^2 && := 15625250002000008000016 \\
12500010000000^2 + 10000004^2 &= 12500010000004^2 && := 156250250000200000080000016
\end{aligned} \tag{3.7}$$

▶ **Division by 4**

$$\begin{aligned}
63000^2 + 502^2 &= 63002^2 \\
6255000^2 + 5002^2 &= 6255002^2 \\
625050000^2 + 50002^2 &= 625050002^2 \\
62500500000^2 + 500002^2 &= 62500500002^2 \\
6250005000000^2 + 5000002^2 &= 6250005000002^2
\end{aligned} \tag{3.8}$$

▶ **Division by 8**

$$\begin{aligned}
31500^2 + 251^2 &= 31501^2 \\
3127500^2 + 2501^2 &= 3127501^2 \\
312525000^2 + 25001^2 &= 312525001^2 \\
31250250000^2 + 250001^2 &= 31250250001^2 \\
3125002500000^2 + 2500001^2 &= 3125002500001^2
\end{aligned} \tag{3.9}$$

- **For $n = 3$; $m = 500, 5000, 50000, ...$ in (3.1):**

$$\begin{aligned}
253000^2 + 3018^2 &= 253018^2 \\
25030000^2 + 30018^2 &= 25030018^2 &&:= 626501801080324 \\
2500300000^2 + 300018^2 &= 2500300018^2 &&:= 6251500180010800324 \\
250003000000^2 + 3000018^2 &= 250003000018^2 &&:= 62501500018000108000324 \\
25000030000000^2 + 30000018^2 &= 25000030000018^2 &&:= 625001500001800001080000324
\end{aligned} \tag{3.10}$$

The first two triples (55, 48, 73) and (2800, 318, 2818) for $n = 3; m = 5$ and, 50 are not written above as they don't give good pattern.

- ▶ **Division by 2**

$$\begin{aligned}
126500^2 + 1509^2 &= 126509^2 \\
12515000^2 + 15009^2 &= 12515009^2 \\
1250150000^2 + 150009^2 &= 1250150009^2 \\
125001500000^2 + 1500009^2 &= 125001500009^2 &&:= 15625375004500027000081 \\
12500015000000^2 + 15000009^2 &= 12500015000009^2 &&:= 156250375000450000270000081
\end{aligned} \tag{3.11}$$

- **For $n = 3$; $m = 8, 80, 800, 8000, 80000, ...$ in (3.1):**

$$\begin{aligned}
644800^2 + 4818^2 &= 644818^2 \\
64048000^2 + 48018^2 &= 64048018^2 \\
6400480000^2 + 480018^2 &= 6400480018^2 &&:= 40966144460817280324 \\
640004800000^2 + 4800018^2 &= 640004800018^2 &&:= 409606144046080172800324 \\
64000048000000^2 + 48000018^2 &= 64000048000018^2 &&:= 4096006144004608001728000324
\end{aligned} \tag{3.12}$$

The first two triples (112, 66, 130) and (6880, 498, 6898) for $n = 3; m = 8$ and, 80 are not written above as they don't give good pattern.

- ▶ **Division by 2**

$$\begin{aligned}
322400^2 + 2409^2 &= 322409^2 \\
32024000^2 + 24009^2 &= 32024009^2 \\
3200240000^2 + 240009^2 &= 3200240009^2 &&:= 10241536115204320081 \\
320002400000^2 + 2400009^2 &= 320002400009^2 &&:= 102401536011520043200081 \\
32000024000000^2 + 24000009^2 &= 32000024000009^2 &&:= 1024001536001152000432000081
\end{aligned} \tag{3.13}$$

- **For $n = 7$; $m = 900, 9000, 90000, ...$ in (3.1):**

$$\begin{aligned}
822600^2 + 12698^2 &= 822698^2\\
81126000^2 + 126098^2 &= 81126098^2\\
8101260000^2 + 1260098^2 &= 8101260098^2\\
810012600000^2 + 12600098^2 &= 810012600098^2 := 656120412317522469609604\\
81000126000000^2 + 126000098^2 &= 81000126000098^2 := 6561020412031752024696009604
\end{aligned} \tag{3.14}$$

The first two triples $(207, 224, 305)$ and $(9360, 1358, 9458)$ for $n = 7; m = 9$ and, 90 are not written above as they don't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
40563000^2 + 63049^2 &= 40563049^2\\
4050630000^2 + 630049^2 &= 4050630049^2\\
405006300000^2 + 6300049^2 &= 405006300049^2\\
40500063000000^2 + 63000049^2 &= 40500063000049^2
\end{aligned} \tag{3.15}$$

The first triple $(411300, 6349, 411349)$ is not written above as it doesn't give good pattern.

- **For $n = 7$; $m = 600, 6000, 60000, ...$ in (3.1):**

$$\begin{aligned}
368400^2 + 8498^2 &= 368498^2\\
36084000^2 + 84098^2 &= 36084098^2\\
3600840000^2 + 840098^2 &= 3600840098^2\\
360008400000^2 + 8400098^2 &= 360008400098^2 := 129606048141121646409604\\
36000084000000^2 + 84000098^2 &= 36000084000098^2 := 1296006048014112016464009604
\end{aligned} \tag{3.16}$$

The first two triples $(120, 182, 218)$ and $(4440, 938, 4538)$ for $n = 7; m = 6$ and, 60 are not written above as they don't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
184200^2 + 4249^2 &= 184249^2\\
18042000^2 + 42049^2 &= 18042049^2\\
1800420000^2 + 420049^2 &= 1800420049^2\\
180004200000^2 + 4200049^2 &= 180004200049^2 := 32401512035280411602401\\
18000042000000^2 + 42000049^2 &= 18000042000049^2 := 324001512003528004116002401
\end{aligned} \tag{3.17}$$

- **For $n = 7;\ m = 666, 6666, 66666, ...$ in (3.1):**

$$\begin{aligned}
452880^2 + 9422^2 &= 452978^2 \\
44528880^2 + 93422^2 &= 44528978^2 \\
4445288880^2 + 933422^2 &= 4445288978^2 \\
444452888880^2 + 9333422^2 &= 444452888978^2 \\
44444528888880^2 + 93333422^2 &= 44444528888978^2
\end{aligned} \tag{3.18}$$

The first two triples (120, 182, 218) and (5280, 1022, 5378) for $n = 7; m = 6$ and, 66 are not written above as they don't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
226440^2 + 4711^2 &= 226489^2 \\
22264440^2 + 46711^2 &= 22264489^2 \\
2222644440^2 + 466711^2 &= 2222644489^2 \\
222226444440^2 + 4666711^2 &= 222226444489^2 \\
22222264444440^2 + 46666711^2 &= 22222264444489^2
\end{aligned} \tag{3.19}$$

- **For $n = 5;\ m = 303, 3003, 30003, ...$ in (3.1):**

$$\begin{aligned}
94839^2 + 3080^2 &= 94889^2 \\
9048039^2 + 30080^2 &= 9048089^2 \\
900480039^2 + 300080^2 &= 900480089^2 &:= 810864390685447921 \\
90004800039^2 + 3000080^2 &= 90004800089^2 &:= 8100864039060854407921 \\
9000048000039^2 + 30000080^2 &= 9000048000089^2 &:= 81000864003906008544007921
\end{aligned} \tag{3.20}$$

The first two triples (39, 80, 89) and (1419, 380, 1469) for $n = 5; m = 3$ and, 33 are not written above as they don't give good pattern.

- **For $n = 7;\ m = 6006, 60006, 600006, ...$ in (3.1):**

$$\begin{aligned}
36156120^2 + 84182^2 &= 36156218^2 \\
3601560120^2 + 840182^2 &= 3601560218^2 \\
360015600120^2 + 8400182^2 &= 360015600218^2 &:= 129611232400326801647524 \\
36000156000120^2 + 84000182^2 &= 36000156000218^2 &:= 1296011232040032068016047524
\end{aligned} \tag{3.21}$$

The first three triples (120, 182, 218), (5280, 1022, 5378) and (375720, 8582, 375818) for $n = 7; m = 6, 66,$ and, 606 are not written above as they don't give good pattern.

▶ **Division by 2**

$$
\begin{aligned}
18078060^2 + 42091^2 &= 18078109^2 \\
1800780060^2 + 420091^2 &= 1800780109^2 \\
180007800060^2 + 4200091^2 &= 180007800109^2 := 32402808100081700411881 \\
18000078000060^2 + 42000091^2 &= 18000078000109^2 := 3240028080100080170004011881
\end{aligned} \tag{3.22}
$$

- **For $n = 8;\ m = 9009, 90009, 900009, \ldots$ in (3.1):**

$$
\begin{aligned}
81306225^2 + 144272^2 &= 81306353^2 \\
8103060225^2 + 1440272^2 &= 8103060353^2 \\
810030600225^2 + 14400272^2 &= 810030600353^2 := 656149573508241603724609 \\
81000306000225^2 + 144000272^2 &= 81000306000353^2 := 6561049572150822216036124609
\end{aligned} \tag{3.23}
$$

The first three triples $(225, 272, 353)$, $(11385, 1712, 11513)$ and $(840825, 14672, 840953)$ for $n = 8; m = 9, 99,$ and, 909 are not written above as they don't give good pattern.

- **For $n = 2;\ m = 33, 333, 3333, 33333, \ldots$ in (3.1):**

$$
\begin{aligned}
1221^2 + 140^2 &= 1229^2 \\
112221^2 + 1340^2 &= 112229^2 \\
11122221^2 + 13340^2 &= 11122229^2 \\
1111222221^2 + 133340^2 &= 1111222229^2 \\
111112222221^2 + 1333340^2 &= 111112222229^2 \\
11111122222221^2 + 13333340^2 &= 11111122222229^2
\end{aligned} \tag{3.24}
$$

The first triple $(21, 20, 29)$ for $n = 2; m = 3$ is not written above as it doesn't give good pattern.

- **For $n = 5;\ m = 33, 333, 3333, 33333, \ldots$ in (3.1):**

$$
\begin{aligned}
1419^2 + 380^2 &= 1469^2 \\
114219^2 + 3380^2 &= 114269^2 \\
11142219^2 + 33380^2 &= 11142269^2 \\
1111422219^2 + 333380^2 &= 1111422269^2 \\
111114222219^2 + 3333380^2 &= 111114222269^2 \\
11111142222219^2 + 33333380^2 &= 11111142222269^2
\end{aligned} \tag{3.25}
$$

The first triple $(39, 80, 89)$ for $n = 5; m = 3$ is not written above as it doesn't give good pattern.

- **For $n = 7;\ m = 333, 3333, 33333, \ldots$ in (3.1):**

$$
\begin{aligned}
115551^2 + 4760^2 &= 115649^2 \\
11155551^2 + 46760^2 &= 11155649^2 \\
1111555551^2 + 466760^2 &= 1111555649^2 \\
111115555551^2 + 4666760^2 &= 111115555649^2 \\
11111155555551^2 + 46666760^2 &= 11111155555649^2
\end{aligned} \tag{3.26}
$$

The first two triples (51, 140, 149) and (1551, 560, 1649) for $n = 5$; $m = 3$ and $m = 33$ are not written above as they don't give good pattern.

- **For $n = 8$; $m = 333, 3333, 33333, \ldots$ in (3.1):**

$$
\begin{aligned}
116217^2 + 5456^2 &= 116345^2 \\
11162217^2 + 53456^2 &= 11162345^2 \\
1111622217^2 + 533456^2 &= 1111622345^2 \\
111116222217^2 + 5333456^2 &= 111116222345^2 \\
11111162222217^2 + 53333456^2 &= 11111162222345^2
\end{aligned} \tag{3.27}
$$

The first two triples (57, 176, 185) and (1617, 656, 1745) for $n = 8$; $m = 3$ and $m = 33$ are not written above as they don't give good pattern.

- **For $n = 4$; $m = 99, 999, 9999, 99999, \ldots$ in (3.1):**

$$
\begin{aligned}
10593^2 + 824^2 &= 10625^2 \\
1005993^2 + 8024^2 &= 1006025^2 &&:= 1012086300625 \\
100059993^2 + 80024^2 &= 100060025^2 &&:= 10012008603000625 \\
10000599993^2 + 800024^2 &= 10000600025^2 &&:= 100012000860030000625 \\
1000005999993^2 + 8000024^2 &= 1000006000025^2 &&:= 1000012000086000300000625 \\
100000059999993^2 + 80000024^2 &= 100000060000025^2 &&:= 10000012000008600003000000625
\end{aligned} \tag{3.28}
$$

The first triple (153, 104, 185) for $n = 4$; $m = 9$ is not written above as it doesn't give good pattern.

- **For $n = 5$; $m = 9, 99, 999, 9999, 99999, \ldots$ in (3.1):**

$$
\begin{aligned}
10791^2 + 1040^2 &= 10841^2 \\
1007991^2 + 10040^2 &= 1008041^2 &&:= 1016146657681 \\
100079991^2 + 100040^2 &= 100080041^2 &&:= 10016014606561681 \\
10000799991^2 + 1000040^2 &= 10000800041^2 &&:= 100016001460065601681 \\
1000007999991^2 + 10000040^2 &= 1000008000041^2 &&:= 1000016000146000656001681 \\
100000079999991^2 + 100000040^2 &= 100000080000041^2 &&:= 10000016000014600006560001681
\end{aligned} \tag{3.29}
$$

The first triple (171, 140, 221) for $n = 5$; $m = 9$ is not written above as it doesn't give good pattern.

- **For $n = 9$; $m = 99, 999, 9999, 99999, ...$ in (3.1):**

$$\begin{aligned}
1015983^2 + 18144^2 &= 1016145^2 &&:= 1032550661025\\
100159983^2 + 180144^2 &= 100160145^2 &&:= 10032054646421025\\
10001599983^2 + 1800144^2 &= 10001600145^2 &&:= 100032005460464021025\\
1000015999983^2 + 18000144^2 &= 1000016000145^2 &&:= 1000032000546004640021025\\
100000159999983^2 + 180000144^2 &= 100000160000145^2 &&:= 10000032000054600046400021025
\end{aligned} \quad (3.30)$$

The first two triples $(243, 324, 405)$ and $(11583, 1944, 11745)$ for $n = 9; m = 9$ and $m = 99$ are not written above as they don't give good pattern.

- **For $m = 2$; $n = 3, 33, 333, 3333, 33333, ...$ in (3.1):**

$$\begin{aligned}
16^2 + 30^2 &= 34^2\\
136^2 + 2310^2 &= 2314^2\\
1336^2 + 223110^2 &= 223114^2\\
13336^2 + 22231110^2 &= 22231114^2\\
133336^2 + 2222311110^2 &= 2222311114^2\\
1333336^2 + 222223111110^2 &= 222223111114^2\\
13333336^2 + 22222231111110^2 &= 22222231111114^2
\end{aligned} \quad (3.31)$$

▶ **Division by 2**

$$\begin{aligned}
8^2 + 15^2 &= 17^2\\
68^2 + 1155^2 &= 1157^2\\
668^2 + 111555,^2 &= 111557^2\\
6668^2 + 11115555^2 &= 11115557^2\\
66668^2 + 1111155555^2 &= 1111155557^2\\
666668^2 + 111111555555^2 &= 111111555557^2\\
6666668^2 + 11111115555555^2 &= 11111115555557^2
\end{aligned} \quad (3.32)$$

- **For $m = 3$; $n = 33, 333, 3333, 33333, ...$ in (3.1):**

$$\begin{aligned}
207^2 + 2376^2 &= 2385^2\\
2007^2 + 223776^2 &= 223785^2\\
20007^2 + 22237776^2 &= 22237785^2\\
200007^2 + 2222377776^2 &= 2222377785^2\\
2000007^2 + 222223777776^2 &= 222223777785^2\\
20000007^2 + 22222237777776^2 &= 22222237777785^2
\end{aligned} \quad (3.33)$$

The first triple $(27, 36, 45)$ for $m = 3; n = 3$ is not written above as it doesn't give good pattern.

- **For $m = 4$; $n = 33, 333, 3333, 33333, ...$ in (3.1):**

$$\begin{aligned}
280^2 + 2442^2 &= 2458^2 \\
2680^2 + 224442^2 &= 224458^2 \\
26680^2 + 22244442^2 &= 22244458^2 \\
266680^2 + 2222444442^2 &= 2222444458^2 \\
2666680^2 + 222224444442^2 &= 222224444458^2 \\
26666680^2 + 22222244444442^2 &= 22222244444458^2
\end{aligned} \tag{3.34}$$

The first triple $(40, 42, 58)$ for $m = 4; n = 3$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
140^2 + 1221^2 &= 1229^2 \\
1340^2 + 112221^2 &= 112229^2 \\
13340^2 + 11122221^2 &= 11122229^2 \\
133340^2 + 1111222221^2 &= 1111222229^2 \\
1333340^2 + 111112222221^2 &= 111112222229^2 \\
13333340^2 + 11111122222221^2 &= 11111122222229^2
\end{aligned} \tag{3.35}$$

- **For $m = 5$; $n = 33, 333, 3333, 33333, ...$ in (3.1):**

$$\begin{aligned}
355^2 + 2508^2 &= 2533^2 \\
3355^2 + 225108^2 &= 225133^2 \\
33355^2 + 22251108^2 &= 22251133^2 \\
333355^2 + 2222511108^2 &= 2222511133^2 \\
3333355^2 + 222225111108^2 &= 222225111133^2 \\
33333355^2 + 22222251111108^2 &= 22222251111133^2
\end{aligned} \tag{3.36}$$

The first triple $(55, 48, 73)$ for $m = 5; n = 3$ is not written above as it doesn't give good pattern.

- **For $m = 2$; $n = 66, 666, 6666, 66666, ...$ in (3.1):**

$$\begin{aligned}
268^2 + 8976^2 &= 8980^2 \\
2668^2 + 889776^2 &= 889780^2 \\
26668^2 + 88897776^2 &= 88897780^2 \\
266668^2 + 8888977776^2 &= 8888977780^2 \\
2666668^2 + 888889777776^2 &= 888889777780^2 \\
26666668^2 + 88888897777776^2 &= 88888897777780^2
\end{aligned} \tag{3.37}$$

The first triple $(28, 96, 100)$ for $m = 2; n = 6$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
134^2 + 4488^2 &= 4490^2\\
1334^2 + 444888^2 &= 444890^2\\
13334^2 + 44448888^2 &= 44448890^2\\
133334^2 + 4444488888^2 &= 4444488890^2\\
1333334^2 + 444444888888^2 &= 444444888890^2\\
13333334^2 + 44444448888888^2 &= 44444448888890^2
\end{aligned} \tag{3.38}$$

- **For $m = 5$; $n = 66, 666, 6666, 66666, ...$ in (3.1):**

$$\begin{aligned}
685^2 + 9372^2 &= 9397^2\\
6685^2 + 893772^2 &= 893797^2\\
66685^2 + 88937772^2 &= 88937797^2\\
666685^2 + 8889377772^2 &= 8889377797^2\\
6666685^2 + 888893777772^2 &= 888893777797^2\\
66666685^2 + 88888937777772^2 &= 88888937777797^2
\end{aligned} \tag{3.39}$$

The first triple (85, 132, 157) for $m = 5; n = 6$ is not written above as it doesn't give good pattern.

- **For $m = 6$; $n = 66, 666, 6666, 66666, ...$ in (3.1):**

$$\begin{aligned}
828^2 + 9504^2 &= 9540^2\\
8028^2 + 895104^2 &= 895140^2\\
80028^2 + 88951104^2 &= 88951140^2\\
800028^2 + 8889511104^2 &= 8889511140^2\\
8000028^2 + 888895111104^2 &= 888895111140^2\\
80000028^2 + 88888951111104^2 &= 88888951111140^2
\end{aligned} \tag{3.40}$$

The first triple (108, 144, 180) for $m = 6; n = 6$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
414^2 + 4752^2 &= 4770^2\\
4014^2 + 447552^2 &= 447570^2\\
40014^2 + 44475552^2 &= 44475570^2\\
400014^2 + 4444755552^2 &= 4444755570^2\\
4000014^2 + 444447555552^2 &= 444447555570^2\\
40000014^2 + 44444475555552^2 &= 44444475555570^2
\end{aligned} \tag{3.41}$$

▶ Division by 4

$$
\begin{aligned}
207^2 + 2376^2 &= 2385^2 \\
2007^2 + 223776^2 &= 223785^2 \\
20007^2 + 22237776^2 &= 22237785^2 \\
200007^2 + 2222377776^2 &= 2222377785^2 \\
2000007^2 + 222223777776^2 &= 222223777785^2 \\
20000007^2 + 22222237777776^2 &= 22222237777785^2
\end{aligned} \tag{3.42}
$$

- **For $m = 5$; $n = 99, 999, 9999, 99999, ...$ in (3.1):**

$$
\begin{aligned}
1015^2 + 20592^2 &= 20617^2 \\
10015^2 + 2005992^2 &= 2006017^2 && := 4024104204289 \\
100015^2 + 200059992^2 &= 200060017^2 && := 40024010402040289 \\
1000015^2 + 20000599992^2 &= 20000600017^2 && := 400024001040020400289 \\
10000015^2 + 2000005999992^2 &= 2000006000017^2 && := 4000024000104000204000289 \\
100000015^2 + 200000059999992^2 &= 200000060000017^2 && := 40000024000010400002040000289
\end{aligned} \tag{3.43}
$$

The first triple (115, 252, 277) for $m = 5; n = 9$ is not written above as it doesn't give good pattern.

- **For $m = 6$; $n = 99, 999, 9999, 99999, ...$ in (3.1):**

$$
\begin{aligned}
1224^2 + 20790^2 &= 20826^2 \\
12024^2 + 2007990^2 &= 2008026^2 && := 4032168416676 \\
120024^2 + 200079990^2 &= 200080026^2 && := 40032016804160676 \\
1200024^2 + 20000799990^2 &= 20000800026^2 && := 400032001680041600676 \\
12000024^2 + 2000007999990^2 &= 2000008000026^2 && := 4000032000168000416000676 \\
120000024^2 + 200000079999990^2 &= 200000080000026^2 && := 40000032000016800004160000676
\end{aligned} \tag{3.44}
$$

The first triple (144, 270, 306) for $m = 6; n = 9$ is not written above as it doesn't give good pattern.

▶ Division by 2

$$
\begin{aligned}
612^2 + 10395^2 &= 10413^2 \\
6012^2 + 1003995^2 &= 1004013^2 && := 1008042104169 \\
60012^2 + 100039995^2 &= 100040013^2 && := 10008004201040169 \\
600012^2 + 10000399995^2 &= 10000400013^2 && := 100008000420010400169 \\
6000012^2 + 1000003999995^2 &= 1000004000013^2 && := 1000008000042000104000169 \\
60000012^2 + 100000039999995^2 &= 100000040000013^2 && := 10000008000004200001040000169
\end{aligned} \tag{3.45}
$$

- **For $m = 9$; $n = 99, 999, 9999, 99999, \ldots$ in (3.1):**

$$\begin{aligned}
1863^2 + 21384^2 &= 21465^2 \\
18063^2 + 2013984^2 &= 2014065^2 \\
180063^2 + 200139984^2 &= 200140065^2 &&:= 40056045618204225 \\
1800063^2 + 20001399984^2 &= 20001400065^2 &&:= 4000560045601820042250 \\
18000063^2 + 2000013999984^2 &= 2000014000065^2 &&:= 4000056000456001820004225 \\
180000063^2 + 200000139999984^2 &= 200000140000065^2 &&:= 400000560000456000182000004225
\end{aligned} \tag{3.46}$$

The first triple $(243, 324, 405)$ for $m = 9; n = 9$ is not written above as it doesn't give good pattern.

3.2 Procedure 2

For all $m, n \in N_+$, $m > n \geq 1$, let's consider the following three functions:

$$\begin{aligned}
F_2(m, n) &:= m^2 - n^2 \\
G_2(m, n) &:= 2mn \\
H_2(m, n) &:= m^2 + n^2
\end{aligned} \tag{3.47}$$

Then we can easily check that

$$\begin{aligned}
F_2(m, n)^2 + G_2(m, n)^2 &= \left(m^2 - n^2\right)^2 + (2mn)^2 \\
&= m^4 - 2m^2n^2 + n^4 + 4m^2n^2 \\
&= m^4 + 2m^2n^2 + n^4 \\
&= \left(m^2 + n^2\right)^2 = H_2(m, n)^2.
\end{aligned}$$

This proves that the triple (F_2, G_2, H_2) is a **Pythagorean triple** for all $m, n \in N_+$, $m > n \geq 1$.

In particular, when $n = 1$, we get $F_2(m, 1) = f_3(m)$, $G_2(m, 1) = g_3(m)$ and $H_2(m, 1) = h_3(m)$, i.e.,

$$\left(F_2(m, 1), G_2(m, 1), H_2(m, 1)\right) = \left(f_3(m), g_3(m), h_3(m)\right).$$

where

$$\begin{aligned}
f_3(m) &:= m^2 - 1 \\
g_3(m) &:= 2m \\
h_3(m) &:= m^2 + 1
\end{aligned} \tag{3.48}$$

The Procedure (3.48) is very famous in the literature [1]. Some patterned examples based on the Procedure (3.48) are studied by author [5]. Some particular cases of Procedure 2 are as follows:

$$\begin{aligned}
(F_2(3, 2), G_2(3, 2), H_2(3, 2)) &\Rightarrow (5, 12, 13) \\
(F_2(4, 2), G_2(4, 2), H_2(4, 2)) &\Rightarrow (12, 16, 20) \\
(F_2(5, 2), G_2(5, 2), H_2(5, 2)) &\Rightarrow (21, 20, 29) \\
(F_2(6, 2), G_2(6, 2), H_2(6, 2)) &\Rightarrow (32, 24, 40) \\
(F_2(9, 3), G_2(9, 3), H_2(9, 3)) &\Rightarrow (72, 54, 90) \\
(F_2(9, 7), G_2(9, 7), H_2(9, 7)) &\Rightarrow (32, 126, 130) \\
(F_2(10, 9), G_2(10, 9), H_2(10, 9)) &\Rightarrow (19, 180, 181)
\end{aligned}$$

Some patterned examples of this Procedure are given below.

3.2.1 Examples

This subsection brings patterns based on Procedure (3.47) given in subsection 2.2. See below some examples of patterns:

- **For $n = 3$; $m = 10, 00, 1000, 10000, ...$ in (3.47):**

$$\begin{aligned}
91^2 + 60^2 &= 109^2 &&:= 11881\\
9991^2 + 600^2 &= 10009^2 &&:= 100180081\\
999991^2 + 6000^2 &= 1000009^2 &&:= 1000018000081\\
99999991^2 + 60000^2 &= 100000009^2 &&:= 10000001800000081
\end{aligned} \tag{3.49}$$

- **For $n = 3$; $m = 20, 200, 2000, 20000, ...$ in (3.47):**

$$\begin{aligned}
391^2 + 120^2 &= 409^2 &&:= 167281\\
39991^2 + 1200^2 &= 40009^2 &&:= 1600720081\\
3999991^2 + 12000^2 &= 4000009^2 &&:= 16000072000081\\
399999991^2 + 120000^2 &= 400000009^2 &&:= 160000007200000081
\end{aligned} \tag{3.50}$$

- **For $n = 3$; $m = 50, 500, 5000, 50000, ...$ in (3.47):**

$$\begin{aligned}
2491^2 + 300^2 &= 2509^2 &&\\
249991^2 + 3000^2 &= 250009^2 &&:= 62504500081\\
24999991^2 + 30000^2 &= 25000009^2 &&:= 625000450000081\\
2499999991^2 + 300000^2 &= 2500000009^2 &&:= 6250000045000000081
\end{aligned} \tag{3.51}$$

- **For $n = 4$; $m = 50, 500, 5000, 50000, ...$ in (3.47):**

$$\begin{aligned}
2484^2 + 400^2 &= 2516^2 &&\\
249984^2 + 4000^2 &= 250016^2 &&:= 62508000256\\
24999984^2 + 40000^2 &= 25000016^2 &&:= 625000800000256\\
2499999984^2 + 400000^2 &= 2500000016^2 &&:= 6250000080000000256\\
249999999984^2 + 4000000^2 &= 250000000016^2 &&:= 62500000008000000000256\\
24999999999984^2 + 40000000^2 &= 25000000000016^2 &&:= 625000000000800000000000256
\end{aligned} \tag{3.52}$$

▶ **Division by 2**

$$\begin{aligned}
1242^2 + 200^2 &= 1258^2 \\
124992^2 + 2000^2 &= 125008^2 \\
12499992^2 + 20000^2 &= 12500008^2 := 156250200000064 \\
1249999992^2 + 200000^2 &= 1250000008^2 := 1562500020000000064 \\
124999999992^2 + 2000000^2 &= 125000000008^2 := 15625000002000000000064 \\
12499999999992^2 + 20000000^2 &= 12500000000008^2 := 156250000000200000000000064
\end{aligned} \tag{3.53}$$

- **Division by 4**

$$\begin{aligned}
62496^2 + 1000^2 &= 62504^2 \\
6249996^2 + 10000^2 &= 6250004^2 := 39062550000016 \\
624999996^2 + 100000^2 &= 625000004^2 := 390625005000000016 \\
62499999996^2 + 1000000^2 &= 62500000004^2 := 3906250000500000000016 \\
6249999999996^2 + 10000000^2 &= 6250000000004^2 := 39062500000050000000000016
\end{aligned} \tag{3.54}$$

Division of first term is excluded as it doesn't give good pattern.

- **Division by 8**

$$\begin{aligned}
31248^2 + 500^2 &= 31252^2 \\
3124998^2 + 5000^2 &= 3125002^2 \\
312499998^2 + 50000^2 &= 312500002^2 := 97656251250000004 \\
31249999998^2 + 500000^2 &= 31250000002^2 := 976562500125000000004 \\
3124999999998^2 + 5000000^2 &= 3125000000002^2 := 9765625000012500000000004
\end{aligned} \tag{3.55}$$

- **Division by 16**

$$\begin{aligned}
1562499^2 + 2500^2 &= 1562501^2 \\
156249999^2 + 25000^2 &= 156250001^2 \\
15624999999^2 + 250000^2 &= 15625000001^2 := 244140625031250000001 \\
1562499999999^2 + 2500000^2 &= 1562500000001^2 := 2441406250003125000000001
\end{aligned} \tag{3.56}$$

We observe that that the first triple (15624, 250, 15626) doesn't give good pattern.

- **For $n = 3;\ m = 90, 900, 9000, 90000, ...$ in (3.47):**

$$\begin{aligned}
8091^2 + 540^2 &= 8109^2 \\
809991^2 + 5400^2 &= 810009^2 := 656114580081 \\
80999991^2 + 54000^2 &= 81000009^2 := 6561001458000081 \\
8099999991^2 + 540000^2 &= 8100000009^2 := 65610000145800000081
\end{aligned} \tag{3.57}$$

- **For $n = 4$; $m = 5, 55, 555, 5555, \ldots$ in (3.47):**

$$\begin{aligned}
9^2 + 40^2 &= 41^2 \\
3009^2 + 440^2 &= 3041^2 \\
308009^2 + 4440^2 &= 308041^2 \\
30858009^2 + 44440^2 &= 30858041^2
\end{aligned} \tag{3.58}$$

3.2.2 Special Examples

This subsection brings special examples of patterns by fixing $n = 1, 2, \ldots, 9$ and varying m in terms 3, 6 and 9.

(i) For $n = 1, 2, \ldots, 9$; $m = 3, 33, 333, 3333, 33333, \ldots$:

- **For $n = 1$; $m = 3, 33, 333, 3333, 33333, \ldots$ in (3.47):**

$$\begin{aligned}
8^2 + 6^2 &= 10^2 \\
1088^2 + 66^2 &= 1090^2 \\
110888^2 + 666^2 &= 110890^2 \\
11108888^2 + 6666^2 &= 11108890^2 \\
1111088888^2 + 66666^2 &= 1111088890^2 \\
111110888888^2 + 666666^2 &= 111110888890^2
\end{aligned} \tag{3.59}$$

In order to get primitive values, let's divide the above pattern by 2, we get a new pattern:

- ▶ **Division by 2**

$$\begin{aligned}
4^2 + 3^2 &= 5^2 \\
544^2 + 33^2 &= 545^2 \\
55444^2 + 333^2 &= 55445^2 \\
5554444^2 + 3333^2 &= 5554445^2 \\
555544444^2 + 33333^2 &= 555544445^2 \\
55555444444^2 + 333333^2 &= 55555444445^2
\end{aligned} \tag{3.60}$$

- **For $n = 2$; $m = 3, 33, 333, 3333, 33333, \ldots$ in (3.47):**

$$\begin{aligned}
5^2 + 12^2 &= 13^2 \\
1085^2 + 132^2 &= 1093^2 \\
110885^2 + 1332^2 &= 110893^2 \\
11108885^2 + 13332^2 &= 11108893^2 \\
1111088885^2 + 133332^2 &= 1111088893^2 \\
111110888885^2 + 1333332^2 &= 111110888893^2
\end{aligned} \tag{3.61}$$

- **For $n=3$; $m=33,333,3333,33333,...$ in (3.47):**

$$\begin{aligned} 1080^2+198^2 &= 1098^2 \\ 110880^2+1998^2 &= 110898^2 \\ 11108880^2+19998^2 &= 11108898^2 \\ 1111088880^2+199998^2 &= 1111088898^2 \\ 111110888880^2+1999998^2 &= 111110888898^2 \end{aligned} \tag{3.62}$$

There is no value for $n=3; m=3$. In order to get primitive values, let's divide the above pattern by 2, we get a new pattern.

▶ **Division by 2**

$$\begin{aligned} 540^2+99^2 &= 549^2 \\ 55440^2+999^2 &= 55449^2 \\ 5554440^2+9999^2 &= 5554449^2 \\ 555544440^2+99999^2 &= 555544449^2 \\ 55555444440^2+999999^2 &= 55555444449^2 \end{aligned} \tag{3.63}$$

- **For $n=4$; $m=33,333,3333,33333,...$ in (3.47):**

$$\begin{aligned} 1073^2+264^2 &= 1105^2 \\ 110873^2+2664^2 &= 110905^2 \\ 11108873^2+26664^2 &= 11108905^2 \\ 1111088873^2+266664^2 &= 1111088905^2 \\ 111110888873^2+2666664^2 &= 111110888905^2 \end{aligned} \tag{3.64}$$

The first triple $(7,24,25)$ for $n=4$; $m=3$ is not written above as it doesn't give good pattern.

- **For $n=5$; $m=33,333,3333,33333,...$ in (3.47):**

$$\begin{aligned} 1064^2+330^2 &= 1114^2 \\ 110864^2+3330^2 &= 110914^2 \\ 11108864^2+33330^2 &= 11108914^2 \\ 1111088864^2+333330^2 &= 1111088914^2 \\ 111110888864^2+3333330^2 &= 111110888914^2 \end{aligned} \tag{3.65}$$

The first triple $(16,30,34)$ for $n=5$; $m=3$ is not written above as it doesn't give good pattern. In order to get primitive values, let's divide the above pattern by 2, we get a new pattern.

▶ **Division by 2**

$$\begin{aligned}
532^2 + 165^2 &= 557^2\\
55432^2 + 1665^2 &= 55457^2\\
5554432^2 + 16665^2 &= 5554457^2\\
555544432^2 + 166665^2 &= 555544457^2\\
55555444432^2 + 1666665^2 &= 55555444457^2
\end{aligned} \tag{3.66}$$

- **For $n = 6;\ m = 33, 333, 3333, 33333, ...$ in (3.47):**

$$\begin{aligned}
1053^2 + 396^2 &= 1125^2\\
110853^2 + 3996^2 &= 110925^2\\
11108853^2 + 39996^2 &= 11108925^2\\
1111088853^2 + 399996^2 &= 1111088925^2\\
111110888853^2 + 3999996^2 &= 111110888925^2
\end{aligned} \tag{3.67}$$

The first triple (27, 36, 45) for $n = 6;\ m = 3$ is not written above as it doesn't give good pattern.

- **For $n = 7;\ m = 33, 333, 3333, 33333, ...$ in (3.47):**

$$\begin{aligned}
1040^2 + 462^2 &= 1138^2\\
110840^2 + 4662^2 &= 110938^2\\
11108840^2 + 46662^2 &= 11108938^2\\
1111088840^2 + 466662^2 &= 1111088938^2\\
111110888840^2 + 4666662^2 &= 111110888938^2
\end{aligned} \tag{3.68}$$

The first triple (40, 42, 58) for $n = 7;\ m = 3$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
520^2 + 231^2 &= 569^2\\
55420^2 + 2331^2 &= 55469^2\\
5554420^2 + 23331^2 &= 5554469^2\\
555544420^2 + 233331^2 &= 555544469^2\\
55555444420^2 + 2333331^2 &= 55555444469^2
\end{aligned} \tag{3.69}$$

- **For $n = 8;\ m = 33, 333, 3333, 33333, ...$ in (3.47):**

$$\begin{aligned}
1025^2 + 528^2 &= 1153^2\\
110825^2 + 5328^2 &= 110953^2\\
11108825^2 + 53328^2 &= 11108953^2\\
1111088825^2 + 533328^2 &= 1111088953^2\\
111110888825^2 + 5333328^2 &= 111110888953^2
\end{aligned} \tag{3.70}$$

The first triple (55, 48, 73) for $n = 8;\ m = 3$ is not written above as it doesn't give good pattern.

- **For $n = 9$; $m = 33, 333, 3333, 33333, ...$ in (3.47):**

$$\begin{aligned}
1008^2 + 594^2 &= 1170^2\\
110808^2 + 5994^2 &= 110970^2\\
11108808^2 + 59994^2 &= 11108970^2\\
1111088808^2 + 599994^2 &= 1111088970^2\\
111110888808^2 + 5999994^2 &= 111110888970^2
\end{aligned} \tag{3.71}$$

The first triple (72, 54, 90) for $n = 9$; $m = 3$ is not written above as it doesn't give good pattern.

► **Division by 2**

$$\begin{aligned}
504^2 + 297^2 &= 585^2\\
55404^2 + 2997^2 &= 55485^2\\
5554404^2 + 29997^2 &= 5554485^2\\
555544404^2 + 299997^2 &= 555544485^2\\
55555444404^2 + 2999997^2 &= 55555444485^2
\end{aligned} \tag{3.72}$$

(ii) For $n = 1, 2, ..., 9$; $m = 6, 66, 666, 6666, 66666, ...$:

- **For $n = 1$; $m = 6, 66, 666, 6666, 66666, ...$ in (3.47):**

$$\begin{aligned}
35^2 + 12^2 &= 37^2\\
4355^2 + 132^2 &= 4357^2\\
443555^2 + 1332^2 &= 443557^2\\
44435555^2 + 13332^2 &= 44435557^2\\
4444355555^2 + 133332^2 &= 4444355557^2\\
444443555555^2 + 1333332^2 &= 444443555557^2
\end{aligned} \tag{3.73}$$

- **For $n = 2$; $m = 6, 66, 666, 6666, 66666, ...$ in (3.47):**

$$\begin{aligned}
32^2 + 24^2 &= 40^2\\
4352^2 + 264^2 &= 4360^2\\
443552^2 + 2664^2 &= 443560^2\\
44435552^2 + 26664^2 &= 44435560^2\\
4444355552^2 + 266664^2 &= 4444355560^2\\
444443555552^2 + 2666664^2 &= 444443555560^2
\end{aligned} \tag{3.74}$$

► **Division by 2**

$$\begin{aligned}
16^2 + 12^2 &= 20^2\\
2176^2 + 132^2 &= 2180^2\\
221776^2 + 1332^2 &= 221780^2\\
22217776^2 + 13332^2 &= 22217780^2\\
2222177776^2 + 133332^2 &= 2222177780^2\\
222221777776^2 + 1333332^2 &= 222221777780^2
\end{aligned} \tag{3.75}$$

▶ **Division by 4**

$$\begin{aligned}
8^2 + 6^2 &= 10^2\\
1088^2 + 66^2 &= 1090^2\\
110888^2 + 666^2 &= 110890^2\\
11108888^2 + 6666^2 &= 11108890^2\\
1111088888^2 + 66666^2 &= 1111088890^2\\
111110888888^2 + 666666^2 &= 111110888890^2
\end{aligned} \tag{3.76}$$

▶ **Division by 8**

$$\begin{aligned}
4^2 + 3^2 &= 5^2\\
544^2 + 33^2 &= 545^2\\
55444^2 + 333^2 &= 55445^2\\
5554444^2 + 3333^2 &= 5554445^2\\
555544444^2 + 33333^2 &= 555544445^2\\
55555444444^2 + 333333^2 &= 55555444445^2
\end{aligned} \tag{3.77}$$

- **For $n = 3$; $m = 66, 666, 6666, 66666, \ldots$ in (3.47):**

$$\begin{aligned}
4347^2 + 396^2 &= 4365^2\\
443547^2 + 3996^2 &= 443565^2\\
44435547^2 + 39996^2 &= 44435565^2\\
4444355547^2 + 399996^2 &= 4444355565^2\\
444443555547^2 + 3999996^2 &= 444443555565^2
\end{aligned} \tag{3.78}$$

The first triple $(27, 36, 45)$ for $n = 3$; $m = 6$ is not written above as it doesn't give good pattern.

- **For $n = 4;\ m = 66, 666, 6666, 66666, ...$ in (3.47):**

$$\begin{aligned}
4340^2 + 528^2 &= 4372^2\\
443540^2 + 5328^2 &= 443572^2\\
44435540^2 + 53328^2 &= 44435572^2\\
4444355540^2 + 533328^2 &= 4444355572^2\\
444443555540^2 + 5333328^2 &= 444443555572^2
\end{aligned} \tag{3.79}$$

The first triple (20, 48, 52) for $n = 4;\ m = 6$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
10^2 + 24^2 &= 26^2\\
2170^2 + 264^2 &= 2186^2\\
221770^2 + 2664^2 &= 221786^2\\
22217770^2 + 26664^2 &= 22217786^2\\
2222177770^2 + 266664^2 &= 2222177786^2\\
222221777770^2 + 2666664^2 &= 222221777786^2
\end{aligned} \tag{3.80}$$

▶ **Division by 4**

$$\begin{aligned}
5^2 + 12^2 &= 13^2\\
1085^2 + 132^2 &= 1093^2\\
110885^2 + 1332^2 &= 110893^2\\
11108885^2 + 13332^2 &= 11108893^2\\
1111088885^2 + 133332^2 &= 1111088893^2\\
111110888885^2 + 1333332^2 &= 111110888893^2
\end{aligned} \tag{3.81}$$

- **For $n = 5;\ m = 6, 66, 666, 6666, 66666, ...$ in (3.47):**

$$\begin{aligned}
4331^2 + 660^2 &= 4381^2\\
443531^2 + 6660^2 &= 443581^2\\
44435531^2 + 66660^2 &= 44435581^2\\
4444355531^2 + 666660^2 &= 4444355581^2\\
444443555531^2 + 6666660^2 &= 444443555581^2
\end{aligned} \tag{3.82}$$

The first triple (11, 60, 61) for $n = 5;\ m = 6$ is not written above as it doesn't give good pattern.

- **For $n = 6$; $m = 6, 66, 666, 6666, 66666, ...$ in (3.47):**

$$\begin{aligned}
4320^2 + 792^2 &= 4392^2\\
443520^2 + 7992^2 &= 443592^2\\
44435520^2 + 79992^2 &= 44435592^2\\
4444355520^2 + 799992^2 &= 4444355592^2\\
444443555520^2 + 7999992^2 &= 444443555592^2
\end{aligned} \tag{3.83}$$

In this case, there is no value for $n = 6$; $m = 6$.

▶ **Division by 2**

$$\begin{aligned}
2160^2 + 396^2 &= 2196^2\\
221760^2 + 3996^2 &= 221796^2\\
22217760^2 + 39996^2 &= 22217796^2\\
2222177760^2 + 399996^2 &= 2222177796^2\\
222221777760^2 + 3999996^2 &= 222221777796^2
\end{aligned} \tag{3.84}$$

▶ **Division by 4**

$$\begin{aligned}
1080^2 + 198^2 &= 1098^2\\
110880^2 + 1998^2 &= 110898^2\\
11108880^2 + 19998^2 &= 11108898^2\\
1111088880^2 + 199998^2 &= 1111088898^2\\
111110888880^2 + 1999998^2 &= 111110888898^2
\end{aligned} \tag{3.85}$$

▶ **Division by 8**

$$\begin{aligned}
540^2 + 99^2 &= 549^2\\
55440^2 + 999^2 &= 55449^2\\
5554440^2 + 9999^2 &= 5554449^2\\
555544440^2 + 99999^2 &= 555544449^2\\
55555444440^2 + 999999^2 &= 55555444449^2
\end{aligned} \tag{3.86}$$

- **For $n = 7$; $m = 6, 66, 666, 6666, 66666, ...$ in (3.47):**

$$\begin{aligned}
4307^2 + 924^2 &= 4405^2\\
443507^2 + 9324^2 &= 443605^2\\
44435507^2 + 93324^2 &= 44435605^2\\
4444355507^2 + 933324^2 &= 4444355605^2\\
444443555507^2 + 9333324^2 &= 444443555605^2
\end{aligned} \tag{3.87}$$

The first triple $(13, 84, 85)$ for $n = 7$; $m = 6$ is not written above as it doesn't give good pattern.

- **For $n = 8$; $m = 66, 666, 6666, 66666, ...$ in (3.47):**

$$\begin{aligned} 4292^2 + 1056^2 &= 4420^2 \\ 443492^2 + 10656^2 &= 443620^2 \\ 44435492^2 + 106656^2 &= 44435620^2 \\ 4444355492^2 + 1066656^2 &= 4444355620^2 \\ 444443555492^2 + 10666656^2 &= 444443555620^2 \end{aligned} \tag{3.88}$$

The first triple (28, 96, 100) for $n = 8$; $m = 6$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned} 2146^2 + 528^2 &= 2210^2 \\ 221746^2 + 5328^2 &= 221810^2 \\ 22217746^2 + 53328^2 &= 22217810^2 \\ 2222177746^2 + 533328^2 &= 2222177810^2 \\ 222221777746^2 + 5333328^2 &= 222221777810^2 \end{aligned} \tag{3.89}$$

▶ **Division by 4**

$$\begin{aligned} 1073^2 + 264^2 &= 1105^2 \\ 110873^2 + 2664^2 &= 110905^2 \\ 11108873^2 + 26664^2 &= 11108905^2 \\ 1111088873^2 + 266664^2 &= 1111088905^2 \\ 111110888873^2 + 2666664^2 &= 111110888905^2 \end{aligned} \tag{3.90}$$

- **For $n = 9$; $m = 66, 666, 6666, 66666, ...$ in (3.47):**

$$\begin{aligned} 4275^2 + 1188^2 &= 4437^2 \\ 443475^2 + 11988^2 &= 443637^2 \\ 44435475^2 + 119988^2 &= 44435637^2 \\ 4444355475^2 + 1199988^2 &= 4444355637^2 \\ 444443555475^2 + 11999988^2 &= 444443555637^2 \end{aligned} \tag{3.91}$$

The first triple (45, 108, 117) for $n = 9$; $m = 6$ is not written above as it doesn't give good pattern.

(iii) For $n = 1, 2, ..., 9;\ m = 9, 99, 999, 9999, 99999, ...$:

- **For $n = 1;\ m = 9, 99, 999, 9999, 99999, ...$ in (3.47):**

$$\begin{aligned}
80^2 + 18^2 &= 82^2 \\
9800^2 + 198^2 &= 9802^2 && := 96079204 \\
998000^2 + 1998^2 &= 998002^2 && := 996007992004 \\
99980000^2 + 19998^2 &= 99980002^2 && := 9996000799920004 \\
9999800000^2 + 199998^2 &= 9999800002^2 && := 99996000079999200004 \\
999998000000^2 + 1999998^2 &= 999998000002^2 && := 999996000007999992000004
\end{aligned} \tag{3.92}$$

- ▶ **Division by 2**

$$\begin{aligned}
40^2 + 9^2 &= 41^2 \\
4900^2 + 99^2 &= 4901^2 && := 24019801 \\
499000^2 + 999^2 &= 499001^2 && := 249001998001 \\
49990000^2 + 9999^2 &= 49990001^2 && := 2499000199980001 \\
4999900000^2 + 99999^2 &= 4999900001^2 && := 24999000019999800001 \\
499999000000^2 + 999999^2 &= 499999000001^2 && := 249999000001999998000001
\end{aligned} \tag{3.93}$$

- **For $n = 2;\ m = 9, 99, 999, 9999, 99999, ...$ in (3.47):**

$$\begin{aligned}
77^2 + 36^2 &= 85^2 \\
9797^2 + 396^2 &= 9805^2 && := 96138025 \\
997997^2 + 3996^2 &= 998005^2 && := 996013980025 \\
99979997^2 + 39996^2 &= 99980005^2 && := 9996001399800025 \\
9999799997^2 + 399996^2 &= 9999800005^2 && := 99996000139998000025 \\
999997999997^2 + 3999996^2 &= 999998000005^2 && := 999996000013999980000025
\end{aligned} \tag{3.94}$$

- **For $n = 3;\ m = 9, 99, 999, 9999, 99999, ...$ in (3.47):**

$$\begin{aligned}
72^2 + 54^2 &= 90^2 \\
9792^2 + 594^2 &= 9810^2 && := 96236100 \\
997992^2 + 5994^2 &= 998010^2 && := 996023960100 \\
99979992^2 + 59994^2 &= 99980010^2 && := 9996002399600100 \\
9999799992^2 + 599994^2 &= 9999800010^2 && := 99996000239996000100 \\
999997999992^2 + 5999994^2 &= 999998000010^2 && := 999996000023999960000100
\end{aligned} \tag{3.95}$$

▶ **Division by 2**

$$\begin{aligned}
4896^2 + 297^2 &= 4905^2 &&:= 24059025\\
498996^2 + 2997^2 &= 499005^2 &&:= 249005990025\\
49989996^2 + 29997^2 &= 49990005^2 &&:= 2499000599900025\\
4999899996^2 + 299997^2 &= 4999900005^2 &&:= 24999000059999000025\\
499998999996^2 + 2999997^2 &= 499999000005^2 &&:= 249999000005999990000025
\end{aligned} \tag{3.96}$$

The first triple $(36, 27, 45)$ is not written above as it doesn't give good pattern.

- **For $n = 4$; $m = 9, 99, 999, 9999, 99999, ...$ in (3.47):**

$$\begin{aligned}
9785^2 + 792^2 &= 9817^2 &&:= 96373489\\
997985^2 + 7992^2 &= 998017^2 &&:= 996037932289\\
99979985^2 + 79992^2 &= 99980017^2 &&:= 9996003799320289\\
9999799985^2 + 799992^2 &= 9999800017^2 &&:= 99996000379993200289\\
999997999985^2 + 7999992^2 &= 999998000017^2 &&:= 999996000037999932000289
\end{aligned} \tag{3.97}$$

The first triple $(65, 72, 97)$ for $n = 4$; $m = 9$ is not written above as it doesn't give good pattern.

- **For $n = 5$; $m = 9, 99, 999, 9999, 99999, ...$ in (3.47):**

$$\begin{aligned}
9776^2 + 990^2 &= 9826^2 &&:= 96550276\\
997976^2 + 9990^2 &= 998026^2 &&:= 996055896676\\
99979976^2 + 99990^2 &= 99980026^2 &&:= 9996005598960676\\
9999799976^2 + 999990^2 &= 9999800026^2 &&:= 99996000559989600676\\
999997999976^2 + 9999990^2 &= 999998000026^2 &&:= 999996000055999896000676
\end{aligned} \tag{3.98}$$

The first triple $(56, 90, 106)$ for $n = 5$; $m = 9$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
4888^2 + 495^2 &= 4913^2 &&\\
498988^2 + 4995^2 &= 499013^2 &&:= 249013974169\\
49989988^2 + 49995^2 &= 49990013^2 &&:= 2499001399740169\\
4999899988^2 + 499995^2 &= 4999900013^2 &&:= 24999001399997400169\\
499998999988^2 + 4999995^2 &= 499999000013^2 &&:= 249999000013999974000169
\end{aligned} \tag{3.99}$$

The first triple $(28^2, 45, 53)$ is not written above as it doesn't give good pattern.

- **For $n = 6$; $m = 99, 999, 9999, 99999, ...$ in (3.47):**

$$\begin{aligned}
9765^2 + 1188^2 &= 9837^2 \\
997965^2 + 11988^2 &= 998037^2 \\
99979965^2 + 119988^2 &= 99980037^2 &&:= 9996007798521369 \\
9999799965^2 + 1199988^2 &= 9999800037^2 &&:= 99996000779985201369 \\
999997999965^2 + 11999988^2 &= 999998000037^2 &&:= 999996000077999852001369
\end{aligned} \tag{3.100}$$

The first triple $(45, 108, 117)$ for $n = 6$; $m = 9$ is not written above as it doesn't give good pattern.

- **For $n = 7$; $m = 99, 999, 9999, 99999, ...$ in (3.47):**

$$\begin{aligned}
9752^2 + 1386^2 &= 9850^2 \\
997952^2 + 13986^2 &= 998050^2 &&:= 996103802500 \\
99979952^2 + 139986^2 &= 99980050^2 &&:= 9996010398002500 \\
9999799952^2 + 1399986^2 &= 9999800050^2 &&:= 99996001039980002500 \\
999997999952^2 + 13999986^2 &= 999998000050^2 &&:= 999996000103999800002500
\end{aligned} \tag{3.101}$$

The first triple $(32, 126, 130)$ for $n = 7$; $m = 9$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
4876^2 + 693^2 &= 4925^2 \\
498976^2 + 6993^2 &= 499025^2 &&:= 249025950625 \\
49989976^2 + 69993^2 &= 49990025^2 &&:= 2499002599500625 \\
4999899976^2 + 699993^2 &= 4999900025^2 &&:= 24999000259995000625 \\
499998999976^2 + 6999993^2 &= 499999000025^2 &&:= 249999000025999950000625
\end{aligned} \tag{3.102}$$

- **For $n = 8$; $m = 99, 999, 9999, 99999, ...$ in (3.47):**

$$\begin{aligned}
9737^2 + 1584^2 &= 9865^2 \\
997937^2 + 15984^2 &= 998065^2 &&:= 996133744225 \\
99979937^2 + 159984^2 &= 99980065^2 &&:= 9996013397404225 \\
9999799937^2 + 1599984^2 &= 9999800065^2 &&:= 99996001339974004225 \\
999997999937^2 + 15999984^2 &= 999998000065^2 &&:= 999996000133999740004225
\end{aligned} \tag{3.103}$$

The first triple $(17, 144, 145)$ for $n = 8$; $m = 9$ is not written above as it doesn't give good pattern.

- **For $n = 9;\ m = 99, 999, 9999, 99999, ...$ in (3.47):**

$$\begin{aligned}
9720^2 + 1782^2 &= 9882^2 \\
997920^2 + 17982^2 &= 998082^2 &&:= 996167678724 \\
99979920^2 + 179982^2 &= 99980082^2 &&:= 9996016796726724 \\
9999799920^2 + 1799982^2 &= 9999800082^2 &&:= 99996001679967206724 \\
999997999920^2 + 17999982^2 &= 999998000082^2 &&:= 999996000167999672006724
\end{aligned} \tag{3.104}$$

In this case, there is no value for $n = 9;\ m = 9$.

- **Division by 2**

$$\begin{aligned}
4860^2 + 891^2 &= 4941^2 \\
498960^2 + 8991^2 &= 499041^2 &&:= 249041919681 \\
49989960^2 + 89991^2 &= 49990041^2 &&:= 2499004199181681 \\
4999899960^2 + 899991^2 &= 4999900041^2 &&:= 24999000419991801681 \\
499998999960^2 + 8999991^2 &= 499999000041^2 &&:= 249999000041999918001681
\end{aligned} \tag{3.105}$$

3.2.3 Fixed Number Multiple Patterns

In Procedure 2 given in (3.47), we observe that the middle value $G_2(m,n) = 2mn$. By fixing the value of $G_2(m,n)$, we get different possibilities for m and n. This is the idea of this subsection. This means having the same middle term, how we can have more Pythagorean triples.

(i) For $G_2(m,n) = 2mn = 110 = 2 \times 55 = 2 \times 5 \times 11$:

In this case, the possible combinations are

$$(m,n) := \{(55,1), (11,5)\}.$$

Based on these combinations, we have following Pythagorean triples:

$$\begin{aligned}
(F_2(55,1), G_2(55,1), H_2(55,1))/2 &\Rightarrow 1512^2 + 55^2 = 1513^2 \\
(F_2(11,5), G_2(11,5), H_2(11,5))/2 &\Rightarrow 48^2 + 55^2 = 73^2
\end{aligned}$$

Let's write patterns based on above two triples.

- **For $n = 1;\ m = 55, 555, 5555, 55555, ...$ in (3.47):**

$$\begin{aligned}
512^2 + 55^2 &= 1513^2 \\
154012^2 + 555^2 &= 154013^2 \\
15429012^2 + 5555^2 &= 15429013^2 \\
1543179012^2 + 55555^2 &= 1543179013^2
\end{aligned} \tag{3.106}$$

We observe that the above pattern is not so beautiful.

- **For $n=5$; $m=11,111,1111,11111,...$ in (3.47):**

$$\begin{aligned} 48^2+55^2 &= 73^2 \\ 6148^2+555^2 &= 6173^2 \\ 617148^2+5555^2 &= 617173^2 \\ 61727148^2+55555^2 &= 61727173^2 \end{aligned} \tag{3.107}$$

We observe that the above pattern is not so beautiful.

(ii) For $G_2(m,n)=2mn=60=2\times 30=2\times 2\times 3\times 5$:

In this case the possible combinations are

$$(m,n):=\{(30,1),(15,2),(10,3),(6,5)\}$$

Based on these combinations, we have following Pythagorean triples:

$$\begin{aligned} (F_2(30,1),G_2(30,1),H_2(30,1)) &\Rightarrow 899^2+60^2=901^2 \\ (F_2(15,2),G_2(15,2),H_2(15,2)) &\Rightarrow 221^2+60^2=229^2 \\ (F_2(10,3),G_2(10,3),H_2(10,3)) &\Rightarrow 91^2+60^2=109^2 \\ (F_2(6,5),G_2(6,5),H_2(6,5)) &\Rightarrow 11^2+60^2=61^2 \end{aligned}$$

Let's write patterns based on above four triples.

- **For $n=1$; $m=30,300,3000,30000,...$ in (3.47):**

$$\begin{aligned} 899^2+60^2 &= 901^2 &&:= 811801 \\ 89999^2+600^2 &= 90001^2 &&:= 8100180001 \\ 8999999^2+6000^2 &= 9000001^2 &&:= 81000018000001 \\ 899999999^2+60000^2 &= 900000001^2 &&:= 810000001800000001 \end{aligned} \tag{3.108}$$

- **For $n=2$; $m=15,155,1555,15555,...$ in (3.47):**

$$\begin{aligned} 221^2+60^2 &= 229^2 \\ 24021^2+620^2 &= 24029^2 \\ 2418021^2+6220^2 &= 2418029^2 \\ 241958021^2+62220^2 &= 241958029^2 \end{aligned} \tag{3.109}$$

We observe that the above pattern is not so beautiful.

- **For $n = 3$; $m = 10, 100, 1000, 10000, ...$ in (3.47):**

$$\begin{aligned}
91^2 + 60^2 &= 109^2 &&:= 11881\\
9991^2 + 600^2 &= 10009^2 &&:= 100180081\\
999991^2 + 6000^2 &= 1000009^2 &&:= 1000018000081\\
99999991^2 + 60000^2 &= 100000009^2 &&:= 10000001800000081
\end{aligned} \tag{3.110}$$

- **For $n = 5$; $m = 66, 666, 6666, 66666, ...$ in (3.47):**

$$\begin{aligned}
4331^2 + 660^2 &= 4381^2\\
443531^2 + 6660^2 &= 443581^2\\
44435531^2 + 66660^2 &= 44435581^2\\
4444355531^2 + 666660^2 &= 4444355581^2
\end{aligned} \tag{3.111}$$

The first triple $(11, 60, 61)$ for $n = 5$; $m = 6$ is not written above as it doesn't give good pattern.

(iii) For $G_2(m, n) = 2mn = 120 = 2 \times 60 = 2 \times 2 \times 2 \times 3 \times 5$:

In this case the possible combinations are

$$(m, n) := \{(60, 1), (30, 2), (20, 3), (15, 4), (12, 5), (10, 6)\}$$

Based on these combinations, we have following Pythagorean triples:

$$\begin{aligned}
(F_2(60, 1), G_2(60, 1), H_2(60, 1)) &\Rightarrow 3599^2 + 120^2 = 3601^2\\
(F_2(30, 2), G_2(30, 2), H_2(30, 2)) &\Rightarrow 896^2 + 120^2 = 904^2\\
(F_2(20, 3), G_2(20, 3), H_2(20, 3)) &\Rightarrow 391^2 + 120^2 = 409^2\\
(F_2(15, 4), G_2(15, 4), H_2(15, 4)) &\Rightarrow 209^2 + 120^2 = 241^2\\
(F_2(12, 5), G_2(12, 5), H_2(12, 5)) &\Rightarrow 119^2 + 120^2 = 169^2\\
(F_2(10, 6), G_2(10, 6), H_2(10, 6)) &\Rightarrow 64^2 + 120^2 = 136^2
\end{aligned}$$

Let's write patterns based on above six triples.

- **For $n = 1$; $m = 60, 600, 6000, 60000, ...$ in (3.47):**

$$\begin{aligned}
3599^2 + 120^2 &= 3601^2 &&:= 12967201\\
359999^2 + 1200^2 &= 360001^2 &&:= 129600720001\\
35999999^2 + 12000^2 &= 36000001^2 &&:= 1296000072000001\\
3599999999^2 + 120000^2 &= 3600000001^2 &&:= 12960000007200000001
\end{aligned} \tag{3.112}$$

- **For $n = 2$; $m = 30, 300, 3000, 30000, \ldots$ in (3.47):**

$$\begin{aligned}
896^2 + 120^2 &= 904^2 &&:= 817216\\
89996^2 + 1200^2 &= 90004^2 &&:= 8100720016\\
8999996^2 + 12000^2 &= 9000004^2 &&:= 81000072000016\\
899999996^2 + 120000^2 &= 900000004^2 &&:= 810000007200000016\\
89999999996^2 + 1200000^2 &= 90000000004^2 &&:= 8100000000720000000016
\end{aligned} \quad (3.113)$$

▶ **Division by 2**

$$\begin{aligned}
448^2 + 60^2 &= 452^2\\
44998^2 + 600^2 &= 45002^2 &&:= 2025180004\\
4499998^2 + 6000^2 &= 4500002^2 &&:= 20250018000004\\
449999998^2 + 60000^2 &= 450000002^2 &&:= 202500001800000004\\
44999999998^2 + 600000^2 &= 45000000002^2 &&:= 2025000000180000000004
\end{aligned} \quad (3.114)$$

▶ **Division by 4**

$$\begin{aligned}
22499^2 + 300^2 &= 22501^2\\
2249999^2 + 3000^2 &= 2250001^2 &&:= 5062504500001\\
224999999^2 + 30000^2 &= 225000001^2 &&:= 50625000450000001\\
22499999999^2 + 300000^2 &= 22500000001^2 &&:= 506250000045000000001
\end{aligned} \quad (3.115)$$

The first triple (224, 30, 226) is not written above as it doesn't give good pattern.

- **For $n = 3$; $m = 20, 200, 2000, 20000, \ldots$ in (3.47):**

$$\begin{aligned}
391^2 + 120^2 &= 409^2 &&:= 167281\\
39991^2 + 1200^2 &= 40009^2 &&:= 1600720081\\
3999991^2 + 12000^2 &= 4000009^2 &&:= 16000072000081\\
399999991^2 + 120000^2 &= 400000009^2 &&:= 160000007200000081
\end{aligned} \quad (3.116)$$

- **For $n = 4$; $m = 15, 155, 1555, 15555, 155555, \ldots$ in (3.47):**

$$\begin{aligned}
209^2 + 120^2 &= 241^2\\
24009^2 + 1240^2 &= 24041^2\\
2418009^2 + 12440^2 &= 2418041^2\\
241958009^2 + 124440^2 &= 241958041^2
\end{aligned} \quad (3.117)$$

We observe that the above pattern is not so beautiful.

- **For $n = 5;\ m = 122, 1222, 12222, 122222, ...$ in (3.47):**

$$\begin{aligned}
14859^2 + 1220^2 &= 14909^2\\
1493259^2 + 12220^2 &= 1493309^2\\
149377259^2 + 122220^2 &= 149377309^2\\
14938217259^2 + 1222220^2 &= 14938217309^2
\end{aligned} \tag{3.118}$$

The first triple (119, 120, 169) for $n = 5;\ m = 12$ is not written above as it doesn't give good pattern.

- **For $n = 6;\ m = 10, 100, 1000, 10000, ...$ in (3.47):**

$$\begin{aligned}
64^2 + 120^2 &:= 136^2\\
9964^2 + 1200^2 &= 10036^2 := 100721296\\
999964^2 + 12000^2 &= 1000036^2 := 1000072001296\\
99999964^2 + 120000^2 &= 100000036^2 := 10000007200001296\\
9999999964^2 + 1200000^2 &= 10000000036^2 := 100000000720000001296
\end{aligned} \tag{3.119}$$

▶ **Division by 2**

$$\begin{aligned}
4982^2 + 600^2 &= 5018^2\\
499982^2 + 6000^2 &= 500018^2 := 250018000324\\
49999982^2 + 60000^2 &= 50000018^2 := 2500001800000324\\
4999999982^2 + 600000^2 &= 5000000018^2 := 25000000180000000324
\end{aligned} \tag{3.120}$$

The first triple (32, 60, 68) for $n = 6;\ m = 10$ is not written above as it doesn't give good pattern.

▶ **Division by 4**

$$\begin{aligned}
2491^2 + 300^2 &= 2509^2\\
249991^2 + 3000^2 &= 250009^2 := 62504500081\\
24999991^2 + 30000^2 &= 25000009^2 := 625000450000081\\
2499999991^2 + 300000^2 &= 2500000009^2 := 6250000045000000081
\end{aligned} \tag{3.121}$$

The first triple (16, 30, 34) for $n = 6;\ m = 10$ is not written above as it doesn't give good pattern.

(iv) For $G_2(m,n) = 2mn = 330 = 2\times 165 = 2\times 3\times 5\times 11$:

In this case the possible combinations are

$$(m,n) := \{(165,1),(55,3),(33,5),(15,11)\}$$

Based on these combinations, we have following Pythagorean triples:

$$\begin{aligned}
&(F_2(165,1), G_2(165,1), H_2(165,1))/2 \Rightarrow 13612^2+165^2 = 13613\\
&(F_2(55,3), G_2(55,3), H_2(55,3))/2 \Rightarrow 1508^2+165^2 = 1517\\
&(F_2(33,5), G_2(33,5), H_2(33,5))/2 \Rightarrow 532^2+165^2 = 557\\
&(F_2(15,11), G_2(15,11), H_2(15,11))/2 \Rightarrow 52^2+165^2 = 173
\end{aligned}$$

Out of four triples, there are two triple don't give a good patterns. These are:

$$\begin{aligned}
&(i)\ n=1;\ m=165: \quad (13612,165,13613) \Rightarrow 13612^2+165^2 = 13613^2\\
&(ii)\ n=11;\ m=15: \quad (52,165,173) \Rightarrow 52^2+165^2 = 173^2
\end{aligned}$$

The pattern for other two triples are as follows:

- **For $n=3;\ m=55,555,5555,55555,...$ in (3.47):**

$$\begin{aligned}
1508^2+165^2 &= 1517^2\\
154008^2+1665^2 &= 154017^2\\
15429008^2+16665^2 &= 15429017^2\\
1543179008^2+166665^2 &= 1543179017^2
\end{aligned} \tag{3.122}$$

This pattern is not so beautiful as others.

- **For $n=5;\ m=33,333,3333,33333,...$ in (3.47):**

$$\begin{aligned}
532^2+165^2 &= 557^2\\
5432^2+1665^2 &= 55457^2\\
5554432^2+16665^2 &= 5554457^2\\
555544432^2+166665^2 &= 555544457^2
\end{aligned} \tag{3.123}$$

- **For $n=82000,820000,8200000,...$ in (3.1):**

In order to get pattern for the triple (13612, 165, 13613), we shall use Procedure 1 given in (3.1) instead of Procedure 2. See below:

$$\begin{aligned}
13448164000^2+164001^2 &= 13448164001^2\\
1344801640000^2+1640001^2 &= 1344801640001^2\\
134480016400000^2+16400001^2 &= 134480016400001^2\\
13448000164000000^2+164000001^2 &= 13448000164000001^2
\end{aligned} \tag{3.124}$$

The first three triples for $\boldsymbol{n = 82, 820, 8200}$ are not written above as they don't give good patterns. These are as follows:

$$\begin{aligned}13612^2 + 165^2 &= 13613^2\\ 1346440^2 + 1641^2 &= 1346441^2\\ 134496400^2 + 16401^2 &= 134496401^2\end{aligned}$$

(v) For $G_2(m,n) = 2mn = 330 = 2 \times 261 = 2 \times 3 \times 3 \times 29$:

In this case the possible combinations are

$$(m,n) := \{(261,1),(87,3),(29,9)\}$$

Based on these combinations, we have following Pythagorean triples:

$$\begin{aligned}(F_2(261,1), G_2(261,1), H_2(261,1))\,/2 &\Rightarrow 34060^2 + 261^2 = 34061^2\\ (F_2(87,3), G_2(87,3), H_2(87,3))\,/2 &\Rightarrow 3780^2 + 261^2 = 3789^2\\ (F_2(29,9), G_2(29,9), H_2(29,9))\,/2 &\Rightarrow 380^2 + 261^2 = 461^2\end{aligned}$$

Out of three triples, there are two triple don't give a good patterns. These are:

$$\begin{aligned}(i)\ n = 1;\ m = 261: \quad (34060, 261, 34061) &\Rightarrow 34060^2 + 261^2 = 34061^2\\ (ii)\ n = 3;\ m = 87: \quad (3780, 261, 3789) &\Rightarrow 3780^2 + 261^2 = 3789^2\end{aligned}$$

- **For $n = 9;\ m = 299, 2999, 29999, 299999, ...$ in (3.47):**

$$\begin{aligned}44660^2 + 2691^2 &= 44741\\ 4496960^2 + 26991^2 &= 4497041\\ 449969960^2 + 269991^2 &= 449970041\\ 44999699960^2 + 2699991^2 &= 44999700041\end{aligned} \tag{3.125}$$

The first triple $(380, 261, 461)$ for $n = 9;\ m = 29$ is not written above as it doesn't give good pattern.

- **For $n = 1300, 1300, 13000, ...$ in (3.1):**

In order to get pattern for the triple (134060, 261, 34061), we shall use Procedure 1 given in (3.1) instead of Procedure 2. See below:

$$\begin{aligned}3382600^2 + 2601^2 &= 3382601^2\\ 338026000^2 + 26001^2 &= 338026001^2\\ 33800260000^2 + 260001^2 &= 33800260001^2\\ 3380002600000^2 + 2600001^2 &= 3380002600001^2\end{aligned} \tag{3.126}$$

The first triple $(34060, 261, 34061)$ for $m = 1;\ n = 130$ is not written above as it doesn't give good pattern.

(vi) For $G_2(m,n) = 2mn = 280 = 2 \times 140 = 2 \times 2 \times 5 \times 7$:

In this case the possible combinations are

$$(m,n) := \{(140,1),(70,2),(35,4),(28,5),(20,7)\}$$

Based on these combinations, we have following Pythagorean triples:

$$\begin{aligned}
(F_2(140,1), G_2(140,1), H_2(140,1)) &\Rightarrow 19599^2 + 280^2 = 19601^2 \\
(F_2(70,2), G_2(70,2), H_2(70,2)) &\Rightarrow 4896^2 + 280^2 = 4904^2 \\
(F_2(35,4), G_2(35,4), H_2(35,4)) &\Rightarrow 1209^2 + 280^2 = 1241^2 \\
(F_2(28,5), G_2(28,5), H_2(28,5)) &\Rightarrow 759^2 + 280^2 = 809^2 \\
(F_2(20,7), G_2(20,7), H_2(20,7)) &\Rightarrow 351^2 + 280^2 = 449^2
\end{aligned}$$

Below are patterns based on above triples:

- **For $n = 1$; $m = 140, 1400, 14000, 140000, \ldots$ in (3.47):**

$$\begin{aligned}
19599^2 + 280^2 &= 19601^2 \\
1959999^2 + 2800^2 &= 1960001^2 := 3841603920001 \\
195999999^2 + 28000^2 &= 196000001^2 := 38416000392000001 \\
19599999999^2 + 280000^2 &= 19600000001^2 := 384160000039200000001
\end{aligned} \tag{3.127}$$

- **For $n = 2$; $m = 70, 700, 7000, 70000, \ldots$ in (3.47):**

$$\begin{aligned}
4896^2 + 280^2 &= 4904^2 \\
489996^2 + 2800^2 &= 490004^2 := 240103920016 \\
48999996^2 + 28000^2 &= 49000004^2 := 2401000392000016 \\
4899999996^2 + 280000^2 &= 4900000004^2 := 24010000039200000016
\end{aligned} \tag{3.128}$$

▶ **Division by 2**

$$\begin{aligned}
2448^2 + 140^2 &= 2452^2 \\
244998^2 + 1400^2 &= 245002^2 := 60025980004 \\
24499998^2 + 14000^2 &= 24500002^2 := 600250098000004 \\
2449999998^2 + 140000^2 &= 2450000002^2 := 6002500009800000004 \\
244999999998^2 + 1400000^2 &= 245000000002^2 := 60025000000980000000004
\end{aligned} \tag{3.129}$$

▶ **Division by 4**

$$
\begin{aligned}
122499^2 + 700^2 &= 122501^2 \\
12249999^2 + 7000^2 &= 12250001^2 &&:= 150062524500001 \\
1224999999^2 + 70000^2 &= 1225000001^2 &&:= 1500625002450000001 \\
122499999999^2 + 700000^2 &= 122500000001^2 &&:= 15006250000245000000001 \\
12249999999999^2 + 7000000^2 &= 12250000000001^2 &&:= 150062500000024500000000001 \qquad (3.130)
\end{aligned}
$$

The first triple $(1224, 70, 1226)$ is not written above as it doesn't give good pattern.

- **For $n = 4$; $m = 350, 3500, 35000, ...$ in (3.47):**

$$
\begin{aligned}
122484^2 + 2800^2 &= 122516^2 \\
12249984^2 + 28000^2 &= 12250016^2 &&:= 150062892000256 \\
1224999984^2 + 280000^2 &= 1225000016^2 &&:= 1500625039200000256 \\
122499999984^2 + 2800000^2 &= 122500000016^2 &&:= 15006250003920000000256 \\
12249999999984^2 + 28000000^2 &= 12250000000016^2 &&:= 150062500000392000000000256 \qquad (3.131)
\end{aligned}
$$

The first triple $(1209, 280, 1241)$ for $n = 4$; $m = 35$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$
\begin{aligned}
61242^2 + 1400^2 &= 61258^2 \\
6124992^2 + 14000^2 &= 6125008^2 \\
612499992^2 + 140000^2 &= 612500008^2 &&:= 375156259800000064 \\
61249999992^2 + 1400000^2 &= 61250000008^2 &&:= 3751562500980000000064 \\
6124999999992^2 + 14000000^2 &= 6125000000008^2 &&:= 37515625000098000000000064 \qquad (3.132)
\end{aligned}
$$

▶ **Division by 4**

$$
\begin{aligned}
3062496^2 + 7000^2 &= 3062504^2 \\
306249996^2 + 70000^2 &= 306250004^2 \\
30624999996^2 + 700000^2 &= 30625000004^2 &&:= 937890625245000000016 \\
3062499999996^2 + 7000000^2 &= 3062500000004^2 &&:= 9378906250024500000000016 \\
306249999999996^2 + 70000000^2 &= 306250000000004^2 &&:= 93789062500002450000000000016 \qquad (3.133)
\end{aligned}
$$

The first triple $(30621, 700, 30629)$ is not written above as it doesn't give good pattern.

▶ **Division by 8**

$$\begin{aligned}
1531248^2 + 3500^2 &= 1531252^2 \\
153124998^2 + 35000^2 &= 153125002^2 \\
15312499998^2 + 350000^2 &= 15312500002^2 \\
1531249999998^2 + 3500000^2 &= 1531250000002^2 := 2344726562506125000000004 \\
153124999999998^2 + 35000000^2 &= 153125000000002^2 := 23447265625000612500000000004
\end{aligned} \quad (3.134)$$

▶ **Division by 16**

$$\begin{aligned}
76562499^2 + 17500^2 &= 76562501^2 \\
7656249999^2 + 175000^2 &= 7656250001^2 \\
765624999999^2 + 1750000^2 &= 765625000001^2 \\
76562499999999^2 + 17500000^2 &= 76562500000001^2 := 5861816406250153125000000001 \\
7656249999999999^2 + 175000000^2 &= 7656250000000001^2 := 586181640625000153125000000000001
\end{aligned} \quad (3.135)$$

The first triple $(765624, 1750, 765626)$ is not written above as it doesn't give good pattern.

- **For $n = 5$; $m = 280, 2800, 28000, ...$ in (3.47):**

$$\begin{aligned}
78375^2 + 2800^2 &= 78425^2 \\
7839975^2 + 28000^2 &= 7840025^2 \\
783999975^2 + 280000^2 &= 784000025^2 := 614656039200000625 \\
78399999975^2 + 2800000^2 &= 78400000025^2 := 6146560003920000000625
\end{aligned} \quad (3.136)$$

The first triple $(759, 280, 809)$ for $n = 5$; $m = 28$ is not written above as it doesn't give good pattern.

- **For $Forn = 7$; $m = 20, 200, 2000, 20000, ...$ in (3.47):**

$$\begin{aligned}
351^2 + 280^2 &= 449^2 \\
39951^2 + 2800^2 &= 40049^2 := 1603922401 \\
3999951^2 + 28000^2 &= 4000049^2 := 16000392002401 \\
399999951^2 + 280000^2 &= 400000049^2 := 160000039200002401
\end{aligned} \quad (3.137)$$

(vii) For $G_2(m, n) = 2mn = 360 = 2 \times 180 = 2 \times 2 \times 2 \times 3 \times 3 \times 5$:

In this case the possible combinations are

$$(m, n) := \{(180, 1), (90, 2), (60, 3), (45, 4), (38, 5), (30, 6), (20, 9), (18, 10), (15, 12)\}$$

Based on these combinations, we have following Pythagorean triples:

$$\begin{aligned}
(F_2(180,1), G_2(180,1), H_2(180,1)) &\Rightarrow 32399^2+360^2 = 32401^2\\
(F_2(90,2), G_2(90,2), H_2(90,2)) &\Rightarrow 8096^2+360^2 = 8104^2\\
(F_2(60,3), G_2(60,3), H_2(60,3)) &\Rightarrow 3591^2+360^2 = 3609^2\\
(F_2(45,4), G_2(45,4), H_2(45,4)) &\Rightarrow 2009^2+360^2 = 2041^2\\
(F_2(36,5), G_2(36,5), H_2(36,5)) &\Rightarrow 1271^2+360^2 = 1321^2\\
(F_2(30,6), G_2(30,6), H_2(30,6)) &\Rightarrow 864^2+360^2 = 936^2\\
(F_2(20,9), G_2(20,9), H_2(20,9)) &\Rightarrow 319^2+360^2 = 481^2\\
(F_2(18,10), G_2(18,10), H_2(18,10)) &\Rightarrow 224^2+360^2 = 424^2\\
(F_2(15,12), G_2(15,12), H_2(15,12)) &\Rightarrow 81^2+360^2 = 369^2
\end{aligned}$$

Let's write patterns based on above triples:

- **For $n=1$; $m=180, 1800, 18000, 180000, ...$ in (3.47):**

$$\begin{aligned}
32399^2+360^2 &= 32401^2\\
3239999^2+3600^2 &= 3240001^2 := 10497606480001\\
323999999^2+36000^2 &= 324000001^2 := 104976000648000001\\
32399999999^2+360000^2 &= 32400000001^2 := 1049760000064800000001
\end{aligned} \quad (3.138)$$

- **For $n=2$; $m=90, 900, 9000, 90000, ...$ in (3.47):**

$$\begin{aligned}
8096^2+360^2 &= 8104^2\\
809996^2+3600^2 &= 810004^2 := 656106480016\\
80999996^2+36000^2 &= 81000004^2 := 6561000648000016\\
8099999996^2+360000^2 &= 8100000004^2 := 65610000064800000016\\
809999999996^2+3600000^2 &= 810000000004^2 := 656100000006480000000016\\
80999999999996^2+36000000^2 &= 81000000000004^2 := 6561000000000648000000000016
\end{aligned} \quad (3.139)$$

- **Division by 2**

$$\begin{aligned}
4048^2+180^2 &= 4052^2\\
404998^2+1800^2 &= 405002^2\\
40499998^2+18000^2 &= 40500002^2 := 1640250162000004\\
4049999998^2+180000^2 &= 4050000002^2 := 16402500016200000004\\
404999999998^2+1800000^2 &= 405000000002^2 := 164025000001620000000004\\
40499999999998^2+18000000^2 &= 40500000000002^2 := 1640250000000162000000000004
\end{aligned} \quad (3.140)$$

- **Division by 4**

$$
\begin{aligned}
202499^2 + 900^2 &= 202501^2 \\
20249999^2 + 9000^2 &= 20250001^2 && := 410062540500001 \\
2024999999^2 + 90000^2 &= 2025000001^2 && := 4100625004050000001 \\
202499999999^2 + 900000^2 &= 202500000001^2 && := 41006250000405000000001 \\
20249999999999^2 + 9000000^2 &= 20250000000001^2 && := 410062500000040500000000001
\end{aligned} \tag{3.141}
$$

The first triple (2024, 90, 2026) is not written above as it doesn't give good pattern.

- **For $n = 3$; $m = 60, 600, 6000, 60000, ...$ in (3.47):**

$$
\begin{aligned}
3591^2 + 360^2 &= 3609^2 \\
359991^2 + 3600^2 &= 360009^2 && := 129606480081 \\
35999991^2 + 36000^2 &= 36000009^2 && := 1296000648000081 \\
3599999991^2 + 360000^2 &= 3600000009^2 && := 12960000064800000081
\end{aligned} \tag{3.142}
$$

- **For $n = 4$; $m = 450, 4500, 45000, ...$ in (3.47):**

$$
\begin{aligned}
202484^2 + 3600^2 &= 202516^2 \\
20249984^2 + 36000^2 &= 20250016^2 \\
2024999984^2 + 360000^2 &= 2025000016^2 && := 4100625064800000256 \\
202499999984^2 + 3600000^2 &= 202500000016^2 && := 41006250006480000000256 \\
20249999999984^2 + 36000000^2 &= 20250000000016^2 && := 410062500000648000000000256
\end{aligned} \tag{3.143}
$$

The first triple (2009, 360, 2041) for $n = 4$; $m = 45$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$
\begin{aligned}
101248^2 + 900^2 &= 101252^2 \\
10124998^2 + 9000^2 &= 10125002^2 \\
1012499998^2 + 90000^2 &= 1012500002^2 && := 1025156254050000004 \\
101249999998^2 + 900000^2 &= 101250000002^2 && := 10251562500405000000004 \\
10124999999998^2 + 9000000^2 &= 10125000000002^2 && := 102515625000040500000000004
\end{aligned} \tag{3.144}
$$

▶ **Division by 4**

$$
\begin{aligned}
5062499^2 + 4500^2 &= 5062501^2 \\
506249999^2 + 45000^2 &= 506250001^2 \\
50624999999^2 + 450000^2 &= 50625000001^2 && := 2562890625101250000001 \\
5062499999999^2 + 4500000^2 &= 5062500000001^2 && := 25628906250010125000000001 \\
506249999999999^2 + 45000000^2 &= 506250000000001^2 && := 256289062500001012500000000001
\end{aligned} \tag{3.145}
$$

The first triple (50624, 450, 50626) is not written above as it doesn't give good pattern.

- **For $n = 5$; $m = 360, 3600, 36000, \ldots$ in (3.47):**

$$\begin{aligned}
129575^2 + 3600^2 &= 129625^2 \\
12959975^2 + 36000^2 &= 12960025^2 \\
1295999975^2 + 360000^2 &= 1296000025^2 := 1679616064800000625 \\
129599999975^2 + 3600000^2 &= 129600000025^2 := 16796160006480000000625 \\
12959999999975^2 + 36000000^2 &= 12960000000025^2 := 167961600000648000000000625
\end{aligned} \qquad (3.146)$$

The first triple $(1271, 360, 1321)$ for $n = 5$; $m = 36$ is not written above as it doesn't give good pattern.

- **For $n = 6$; $m = 30, 300, 3000, 30000, \ldots$ in (3.47):**

$$\begin{aligned}
864^2 + 360^2 &= 936^2 \\
89964^2 + 3600^2 &= 90036^2 := 8106481296 \\
8999964^2 + 36000^2 &= 9000036^2 := 81000648001296 \\
899999964^2 + 360000^2 &= 900000036^2 := 810000064800001296
\end{aligned} \qquad (3.147)$$

► **Division by 2**

$$\begin{aligned}
44982^2 + 1800^2 &= 45018^2 \\
4499982^2 + 18000^2 &= 4500018^2 := 20250162000324 \\
449999982^2 + 180000^2 &= 450000018^2 := 202500016200000324 \\
44999999982^2 + 1800000^2 &= 45000000018^2 := 2025000001620000000324 \\
4499999999982^2 + 18000000^2 &= 4500000000018^2 := 20250000000162000000000324
\end{aligned} \qquad (3.148)$$

The first triple $(432, 180, 468)$ is not written above as it doesn't give good pattern.

► **Division by 4**

$$\begin{aligned}
22491^2 + 900^2 &= 22509^2 \\
2249991^2 + 9000^2 &= 2250009^2 := 5062540500081 \\
224999991^2 + 90000^2 &= 225000009^2 := 50625004050000081 \\
22499999991^2 + 900000^2 &= 22500000009^2 := 506250000405000000081 \\
2249999999991^2 + 9000000^2 &= 2250000000009^2 := 5062500000040500000000081
\end{aligned} \qquad (3.149)$$

- **For $n = 9$; $m = 20, 200, 2000, 20000, \ldots$ in (3.47):**

$$\begin{aligned}
319^2 + 360^2 &= 481^2 \\
39919^2 + 3600^2 &= 40081^2 := 1606486561 \\
3999919^2 + 36000^2 &= 4000081^2 := 16000648006561 \\
399999919^2 + 360000^2 &= 400000081^2 := 160000064800006561
\end{aligned} \qquad (3.150)$$

- **For $n = 10$; $m = 180, 1800, 18000, ...$ in (3.47):**

$$\begin{aligned}
32300^2 + 3600^2 &= 32500^2 \\
3239900^2 + 36000^2 &= 3240100^2 \\
323999900^2 + 360000^2 &= 324000100^2 := 104976064800010000 \\
32399999900^2 + 3600000^2 &= 32400000100^2 := 1049760006480000010000 \\
3239999999900^2 + 36000000^2 &= 3240000000100^2 := 10497600000648000000010000
\end{aligned} \tag{3.151}$$

The first triple (224, 360, 424) for $n = 10$; $m = 18$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
16150^2 + 1800^2 &= 16250^2 \\
1619950^2 + 18000^2 &= 1620050^2 := 2624562002500 \\
161999950^2 + 180000^2 &= 162000050^2 := 26244016200002500 \\
16199999950^2 + 1800000^2 &= 16200000050^2 := 262440001620000002500 \\
1619999999950^2 + 18000000^2 &= 1620000000050^2 := 2624400000162000000002500
\end{aligned} \tag{3.152}$$

▶ **Division by 4**

$$\begin{aligned}
8075^2 + 900^2 &= 8125^2 \\
809975^2 + 9000^2 &= 810025^2 := 656140500625 \\
80999975^2 + 90000^2 &= 81000025^2 := 6561004050000625 \\
8099999975^2 + 900000^2 &= 8100000025^2 := 65610000405000000625 \\
809999999975^2 + 9000000^2 &= 810000000025^2 := 656100000040500000000625
\end{aligned} \tag{3.153}$$

- **For $n = 12$; $m = 1500, 15000, ...$ in (3.47):**

$$\begin{aligned}
2249856^2 + 36000^2 &= 2250144^2 \\
224999856^2 + 360000^2 &= 225000144^2 := 50625064800020736 \\
22499999856^2 + 3600000^2 &= 22500000144^2 := 506250006480000020736 \\
2249999999856^2 + 36000000^2 &= 2250000000144^2 := 5062500000648000000020736 \\
224999999999856^2 + 360000000^2 &= 225000000000144^2 := 50625000000064800000000020736
\end{aligned} \tag{3.154}$$

The first two triples (81, 360, 369) and (22356, 3600, 22644) for $n = 12$; $m = 15$ and 150 are not written above as they don't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
1124928^2 + 18000^2 &= 1125072^2 \\
112499928^2 + 180000^2 &= 112500072^2 \\
11249999928^2 + 1800000^2 &= 11250000072^2 &:= 126562501620000005184 \\
1124999999928^2 + 18000000^2 &= 1125000000072^2 &:= 1265625000162000000005184 \\
112499999999928^2 + 180000000^2 &= 112500000000072^2 &:= 12656250000016200000000005184
\end{aligned} \quad (3.155)$$

- **Division by 4**

$$\begin{aligned}
562464^2 + 9000^2 &= 562536^2 \\
56249964^2 + 90000^2 &= 56250036^2 \\
5624999964^2 + 900000^2 &= 5625000036^2 &:= 31640625405000001296 \\
562499999964^2 + 9000000^2 &= 562500000036^2 &:= 316406250040500000001296 \\
56249999999964^2 + 90000000^2 &= 56250000000036^2 &:= 3164062500004050000000001296
\end{aligned} \quad (3.156)$$

- **Division by 8**

$$\begin{aligned}
281232^2 + 4500^2 &= 281268^2 \\
28124982^2 + 45000^2 &= 28125018^2 \\
2812499982^2 + 450000^2 &= 2812500018^2 \\
281249999982^2 + 4500000^2 &= 281250000018^2 &:= 79101562510125000000324 \\
28124999999982^2 + 45000000^2 &= 28125000000018^2 &:= 791015625001012500000000324
\end{aligned} \quad (3.157)$$

- **Division by 16**

$$\begin{aligned}
14062491^2 + 22500^2 &= 14062509^2 \\
1406249991^2 + 225000^2 &= 1406250009^2 \\
140624999991^2 + 2250000^2 &= 140625000009^2 &:= 19775390627531250000081 \\
14062499999991^2 + 22500000^2 &= 14062500000009^2 &:= 197753906250253125000000081
\end{aligned} \quad (3.158)$$

The first triple (140616, 2250, 140634) is not written as it doesn't give good pattern.

(viii) For $G_2(m,n) = 2mn = 150 = 2 \times 75 = 2 \times 3 \times 5 \times 5$:

In this case, the possible combinations are

$$(m,n) := \{(15,5),(25,3),(75,1)\}.$$

Based on these combinations, we have following Pythagorean triples:

$$
\begin{aligned}
(F_2(75,1), G_2(75,1), H_2(75,1)) &\Rightarrow 5624^2 + 150^2 = 5626^2\\
(F_2(25,3), G_2(25,3), H_2(25,3)) &\Rightarrow 616^2 + 150^2 = 634^2\\
(F_2(15,5), G_2(15,5), H_2(15,5)) &\Rightarrow 200^2 + 150^2 = 250^2
\end{aligned}
$$

We observe that all the numbers appearing above are even numbers. Obviously, they can be divided by 2. Dividing by 2 the above value we get following triples:

$$
\begin{aligned}
(F_2(75,1), G_2(75,1), H_2(75,1))\,/2 &\Rightarrow 2812^2 + 75^2 = 2813^2\\
(F_2(25,3), G_2(25,3), H_2(25,3))\,/2 &\Rightarrow 308^2 + 75^2 = 317^2\\
(F_2(15,5), G_2(15,5), H_2(15,5))\,/2 &\Rightarrow 100^2 + 75^2 = 125^2
\end{aligned}
$$

Let's write patterns based on above triples:

- **For $n = 1$; $m = 750, 7500, 75000, ...$ in (3.47):**

$$
\begin{aligned}
562499^2 + 1500^2 &= 562501^2\\
56249999^2 + 15000^2 &= 56250001^2 &&:= 3164062612500001\\
5624999999^2 + 150000^2 &= 5625000001^2 &&:= 31640625011250000001\\
562499999999^2 + 1500000^2 &= 562500000001^2 &&:= 316406250001125000000001\\
56249999999999^2 + 15000000^2 &= 56250000000001^2 &&:= 3164062500000112500000000001
\end{aligned} \tag{3.159}
$$

The first triple $(5624, 150, 5626)$ for $n = 1$; $m = 75$ is not written above as it doesn't give good pattern.

The first term for $m = 75$ is not written above. It is ^{2}.

- **For $n = 3$; $m = 250, 2500, 25000, ...$ in (3.47):**

$$
\begin{aligned}
62491^2 + 1500^2 &= 62509^2\\
6249991^2 + 15000^2 &= 6250009^2\\
624999991^2 + 150000^2 &= 625000009^2 &&:= 390625011250000081\\
62499999991^2 + 1500000^2 &= 62500000009^2 &&:= 3906250001125000000081\\
6249999999991^2 + 15000000^2 &= 6250000000009^2 &&:= 39062500000112500000000081
\end{aligned} \tag{3.160}
$$

The first triple $(616, 150, 634)$ for $n = 3$; $m = 25$ is not written above as it doesn't give good pattern.

- **For $n = 5$; $m = 15, 150, 1500, 15000, ...$ in (3.47):**

$$
\begin{aligned}
22475^2 + 1500^2 &= 22525^2\\
2249975^2 + 15000^2 &= 2250025^2\\
224999975^2 + 150000^2 &= 225000025^2 &&:= 50625011250000625\\
22499999975^2 + 1500000^2 &= 22500000025^2 &&:= 506250001125000000625\\
2249999999975^2 + 15000000^2 &= 2250000000025^2 &&:= 5062500000112500000000625
\end{aligned} \tag{3.161}
$$

The first triple $(200, 150, 250)$ for $n = 5$; $m = 25$ is not written above as it doesn't give good pattern.

(ix) For $G_2(m,n) = 2mn = 444 = 2 \times 222 = 2 \times 2 \times 3 \times 37$:

In this case, the possible combinations are

$$(m,n) := \{(37,6),(74,3),(111,2),(222,1)\}.$$

Based on these combinations, we have following Pythagorean triples:

$$\begin{aligned}
&(F_2(222,1), G_2(222,1), H_2(222,1)) \Rightarrow 49283^2 + 444^2 = 49285^2 \\
&(F_2(111,2), G_2(111,2), H_2(111,2)) \Rightarrow 12317^2 + 444^2 = 12325^2 \\
&(F_2(74,3), G_2(74,3), H_2(74,3)) \quad \Rightarrow 5467^2 + 444^2 = 5485^2 \\
&(F_2(37,6), G_2(37,6), H_2(37,6)) \quad \Rightarrow 1333^2 + 444^2 = 1405^2
\end{aligned}$$

By no means can we say that this process is complete as there are more Pythagorean triples with 444, for example: $333^2 + 444^2 = 555^2$.

Remark 7. *The possibilities appearing in (3.170) don't include the one given in (??) and (??).*

Let's write patterns based on above triples:

- **For $n = 1$; $m = 2220, 22200, 222000, ...$ in (3.47):**

$$\begin{aligned}
4928399^2 + 4440^2 &= 4928401^2 \\
492839999^2 + 44400^2 &= 492840001^2 \\
49283999999^2 + 444000^2 &= 49284000001^2 := 2428912656098568000001 \\
4928399999999^2 + 4440000^2 &= 4928400000001^2 := 24289126560009856800000001 \\
492839999999999^2 + 44400000^2 &= 492840000000001^2 := 242891265600000985680000000001
\end{aligned} \quad (3.162)$$

The first triple (49283, 444, 49285) for $n = 1$; $m = 222$ is not written above as it doesn't give good pattern.

- **For $n = 2$; $m = 1110, 11100, 111000, ...$ in (3.47):**

$$\begin{aligned}
1232096^2 + 4440^2 &= 1232104^2 \\
123209996^2 + 44400^2 &= 123210004^2 \\
12320999996^2 + 444000^2 &= 12321000004^2 := 151807041098568000016 \\
1232099999996^2 + 4440000^2 &= 1232100000004^2 := 1518070410009856800000016 \\
123209999999996^2 + 44400000^2 &= 123210000000004^2 := 15180704100000985680000000016
\end{aligned} \quad (3.163)$$

The first triple (12317, 444, 12325) for $n = 2$; $m = 111$ is not written above as it doesn't give good pattern.

► **Division by 2**

$$
\begin{aligned}
616048^2 + 2220^2 &= 616052^2 \\
61604998^2 + 22200^2 &= 61605002^2 \\
6160499998^2 + 222000^2 &= 6160500002^2 \\
616049999998^2 + 2220000^2 &= 616050000002^2 := 379517602502464200000004 \\
61604999999998^2 + 22200000^2 &= 61605000000002^2 := 37951760250002464200000000004
\end{aligned} \tag{3.164}
$$

▶ **Division by 4**

$$
\begin{aligned}
308024^2 + 1110^2 &= 308026^2 \\
30802499^2 + 11100^2 &= 30802501^2 \\
3080249999^2 + 111000^2 &= 3080250001^2 \\
308024999999^2 + 1110000^2 &= 308025000001^2 := 94879400625616050000001 \\
30802499999999^2 + 11100000^2 &= 30802500000001^2 := 948794006250061605000000001
\end{aligned} \tag{3.165}
$$

- **For $n = 3;\ m = 740, 7400, 74000, ...$ in (3.47):**

$$
\begin{aligned}
547591^2 + 4440^2 &= 547609^2 \\
54759991^2 + 44400^2 &= 54760009^2 \\
5475999991^2 + 444000^2 &= 5476000009^2 := 29986576098568000081 \\
547599999991^2 + 4440000^2 &= 547600000009^2 := 2998657600098568000000081 \\
54759999999991^2 + 44400000^2 &= 54760000000009^2 := 299865760000098568000000000081
\end{aligned} \tag{3.166}
$$

The first triple (5467, 444, 5485) for $n = 3;\ m = 74$ is not written above as it doesn't give good pattern.

- **For $n = 6;\ m = 37, 370, 3700, 37000, ...$ in (3.47):**

$$
\begin{aligned}
136864^2 + 4440^2 &= 136936^2 \\
13689964^2 + 44400^2 &= 13690036^2 \\
1368999964^2 + 444000^2 &= 1369000036^2 := 1874161098568001296 \\
136899999964^2 + 4440000^2 &= 136900000036^2 := 18741610009856800001296 \\
13689999999964^2 + 44400000^2 &= 13690000000036^2 := 187416100000985680000001296
\end{aligned} \tag{3.167}
$$

The first triple (1333, 444, 1405) for $n = 6;\ m = 37$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$
\begin{aligned}
68448^2 + 740^2 &= 68452^2 \\
6844998^2 + 7400^2 &= 6845002^2 \\
684499998^2 + 74000^2 &= 684500002^2 & := 468540252738000004 \\
68449999998^2 + 740000^2 &= 68450000002^2 & :=, 4685402500273800000004 \\
6844999999998^2 + 7400000^2 &=, 6845000000002^2 & := 46854025000027380000000004
\end{aligned} \tag{3.168}
$$

▶ **Division by 4**

$$
\begin{aligned}
3422499^2 + 3700^2 &= 3422501^2 \\
342249999^2 + 37000^2 &= 342250001^2 \\
34224999999^2 + 370000^2 &= 34225000001^2 & := 1171350625068450000001 \\
3422499999999^2 + 3700000^2 &= 3422500000001^2 & := 11713506250006845000000001 \\
342249999999999^2 + 37000000^2 &= 342250000000001^2 & := 117135062500000684500000000001
\end{aligned} \tag{3.169}
$$

The first triple $(34224, 370, 34226)$ is not written above as it doesn't give good pattern.

3.3 Procedure 3

For all $m, n \in N_+,\ \ 2m > n \geq 1$, let's consider the following three functions:

$$
\begin{aligned}
F_3(m,n) &:= 4m^2 - n^2 \\
G_3(m,n) &:= 4mn \\
H_3(m,n) &:= 4m^2 + n^2
\end{aligned} \tag{3.170}
$$

Then we can easily check that

$$
\begin{aligned}
F_3(m,n)^2 + G_3(m,n)^2 &= \left(4m^2 - n^2\right)^2 + (4mn)^2 \\
&= 16m^4 - 8m^2n^2 + n^4 + 16m^2n^2 \\
&= 16m^4 + 8m^2n^2 + n^4 \\
&= \left(4m^2 + n^2\right)^2 = H_3(m,n)^2.
\end{aligned}
$$

This proves that the triple (F_3, G_3, H_3) is a **Pythagorean triple** for all $m, n \in N_+,\ \ 2m > n \geq 1$. In particular, when $n = 1$, we get $F_3(m,1) = f_4(m),\ \ G_3(m,1) = g_4(m)$ and $H_3(m,1) = h_4(m)$, i.e.,

$$
(F_3(m,1), G_3(m,1), H_3(m,1)) = (f_4(m), g_4(m), h_4(m)),
$$

where

$$
\begin{aligned}
f_4(m) &:= 4m^2 - 1 \\
g_4(m) &:= 4m \\
h_4(m) &:= 4m^2 + 1
\end{aligned} \tag{3.171}
$$

Some patterned examples based on the Procedure (3.171) are studied by author [5]. Some particular cases of Procedure 3 are as follows:

$$\begin{aligned}
&(F_3(2,1), G_3(2,1), H_3(2,1)) &&\Rightarrow (15,8,17)\\
&(F_3(3,1), G_3(3,1), H_3(3,1)) &&\Rightarrow (35,12,37)\\
&(F_3(3,2), G_3(3,2), H_3(3,2)) &&\Rightarrow (32,24,40)\\
&(F_3(5,2), G_3(5,2), H_3(5,2)) &&\Rightarrow (96,40,104)\\
&(F_3(6,2), G_3(6,2), H_3(6,2)) &&\Rightarrow (140,48,148)\\
&(F_3(9,3), G_3(9,3), H_3(9,3)) &&\Rightarrow (315,108,333)\\
&(F_3(9,7), G_3(9,7), H_3(9,7)) &&\Rightarrow (275,252,373)\\
&(F_3(10,9), G_3(10,9), H(10,9)) &&\Rightarrow (319,360,481)
\end{aligned}$$

Below there are some patterned examples of this Procedure.

3.3.1 Examples

This subsection brings patterns based on Procedure (3.170) given in subsection 6.2.3. Below are some examples of patterns:

(i) For $n = 1, 2, 3, 4, 5;\ m = 3, 33, 333, 3333, 33333, ...$:

- **For $n = 1;\ m = 3, 33, 333, 3333, 33333, ...$ in (3.170):**

$$\begin{aligned}
35^2 + 12^2 &= 37^2\\
4355^2 + 132^2 &= 4357^2\\
443555^2 + 1332^2 &= 443557^2\\
44435555^2 + 13332^2 &= 44435557^2\\
4444355555^2 + 133332^2 &= 4444355557^2
\end{aligned} \tag{3.172}$$

- **For $n = 2;\ m = 3, 33, 333, 3333, 33333, ...$ in (3.170):**

$$\begin{aligned}
32^2 + 24^2 &= 40^2\\
4352^2 + 264^2 &= 4360^2\\
443552^2 + 2664^2 &= 443560^2\\
44435552^2 + 26664^2 &= 44435560^2\\
4444355552^2 + 266664^2 &= 4444355560^2
\end{aligned} \tag{3.173}$$

▶ **Division by 2**

$$
\begin{aligned}
16^2 + 12^2 &= 20^2\\
2176^2 + 132^2 &= 2180^2\\
221776^2 + 1332^2 &= 221780^2\\
22217776^2 + 13332^2 &= 22217780^2\\
2222177776,^2 + 133332^2 &= 2222177780^2\\
222221777776^2 + 1333332^2 &= 222221777780^2
\end{aligned} \tag{3.174}
$$

► **Division by 4**

$$
\begin{aligned}
8^2 + 6^2 &= 10^2\\
1088^2 + 66^2 &= 1090^2\\
110888^2 + 666^2 &= 110890^2\\
11108888^2 + 6666^2 &= 11108890^2\\
1111088888^2 + 66666^2 &= 1111088890^2\\
111110888888^2 + 666666^2 &= 111110888890^2
\end{aligned} \tag{3.175}
$$

► **Division by 8**

$$
\begin{aligned}
4^2 + 3^2 &= 5^2\\
544^2 + 33^2 &= 545^2\\
55444^2 + 333^2 &= 55445^2\\
5554444^2 + 3333^2 &= 5554445^2\\
555544444^2 + 33333^2 &= 555544445^2\\
55555444444^2 + 333333^2 &= 55555444445^2
\end{aligned} \tag{3.176}
$$

- **For $n = 3;\ m = 33, 333, 3333, 33333, \ldots$ in (3.170):**

$$
\begin{aligned}
4347^2 + 396^2 &= 4365^2\\
443547^2 + 3996^2 &= 443565^2\\
44435547^2 + 39996^2 &= 44435565^2\\
4444355547^2 + 399996^2 &= 4444355565^2
\end{aligned} \tag{3.177}
$$

The first triple $(27, 36, 45)$ for $n = 3;\ m = 3$ is not written above as it doesn't give good pattern.

- **For $n = 4;\ m = 33, 333, 3333, 33333, \ldots$ in (3.170):**

$$\begin{aligned} 4340^2 + 528^2 &= 4372^2 \\ 443540^2 + 5328^2 &= 443572^2 \\ 44435540^2 + 53328^2 &= 44435572^2 \\ 4444355540^2 + 533328^2 &= 4444355572^2 \end{aligned} \tag{3.178}$$

The first triple (20, 48, 52) for $n = 4;\ m = 3$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned} 2170^2 + 264^2 &= 2186^2 \\ 221770^2 + 2664^2 &= 221786^2 \\ 22217770^2 + 26664^2 &= 22217786^2 \\ 2222177770^2 + 266664^2 &= 2222177786^2 \\ 222221777770^2 + 2666664^2 &= 222221777786^2 \end{aligned} \tag{3.179}$$

▶ **Division by 4**

$$\begin{aligned} 5^2 + 12^2 &= 13^2 \\ 1085^2 + 132^2 &= 1093^2 \\ 110885^2 + 1332^2 &= 110893^2 \\ 11108885^2 + 13332^2 &= 11108893^2 \\ 1111088885^2 + 133332^2 &= 1111088893^2 \\ 111110888885^2 + 1333332^2 &= 111110888893^2 \end{aligned} \tag{3.180}$$

- **For $n = 5;\ m = 33, 333, 3333, 33333, \ldots$ in (3.170):**

$$\begin{aligned} 4331^2 + 660^2 &= 4381^2 \\ 443531^2 + 6660^2 &= 443581^2 \\ 44435531^2 + 66660^2 &= 44435581^2 \\ 4444355531^2 + 666660^2 &= 4444355581^2. \end{aligned} \tag{3.181}$$

The first triple (11, 60, 61) for $n = 5;\ m = 3$ is not written above as it doesn't give good pattern.

(ii) For $n = 1, 2, 3, ..., 11;\ m = 6, 66, 666, 6666, 66666, ...$:

- **For $n = 1;\ m = 6, 66, 666, 6666, 66666, ...$ in (3.170):**

$$\begin{aligned}
143^2 + 24^2 &= 145^2\\
17423^2 + 264^2 &= 17425^2\\
1774223^2 + 2664^2 &= 1774225^2\\
177742223^2 + 26664^2 &= 177742225^2\\
17777422223^2 + 266664^2 &= 17777422225^2
\end{aligned} \tag{3.182}$$

- **For $n = 2;\ m = 66, 666, 6666, 66666, ...$ in (3.170):**

$$\begin{aligned}
17420^2 + 528^2 &= 17428^2\\
1774220^2 + 5328^2 &= 1774228^2\\
177742220^2 + 53328^2 &= 177742228^2\\
17777422220^2 + 533328^2 &= 17777422228^2
\end{aligned} \tag{3.183}$$

The first triple (140, 48, 148) for $n = 2;\ m = 6$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
70^2 + 24^2 &= 74^2\\
8710^2 + 264^2 &= 8714^2\\
887110^2 + 2664^2 &= 887114^2\\
88871110^2 + 26664^2 &= 88871114^2\\
8888711110^2 + 266664^2 &= 8888711114^2\\
888887111110^2 + 2666664^2 &= 888887111114^2
\end{aligned} \tag{3.184}$$

▶ **Division by 4**

$$\begin{aligned}
35^2 + 12^2 &= 37^2\\
4355^2 + 132^2 &= 4357^2\\
443555^2 + 1332^2 &= 443557^2\\
44435555^2 + 13332^2 &= 44435557^2\\
4444355555^2 + 133332^2 &= 4444355557^2\\
444443555555^2 + 1333332^2 &= 444443555557^2
\end{aligned} \tag{3.185}$$

- **For $n = 3$; $m = 66, 666, 6666, 66666, \ldots$ in (3.170):**

$$\begin{aligned}
17415^2 + 792^2 &= 17433^2\\
1774215^2 + 7992^2 &= 1774233^2\\
177742215^2 + 79992^2 &= 177742233^2\\
17777422215^2 + 799992^2 &= 17777422233^2
\end{aligned} \tag{3.186}$$

The first triple (135, 72, 153) for $n = 3$; $m = 6$ is not written above as it doesn't give good pattern.

- **For $n = 4$; $m = 66, 666, 6666, 66666, \ldots$ in (3.170):**

$$\begin{aligned}
17408^2 + 1056^2 &= 17440^2\\
1774208^2 + 10656^2 &= 1774240^2\\
177742208^2 + 106656^2 &= 177742240^2\\
17777422208^2 + 1066656^2 &= 17777422240^2
\end{aligned} \tag{3.187}$$

The first triple (128, 96, 160) for $n = 4$; $m = 6$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
8704^2 + 528^2 &= 8720^2\\
887104^2 + 5328^2 &= 887120^2\\
88871104^2 + 53328^2 &= 88871120^2\\
8888711104^2 + 533328^2 &= 8888711120^2\\
888887111104^2 + 5333328^2 &= 888887111120^2
\end{aligned} \tag{3.188}$$

▶ **Division by 4**

$$\begin{aligned}
32^2 + 24^2 &= 40^2\\
4352^2 + 264^2 &= 4360^2\\
443552^2 + 2664^2 &= 443560^2\\
44435552^2 + 26664^2 &= 44435560^2\\
4444355552^2 + 266664^2 &= 4444355560^2\\
444443555552^2 + 2666664^2 &= 444443555560^2
\end{aligned} \tag{3.189}$$

▶ **Division by 8**

$$\begin{aligned}
16^2 + 12^2 &= 20^2\\
2176^2 + 132^2 &= 2180^2\\
221776^2 + 1332^2 &= 221780^2\\
22217776^2 + 13332^2 &= 22217780^2\\
2222217777776^2 + 1333332^2 &= 2222217777780^2
\end{aligned} \tag{3.190}$$

▶ **Division by 16**

$$\begin{aligned} 8^2+6^2 &= 10^2 \\ 1088^2+66^2 &= 1090^2 \\ 110888^2+666^2 &= 110890^2 \\ 11108888^2+6666^2 &= 11108890^2 \\ 1111088888^2+66666^2 &= 1111088890^2 \\ 111110888888^2+666666^2 &= 111110888890^2 \end{aligned} \tag{3.191}$$

▶ **Division by 32**

$$\begin{aligned} 4^2+3^2 &= 5^2 \\ 544^2+33^2 &= 545^2 \\ 55444^2+333^2 &= 55445^2 \\ 5554444^2+3333^2 &= 5554445^2 \\ 555544444^2+33333^2 &= 555544445^2 \\ 55555444444^2+333333^2 &= 55555444445^2 \end{aligned} \tag{3.192}$$

- **For $n=5;\ m=666, 6666, 66666, ...$ in (3.170):**

$$\begin{aligned} 1774199^2+13320^2 &= 1774249^2 \\ 177742199^2+133320^2 &= 177742249^2 \\ 17777422199^2+1333320^2 &= 17777422249^2 \\ 1777774222199^2+13333320^2 &= 1777774222249^2 \end{aligned} \tag{3.193}$$

The first two triple (119, 120, 169) and (17399, 1320, 17449) for $n=5;\ m=6$ and 66 are not written above as they don't give good pattern.

- **For $n=6;\ m=666, 6666, 66666, ...$ in (3.170):**

$$\begin{aligned} 1774188^2+15984^2 &= 1774260^2 \\ 177742188^2+159984^2 &= 177742260^2 \\ 17777422188^2+1599984^2 &= 17777422260^2 \\ 1777774222188^2+15999984^2 &= 1777774222260^2 \end{aligned} \tag{3.194}$$

The first two triple (108, 144, 180) and (17388, 1584, 17460) for $n=6;\ m=6$ and 66 are not written above as they don't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
887094^2 + 7992^2 &= 887130^2 \\
88871094^2 + 79992^2 &= 88871130^2 \\
8888711094^2 + 799992^2 &= 8888711130^2 \\
888887111094^2 + 7999992^2 &= 888887111130^2
\end{aligned} \tag{3.195}$$

▶ **Division by 4**

$$\begin{aligned}
4347^2 + 396^2 &= 4365^2 \\
443547^2 + 3996^2 &= 443565^2 \\
44435547^2 + 39996^2 &= 44435565^2 \\
4444355547^2 + 399996^2 &= 4444355565^2 \\
444443555547^2 + 3999996^2 &= 444443555565^2
\end{aligned} \tag{3.196}$$

- **For $n = 7$; $m = 666, 6666, 66666, ...$ in (3.170):**

$$\begin{aligned}
1774175^2 + 18648^2 &= 1774273^2 \\
177742175^2 + 186648^2 &= 177742273^2 \\
17777422175^2 + 1866648^2 &= 17777422273^2 \\
1777774222175^2 + 18666648^2 &= 1777774222273^2
\end{aligned} \tag{3.197}$$

The first two triple $(95, 168, 193)$ and $(17375, 1848, 17473)$ for $n = 7$; $m = 6$ and 66 are not written above as they don't give good pattern.

- **For $n = 8$ $m = 666, 6666, 66666, ...$ in (3.170):**

$$\begin{aligned}
1774160^2 + 21312^2 &= 1774288^2 \\
177742160^2 + 213312^2 &= 177742288^2 \\
17777422160^2 + 2133312^2 &= 17777422288^2 \\
1777774222160^2 + 21333312^2 &= 1777774222288^2
\end{aligned} \tag{3.198}$$

The first two triple $(80, 192, 208)$ and $(17360, 2112, 17488)$ for $n = 8$; $m = 6$ and 66 are not written above as they don't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
8680^2 + 1056^2 &= 8744^2 \\
887080^2 + 10656^2 &= 887144^2 \\
88871080^2 + 106656^2 &= 88871144^2 \\
8888711080^2 + 1066656^2 &= 8888711144^2 \\
888887111080^2 + 10666656^2 &= 888887111144^2
\end{aligned} \tag{3.199}$$

▶ **Division by 4**

$$\begin{aligned} 4340^2 + 528^2 &= 4372^2 \\ 443540^2 + 5328^2 &= 443572^2 \\ 44435540^2 + 53328^2 &= 44435572^2 \\ 4444355540^2 + 533328^2 &= 4444355572^2 \\ 444443555540^2 + 5333328^2 &= 444443555572^2 \end{aligned} \tag{3.200}$$

▶ **Division by 8**

$$\begin{aligned} 10^2 + 24^2 &= 26^2 \\ 2170^2 + 264^2 &= 2186^2 \\ 221770^2 + 2664^2 &= 221786^2 \\ 22217770^2 + 26664^2 &= 22217786^2 \\ 2222177770^2 + 266664^2 &= 2222177786^2 \\ 222221777770^2 + 2666664^2 &= 222221777786^2 \end{aligned} \tag{3.201}$$

▶ **Division by 16**

$$\begin{aligned} 5^2 + 12^2 &= 13^2 \\ 1085^2 + 132^2 &= 1093^2 \\ 110885^2 + 1332^2 &= 110893^2 \\ 11108885^2 + 13332^2 &= 11108893^2 \\ 1111088885,^2 + 133332^2 &= 1111088893^2 \\ 111110888885^2 + 1333332^2 &= 111110888893^2 \end{aligned} \tag{3.202}$$

- **For $n = 9;\ m = 666, 6666, 66666, ...$ in (3.170):**

$$\begin{aligned} 1774143^2 + 23976^2 &= 1774305^2 \\ 177742143^2 + 239976^2 &= 177742305^2 \\ 17777422143^2 + 2399976^2 &= 17777422305^2 \\ 1777774222143^2 + 23999976^2 &= 1777774222305^2 \end{aligned} \tag{3.203}$$

The first two triple (63, 216, 225) and (17343, 2376, 17505) for $n = 9;\ m = 6$ and 66 are not written above as they don't give good pattern.

- **For $n = 10$; $m = 666, 6666, 66666, ...$ in (3.170):**

$$\begin{aligned} 1774124^2 + 26640^2 &= 1774324^2 \\ 177742124^2 + 266640^2 &= 177742324^2 \\ 17777422124^2 + 2666640^2 &= 17777422324^2 \\ 1777774222124^2 + 26666640^2 &= 1777774222324^2 \end{aligned} \tag{3.204}$$

The first two triple (44, 240, 244) and (17324, 2640, 17524) for $n = 10$; $m = 6$ and 66 are not written above as they don't give good pattern.

▶ **Division by 2**

$$\begin{aligned} 8662^2 + 1320^2 &= 8762^2 \\ 887062^2 + 13320^2 &= 887162^2 \\ 88871062^2 + 133320^2 &= 88871162^2 \\ 8888711062^2 + 1333320^2 &= 8888711162^2 \\ 888887111062^2 + 13333320^2 &= 888887111162^2 \end{aligned} \tag{3.205}$$

▶ **Division by 4**

$$\begin{aligned} 4331^2 + 660^2 &= 4381^2 \\ 443531^2 + 6660^2 &= 443581^2 \\ 44435531^2 + 66660^2 &= 44435581^2 \\ 4444355531^2 + 666660^2 &= 4444355581^2 \\ 444443555531^2 + 6666660^2 &= 444443555581^2 \end{aligned} \tag{3.206}$$

- **For $n = 11$; $m = 666, 6666, 66666, ...$ in (3.170):**

$$\begin{aligned} 1774103^2 + 29304^2 &= 1774345^2 \\ 177742103^2 + 293304^2 &= 177742345^2 \\ 17777422103^2 + 2933304^2 &= 17777422345^2 \\ 1777774222103^2 + 29333304^2 &= 1777774222345^2 \end{aligned} \tag{3.207}$$

The first two triple (23, 264, 265) and (17303, 2904, 17545) for $n = 11$; $m = 6$ and 66 are not written above as they don't give good pattern.

(iii) For $n = 1, 2, 3, ..., 17$; $m = 9, 99, 999, 9999, 99999, ...$:

- **For $n = 1$; $m = 9, 99, 999, 9999, 99999, ...$ in (3.170):**

$$\begin{aligned}
323^2 + 36^2 &= 325^2 \\
39203^2 + 396^2 &= 39205^2 \\
3992003^2 + 3996^2 &= 3992005^2 &&:= 15936103920025 \\
399920003^2 + 39996^2 &= 399920005^2 &&:= 159936010399200025 \\
39999200003^2 + 399996^2 &= +39999200005^2 &&:= 1599936001039992000025 \\
3999992000003^2 + 3999996^2 &= 3999992000005^2 &&:= 15999936000103999920000025
\end{aligned} \tag{3.208}$$

- **For $n = 2$; $m = 9, 99, 999, 9999, 99999, ...$ in (3.170):**

$$\begin{aligned}
320^2 + 72^2 &= 328^2 \\
39200^2 + 792^2 &= 39208^2 \\
3992000^2 + 7992^2 &= 3992008^2 &&:= 15936127872064 \\
399920000^2 + 79992^2 &= 399920008^2 &&:= 159936012798720064 \\
39999200000^2 + 799992^2 &= 39999200008^2 &&:= 1599936001279987200064 \\
3999992000000^2 + 7999992^2 &= 3999992000008^2 &&:= 15999936000127999872000064
\end{aligned} \tag{3.209}$$

▶ Division by 2

$$\begin{aligned}
160^2 + 36^2 &= 164^2 \\
19600^2 + 396^2 &= 19604^2 \\
1996000^2 + 3996^2 &= 1996004^2 &&:= 3984031968016 \\
199960000^2 + 39996^2 &= 199960004^2 &&:= 39984003199680016 \\
19999600000^2 + 399996,^2 &= 19999600004^2 &&:= 399984000319996800016 \\
1999996000000^2 + 3999996^2 &= 1999996000004,^2 &&:= 3999984000031999968000016
\end{aligned} \tag{3.210}$$

▶ Division by 4

$$\begin{aligned}
80^2 + 18^2 &= 82^2 \\
9800^2 + 198^2 &= 9802^2 &&:= 96079204 \\
998000^2 + 1998^2 &= 998002^2 &&:= 996007992004 \\
99980000^2 + 19998^2 &= 99980002^2 &&:= 9996000799920004 \\
9999800000^2 + 199998^2 &= 9999800002^2 &&:= 99996000079999200004 \\
999998000000^2 + 1999998^2 &= 999998000002^2 &&:= 999996000007999992000004
\end{aligned} \tag{3.211}$$

▶ **Division by 8**

$$\begin{aligned}
40^2+9^2 &= 41^2 \\
4900^2+99^2 &= 4901^2 &&:= 24019801 \\
499000^2+999^2 &= 499001^2 &&:= 249001998001 \\
49990000^2+9999^2 &= 49990001^2 &&:= 2499000199980001 \\
4999900000^2+99999^2 &= 4999900001^2 &&:= 24999000019999800001 \\
499999000000^2+999999^2 &= 499999000001^2 &&:= 249999000001999998000001
\end{aligned} \tag{3.212}$$

- **For $n=3;\ m=99,999,9999,99999,...$ in (3.170):**

$$\begin{aligned}
39195^2+1188^2 &= 39213^2 \\
3991995^2+11988^2 &= 3992013^2 &&:= 15936167792169 \\
399919995^2+119988^2 &= 399920013^2 &&:= 159936016797920169 \\
39999199995^2+1199988^2 &= 39999200013^2 &&:= 1599936001679979200169 \\
3999991999995^2+11999988^2 &= 3999992000013^2 &&:= 15999936000167999792000169
\end{aligned} \tag{3.213}$$

The first triple $(315,108,333)$ for $n=3;\ m=9$ is not written above as it doesn't give good pattern.

- **For $n=4;\ m=99,999,9999,99999,...$ in (3.170):**

$$\begin{aligned}
39188^2+1584^2 &= 39220^2 \\
3991988^2+15984^2 &= 3992020^2 &&:= 15936223680400 \\
399919988^2+159984^2 &= 399920020^2 &&:= 159936022396800400 \\
39999199988^2+1599984^2 &= 39999200020^2 &&:= 1599936002239968000400 \\
3999991999988^2+15999984^2 &= 3999992000020^2 &&:= 15999936000223999680000400
\end{aligned} \tag{3.214}$$

The first triple $(308,144,340)$ for $n=4;\ m=9$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
19594^2+792^2 &= 19610^2 \\
1995994^2+7992^2 &= 1996010^2 &&:= 3984055920100 \\
199959994^2+79992^2 &= 199960010^2 &&:= 39984005599200100 \\
19999599994^2+799992^2 &= 19999600010^2 &&:= 399984000559992000100 \\
1999995999994^2+7999992^2 &= 1999996000010^2 &&:= 3999984000055999920000100
\end{aligned} \tag{3.215}$$

▶ **Division by 4**

$$
\begin{aligned}
77^2 + 36^2 &= 85^2 \\
9797^2 + 396^2 &= 9805^2 := 96138025 \\
997997^2 + 3996^2 &= 998005^2 := 996013980025 \\
99979997^2 + 39996^2 &= 99980005^2 := 9996001399800025 \\
9999799997^2 + 399996^2 &= 9999800005^2 := 99996000139998000025 \\
999997999997^2 + 3999996^2 &= 999998000005^2 := 999996000013999980000025
\end{aligned} \quad (3.216)
$$

- **For $n = 5;\ m = 99, 999, 9999, 99999, ...$ in (3.170):**

$$
\begin{aligned}
39179^2 + 1980^2 &= 39229^2 \\
3991979^2 + 19980^2 &= 3992029^2 := 15936295536841 \\
399919979^2 + 199980^2 &= 399920029^2 := 159936029595360841 \\
39999199979^2 + 1999980^2 &= 39999200029^2 := 1599936002959953600841 \\
3999991999979^2 + 19999980^2 &= 3999992000029^2 := 15999936000295999536000841
\end{aligned} \quad (3.217)
$$

The first triple $(299, 180, 349)$ for $n = 5;\ m = 9$ is not written above as it doesn't give good pattern.

- **For $n = 6;\ m = 99, 999, 9999, 99999, ...$ in (3.170):**

$$
\begin{aligned}
39168^2 + 2376^2 &= 39240^2 \\
3991968^2 + 23976^2 &= 3992040^2 := 15936383361600 \\
399919968^2 + 239976^2 &= 399920040^2 := 159936038393601600 \\
39999199968^2 + 2399976^2 &= 39999200040^2 := 1599936003839936001600 \\
3999991999968^2 + 23999976^2 &= 3999992000040^2 := 15999936000383999360001600
\end{aligned} \quad (3.218)
$$

The first triple $(288, 216, 360)$ for $n = 6;\ m = 9$ is not written above as it doesn't give good pattern.

- **Division by 2**

$$
\begin{aligned}
19584^2 + 1188^2 &= 19620^2 \\
1995984^2 + 11988^2 &= 1996020^2 := 3984095840400 \\
199959984^2 + 119988^2 &= 199960020^2 := 39984009598400400 \\
19999599984^2 + 1199988^2 &= 19999600020^2 := 399984000959984000400 \\
1999995999984^2 + 11999988^2 &= 1999996000020^2 := 3999984000095999840000400
\end{aligned} \quad (3.219)
$$

- **Division by 4**

$$\begin{aligned}
72^2 + 54^2 &= 90^2 \\
9792^2 + 594^2 &= 9810^2 &&:= 96236100 \\
997992^2 + 5994^2 &= 998010^2 &&:= 996023960100 \\
99979992^2 + 59994^2 &= 99980010^2 &&:= 9996002399600100 \\
9999799992^2 + 599994^2 &= 9999800010^2 &&:= 99996000239996000100 \\
999997999992^2 + 5999994^2 &= 999998000010^2 &&:= 999996000023999960000100
\end{aligned} \tag{3.220}$$

▶ **Division by 8**

$$\begin{aligned}
4896^2 + 297^2 &= 4905^2 &&:= 24059025 \\
498996^2 + 2997^2 &= 499005^2 &&:= 249005990025 \\
49989996^2 + 29997^2 &= 49990005^2 &&:= 2499000599900025 \\
4999899996^2 + 299997^2 &= 4999900005^2 &&:= 24999000059999000025 \\
499998999996^2 + 2999997^2 &= 499999000005^2 &&:= 249999000005999990000025
\end{aligned} \tag{3.221}$$

- **For $n = 7;\ m = 99, 999, 9999, 99999, ...$ in (3.170):**

$$\begin{aligned}
39155^2 + 2772^2 &= 39253^2 \\
3991955^2 + 27972^2 &= 3992053^2 \\
399919955^2 + 279972^2 &= 399920053^2 &&:= 159936048791522809 \\
39999199955^2 + 2799972^2 &= 39999200053^2 &&:= 1599936004879915202809 \\
3999991999955^2 + 27999972^2 &= 3999992000053^2 &&:= 15999936000487999152002809
\end{aligned} \tag{3.222}$$

The first triple (275, 252, 373) for $n = 7;\ m = 9$ is not written above as it doesn't give good pattern.

- **For $n = 8;\ m = 99, 999, 9999, 99999, ...$ in (3.170):**

$$\begin{aligned}
39140^2 + 3168^2 &= 39268^2 \\
3991940^2 + 31968^2 &= 3992068^2 \\
399919940^2 + 319968^2 &= 399920068^2 &&:= 159936060789124624 \\
39999199940^2 + 3199968^2 &= 39999200068^2 &&:= 1599936006079891204624 \\
3999991999940^2 + 31999968^2 &= 3999992000068^2 &&:= 15999936000607998912004624
\end{aligned} \tag{3.223}$$

The first triple (260, 288, 388) for $n = 8;\ m = 9$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
19570^2 + 1584^2 &= 19634^2 \\
1995970^2 + 15984^2 &= 1996034^2 \\
199959970^2 + 159984^2 &= 199960034^2 &&:= 39984015197281156 \\
19999599970^2 + 1599984^2 &= 19999600034^2 &&:=, 399984001519972801156 \\
1999995999970^2 + 15999984^2 &= 1999996000034^2 &&:= 3999984000151999728001156
\end{aligned} \tag{3.224}$$

▶ **Division by 4**

$$\begin{aligned}
9785^2 + 792^2 &= 9817^2 \\
997985^2 + 7992^2 &= 998017^2 && := 996037932289 \\
99979985^2 + 79992^2 &= 99980017^2 && := 9996003799320289 \\
9999799985^2 + 799992^2 &= 9999800017^2 && := 99996000379993200289 \\
999997999985^2 + 7999992^2 &= 999998000017^2 && := 999996000037999932000289
\end{aligned} \tag{3.225}$$

- **For $n = 9;\ m = 999, 9999, 99999, ...$ in (3.170):**

$$\begin{aligned}
3991923^2 + 35964^2 &= 3992085^2 \\
399919923^2 + 359964^2 &= 399920085^2 && := 159936074386407225 \\
39999199923^2 + 3599964^2 &= 39999200085^2 && := 1599936007439864007225 \\
3999991999923^2 + 35999964^2 &= 3999992000085^2 && := 15999936000743998640007225
\end{aligned} \tag{3.226}$$

The first two triple (243, 324, 405) and (39123, 3564, 39285) for $n = 9;\ m = 9$ and 99 are not written above as they don't give good pattern.

- **For $n = 10;\ m = 999, 9999, 99999, ...$ in (3.170):**

$$\begin{aligned}
3991904^2 + 39960^2 &= 3992104^2 \\
399919904^2 + 399960^2 &= 39920104^2 && := 159936089583370816 \\
39999199904^2 + 3999960^2 &= 39999200104^2 && := 1599936008959833610816 \\
3999991999904^2 + 39999960^2 &= 3999992000104^2 && := 15999936000895998336010816
\end{aligned} \tag{3.227}$$

The first two triple (224, 360, 424) and (39104, 3960, 39304) for $n = 10;\ m = 9$ and 99 are not written above as they don't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
19552^2 + 1980^2 &= 19652^2 \\
1995952^2 + 19980^2 &= 1996052^2 \\
199959952^2 + 199980^2 &= 199960052^2 && := 39984022395842704 \\
19999599952^2 + 1999980^2 &= 19999600052^2 && := 3999840022399584027 04 \\
1999995999952^2 + 19999980^2 &= 1999996000052^2 && := 399998400022399958 4002704
\end{aligned} \tag{3.228}$$

▶ **Division by 4**

$$
\begin{aligned}
9776^2 + 990^2 &= 9826^2 \\
997976^2 + 9990^2 &= 998026^2 &&:= 996055896676 \\
99979976^2 + 99990^2 &= 99980026^2 &&:= 9996005598960676 \\
9999799976^2 + 999990^2 &= 9999800026^2 &&:= 99996000559989600676 \\
999997999976^2 + 9999990^2 &= 999998000026^2 &&:= 999996000055999896000676
\end{aligned} \tag{3.229}
$$

▶ **Division by 8**

$$
\begin{aligned}
4888^2 + 495^2 &= 4913^2 \\
498988^2 + 4995^2 &= 499013^2 &&:= 249013974169 \\
49989988^2 + 49995^2 &= 49990013^2 &&:= 2499001399740169 \\
4999899988^2 + 499995^2 &= 4999900013^2 &&:= 24999000139997400169 \\
499998999988^2 + 4999995^2 &= 499999000013^2 &&:= 249999000013999974000169
\end{aligned} \tag{3.230}
$$

- **For $n = 11;\ m = 999, 9999, 99999, ...$ in (3.170):**

$$
\begin{aligned}
3991883^2 + 43956^2 &= 3992125^2 \\
399919883^2 + 439956^2 &= 399920125^2 &&:= 159936106380015625 \\
39999199883^2 + 4399956^2 &= 39999200125^2 &&:= 1599936010639800015625 \\
3999991999883^2 + 43999956^2 &= 3999992000125^2 &&:= 15999936001063998000015625
\end{aligned} \tag{3.231}
$$

The first two triple (203, 396, 445) and (39083, 4356, 39325) for $n = 11;\ m = 9$ and 99 are not written above as they don't give good pattern.

- **For $n = 12;\ m = 999, 9999, 99999, ...$ in (3.170):**

$$
\begin{aligned}
3991860^2 + 47952^2 &= 3992148^2 \\
399919860^2 + 479952^2 &= 399920148^2 \\
39999199860^2 + 4799952^2 &= 39999200148^2 &&:= 1599936012479763221904 \\
3999991999860^2 + 47999952^2 &= 3999992000148^2 &&:= 15999936001247997632021904
\end{aligned} \tag{3.232}
$$

The first two triple (180, 432, 468) and (39060, 4752, 39348) for $n = 12;\ m = 9$ and 99 are not written above as they don't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
19530^2+2376^2 &= 19674^2\\
1995930^2+23976^2 &= 1996074^2\\
199959930^2+239976^2 &= 199960074^2 &:= 39984031194085476\\
19999599930^2+2399976^2 &= 19999600074^2 &:= 3999840031199940805476\\
1999995999930^2+23999976^2 &= 1999996000074^2 &:= 399998400031199994080054767
\end{aligned} \tag{3.233}$$

▶ **Division by 4**

$$\begin{aligned}
9765^2+1188^2 &= 9837^2\\
997965^2+11988^2 &= 998037^2\\
99979965^2+119988^2 &= 99980037^2 &:= 9996007798521369\\
9999799965^2+1199988^2 &= 9999800037^2 &:= 99996000779985201369\\
999997999965^2+11999988^2 &= 999998000037^2 &:= 999996000077999852001369
\end{aligned} \tag{3.234}$$

- **For $n=13$; $m=999,9999,99999,...$ in (3.170):**

$$\begin{aligned}
3991835^2+51948^2 &= 3992173^2\\
399919835^2+519948^2 &= 399920173^2\\
39999199835^2+5199948^2 &= 39999200173^2 &:= 1599936014479723229929\\
3999991999835^2+51999948^2 &= 3999992000173^2 &:= 15999936001447997232029929
\end{aligned} \tag{3.235}$$

The first two triple (155, 468, 493) and (39035, 5148, 39373) for $n=13$; $m=9$ and 99 are not written above as they don't give good pattern.

- **For $n=14$; $m=999,9999,99999,...$ in (3.170):**

$$\begin{aligned}
3991808^2+55944^2 &= 3992200^2\\
399919808^2+559944^2 &= 399920200^2\\
39999199808^2+5599944^2 &= 39999200200^2 &:= 1599936016639680040000\\
3999991999808^2+55999944^2 &= 3999992000200^2 &:= 15999936001663996800040000
\end{aligned} \tag{3.236}$$

The first two triple (128, 504, 520) and (39008, 5544, 39400) for $n=14$; $m=9$ and 99 are not written above as they don't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
19504^2+2772^2 &= 19700^2\\
1995904^2+27972^2 &= 1996100^2 &:= 3984415210000\\
199959904^2+279972,^2 &= 199960100^2 &:= 39984041592010000\\
19999599904^2+2799972^2 &= 19999600100^2 &:= 399984004159920010000\\
1999995999904^2+27999972^2 &= 1999996000100^2 &:= 9999840004159992000010000
\end{aligned} \tag{3.237}$$

▶ **Division by 4**

$$
\begin{aligned}
9752^2 + 1386^2 &= 9850^2 \\
997952^2 + 13986,^2 &= 998050^2 &&:= 996103802500 \\
99979952^2 + 139986^2 &= 99980050^2 &&:= 9996010398002500 \\
9999799952^2 + 1399986^2 &= 9999800050^2 &&:= 99996001039980002500 \\
999997999952^2 + 13999986^2 &= 999998000050^2 &&:= 999996000103999800002500
\end{aligned} \tag{3.238}
$$

▶ **Division by 8**

$$
\begin{aligned}
4876^2 + 693^2 &= 4925^2 \\
498976^2 + 6993^2 &= 499025^2 &&:= 249025950625 \\
49989976^2 + 69993^2 &= 49990025^2 &&:= 2499002599500625 \\
4999899976^2 + 699993^2 &= 4999900025^2 &&:= 24999000259995000625 \\
499998999976^2 + 6999993^2 &= 499999000025^2 &&:= 249999000025999950000625
\end{aligned} \tag{3.239}
$$

- **For $n = 15;\ m = 999, 9999, 99999, ...$ in (3.170):**

$$
\begin{aligned}
3991779^2 + 59940^2 &= 3992229^2 \\
399919779^2 + 599940^2 &= 399920229^2 \\
39999199779^2 + 5999940^2 &= 39999200229^2 &&:= 1599936018959633652441 \\
3999991999779^2 + 59999940^2 &= 3999992000229^2 &&:= 15999936001895996336052441
\end{aligned} \tag{3.240}
$$

The first two triple $(99, 540, 549)$ and $(38979, 5940, 39429)$ for $n = 15;\ m = 9$ and 99 are not written above as they don't give good pattern.

- **For $n = 16;\ m = 999, 9999, 99999, ...$ in (3.170):**

$$
\begin{aligned}
3991748^2 + 63936^2 &= 3992260^2 \\
399919748^2 + 639936^2 &= 399920260^2 \\
39999199748^2 + 6399936^2 &= 39999200260^2 &&:= 1599936021439584067600 \\
3999991999748^2 + 63999936^2 &= 3999992000260^2 &&:= 15999936002143995840067600
\end{aligned} \tag{3.241}
$$

The first two triple $68^2, 576, 580)$ and $(38948, 6336, 39460)$ for $n = 16;\ m = 9$ and 99 are not written above as they don't give good pattern.

▶ **Division by 2**

$$
\begin{aligned}
19474^2 + 3168^2 &= 19730^2 \\
1995874^2 + 31968^2 &= 1996130^2 \\
199959874^2 + 319968^2 &= 199960130^2 := 39984053589616900 \\
19999599874^2 + 3199968^2 &= 19999600130^2 := 399984005359896016900 \\
1999995999874^2 + 31999968^2 &= 1999996000130^2 := 3999984000535998960016900
\end{aligned} \tag{3.242}
$$

▶ **Division by 4**

$$
\begin{aligned}
9737^2 + 1584^2 &= 9865^2 \\
997937^2 + 15984^2 &= 998065^2 := 996133744225 \\
99979937^2 + 159984^2 &= 99980065^2 := 9996013397404225 \\
9999799937^2 + 1599984^2 &= 9999800065^2 := 99996001339974004225 \\
999997999937^2 + 15999984^2 &= 999998000065^2 :=, 999996000133999740004225
\end{aligned} \tag{3.243}
$$

- **For $n = 17;\ m = 999, 9999, 99999, \ldots$ in (3.170):**

$$
\begin{aligned}
3991715^2 + 67932^2 &= 3992293^2 \\
399919715^2 + 679932^2 &= 399920293^2 \\
39999199715^2 + 6799932^2 &= 39999200293^2 := 1599936024079531285849 \\
3999991999715^2 + 67999932^2 &= 3999992000293^2 := 15999936002407995312085849
\end{aligned} \tag{3.244}
$$

The first two triple 35, 612, 613) and (38915, 6732, 39493) for $n = 17;\ m = 9$ and 99 are not written above as they don't give good pattern.

3.4 Procedure 4

For all $m, n \in N_+,\ \ 4\,m > n \geq 1$, let's consider the following three functions:

$$
\begin{aligned}
F_4(m,n) &:= 16m^2 - n^2 \\
G_4(m,n) &:= 8mn \\
H_4(m,n) &:= 16m^2 + n^2
\end{aligned} \tag{3.245}
$$

Then we can easily check that

$$
\begin{aligned}
F_4(m,n)^2 + G_4(m,n)^2 &= \left(16m^2 - n^2\right)^2 + (8mn)^2 \\
&= 256m^4 - 32m^2n^2 + n^4 + 64m^2n^2 \\
&= 256m^4 + 32m^2n^2 + n^4 \\
&= \left(16m^2 + n^2\right)^2 = H_4(m,n)^2.
\end{aligned}
$$

This proves that the triple (F_4, G_4, H_4) is a **Pythagorean triple** for all $m, n \in N_+,\ \ 4\,m > n \geq 1$. When $n = 1$, we get $F_4(m,1) = f_5(m),\ \ G_4(m,1) = g_5(m)$ and $H_4(m,1) = h_5(m)$, i.e.,

$$(F_4(m,1), G_4(m,1), H_4(m,1)) = (f_5(m), g_5(m), h_5(m)),$$

where

$$\begin{aligned} f_5(m) &:= 16m^2 - 1 \\ g_5(m) &:= 8m \\ h_5(m) &:= 16m^2 + 1 \end{aligned} \tag{3.246}$$

Some patterned examples based on the Procedure (3.246) are studied by author [5]. Some particular cases of Procedure 4 are as follows:

$$\begin{aligned}
(F_4(2,1), G_4(2,1), H_4(2,1)) &\Rightarrow (63, 16, 65) \\
(F_4(3,1), G_4(3,1), H_4(3,1)) &\Rightarrow (143, 24, 145) \\
(F_4(3,2), G_4(3,2), H_4(3,2)) &\Rightarrow (140, 48, 148) \\
(F_4(5,2), G_4(5,2), H_4(5,2)) &\Rightarrow (396, 80, 404) \\
(F_4(6,2), G_4(6,2), H_4(6,2)) &\Rightarrow (572, 96, 580) \\
(F_4(9,3), G_4(9,3), H_4(9,3)) &\Rightarrow (1287, 216, 1305) \\
(F_4(9,7), G_4(9,7), H_4(9,7)) &\Rightarrow (1247, 504, 1345) \\
(F_4(10,9), G_4(10,9), H_4(10,9)) &\Rightarrow (1519, 720, 1681)
\end{aligned}$$

If we double the above values m we get the same results for the triples (F_3, G_3, H_3). See below:

$$\begin{aligned}
(F_4(2,1), G_4(2,1), H_4(2,1)) &= (F_3(4,1), G_3(4,1), H_3(4,1)) \Rightarrow (63, 16, 65) \\
(F_4(3,1), G_4(3,1), H_4(3,1)) &= (F_3(6,1), G_3(6,1), H_3(6,1)) \Rightarrow (143, 24, 145) \\
(F_4(3,2), G_4(3,2), H_4(3,2)) &= (F_3(6,2), G_3(6,2), H_3(6,2)) \Rightarrow (140, 48, 148) \\
(F_4(5,2), G_4(5,2), H_4(5,2)) &= (F_3(10,2), G_3(10,2), H_3(10,2)) \Rightarrow (396, 80, 404) \\
(F_4(6,2), G_4(6,2), H_4(6,2)) &= (F_3(12,2), G_3(12,2), H_3(12,2)) \Rightarrow (572, 96, 580) \\
(F_4(9,3), G_4(9,3), H_4(9,3)) &= (F_3(18,3), G_3(18,3), H_3(18,3)) \Rightarrow (1287, 216, 1305) \\
(F_4(9,7), G_4(9,7), H_4(9,7)) &= (F_3(18,7), G_3(18,7), H_3(18,7)) \Rightarrow (1247, 504, 1345) \\
(F_4(10,9), G_4(10,9), H_4(10,9)) &= (F_3(20,9), G_3(20,9), H_3(20,9)) \Rightarrow (1519, 720, 1681)
\end{aligned}$$

Equivalently, we have

$$(F_4(m,n), G_4(m,n), H_4(m,n)) = (F_3(2m,n), G_3(2m,n), H_3(2m,n))$$

Below are patterned examples of this Procedure.

This subsection brings patterns based on Procedure (3.245) given in subsection 3.4. Below are some examples of patterns:

(i) For $n = 1, 2, 3, 4, 5;\ m = 6, 60, 600, 6000, 60000, ...$:

- **For $n = 1;\ m = 60, 600, 6000, 60000, ...$ in (3.245):**

$$\begin{aligned}
57599^2 + 480^2 &= 57601^2 \\
5759999^2 + 4800^2 &= 5760001^2 := 33177611520001 \\
575999999^2 + 48000^2 &= 576000001^2 := 331776001152000001 \\
57599999999^2 + 480000^2 &= 57600000001^2 := 3317760000115200000001 \\
5759999999999^2 + 4800000^2 &= 5760000000001^2 := 33177600000011520000000001
\end{aligned} \tag{3.247}$$

The first triple $(575, 48, 577)$ for $n = 1;\ m = 6$ is not written above as it doesn't give good pattern.

- **For $n = 2;\ m = 60, 600, 6000, 60000, ...$ in (3.245):**

$$\begin{aligned}
57596^2 + 960^2 &= 57604^2 \\
5759996^2 + 9600^2 &= 5760004^2 &&:= 33177646080016 \\
575999996^2 + 96000^2 &= 576000004^2 &&:= 331776004608000016 \\
57599999996^2 + 960000^2 &= 57600000004^2 &&:= 3317760000460800000016 \\
5759999999996^2 + 9600000^2 &= 5760000000004^2 &&:= 33177600000046080000000016
\end{aligned} \quad (3.248)$$

The first triple $(572, 96, 580)$ for $n = 2;\ m = 6$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
28798^2 + 480^2 &= 28802^2 \\
2879998^2 + 4800^2 &= 2880002^2 &&:= 16588823040008 \\
287999998^2 + 48000^2 &= 288000002^2 &&:= 165888002304000008 \\
28799999998^2 + 480000^2 &= 28800000002^2 &&:= 1658880000230400000008 \\
2879999999998^2 + 4800000^2 &= 2880000000002^2 &&:= 16588800000023040000000008 \\
287999999999998^2 + 48000000^2 &= 288000000000002^2 &&:= 165888000000002304000000000008
\end{aligned} \quad (3.249)$$

▶ **Division by 4**

$$\begin{aligned}
14399^2 + 240^2 &= 14401^2 \\
1439999^2 + 2400^2 &= 1440001^2 &&:= 4147205760002 \\
143999999^2 + 24000^2 &= 144000001^2 &&:= 41472000576000002 \\
14399999999^2 + 240000^2 &= 14400000001^2 &&:= 414720000057600000002 \\
1439999999999^2 + 2400000^2 &= 1440000000001^2 &&:= 4147200000005760000000002 \\
143999999999999^2 + 24000000^2 &= 144000000000001^2 &&:= 41472000000000576000000000002
\end{aligned} \quad (3.250)$$

- **For $n = 3;\ m = 60, 600, 6000, 60000, ...$ in (3.245):**

$$\begin{aligned}
57591^2 + 1440^2 &= 57609^2 \\
5759991^2 + 14400^2 &= 5760009^2
\end{aligned} \quad (3.251)$$

$$\begin{aligned}
575999991^2 + 144000^2 &= 576000009^2 &&:= 331776010368000081 \\
57599999991^2 + 1440000^2 &= 57600000009^2 &&:= 3317760001036800000081 \\
5759999999991^2 + 14400000^2 &= 5760000000009^2 &&:= 33177600000103680000000081
\end{aligned} \quad (3.252)$$

The first triple $(567, 144, 585)$ for $n = 3;\ m = 6$ is not written above as it doesn't give good pattern.

- **For $n = 4$; $m = 60, 600, 6000, 60000, ...$ in (3.245):**

$$\begin{aligned} 57584^2 + 1920^2 &= 57616^2 \\ 5759984^2 + 19200^2 &= 5760016^2 \\ 575999984^2 + 192000^2 &= 576000016^2 &:= 331776018432000256 \\ 57599999984^2 + 1920000^2 &= 57600000016^2 &:= 3317760001843200000256 \\ 5759999999984^2 + 19200000^2 &= 5760000000016^2 &:= 33177600000184320000000256 \end{aligned} \tag{3.253}$$

The first triple $(560, 192, 592)$ for $n = 4$; $m = 6$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned} 28792^2 + 960^2 &= 28808^2 \\ 2879992^2 + 9600^2 &= 2880008^2 &:= 4147223040032 \\ 287999992^2 + 96000^2 &= 288000008^2 &:= 41472002304000032 \\ 28799999992^2 + 960000^2 &= 28800000008^2 &:= 414720000230400000032 \\ 2879999999992^2 + 9600000^2 &= 2880000000008^2 &:= 4147200000023040000000032 \\ 287999999999992^2 + 96000000^2 &= 288000000000008^2 &:= 41472000000002304000000000032 \end{aligned} \tag{3.254}$$

▶ **Division by 4**

$$\begin{aligned} 14396^2 + 480^2 &= 14404^2 \\ 1439996^2 + 4800^2 &= 1440004^2 &:= 4147223040032 \\ 143999996^2 + 48000^2 &= 144000004^2 &:= 41472002304000032 \\ 14399999996^2 + 480000^2 &= 14400000004^2 &:= 414720000230400000032 \\ 1439999999996^2 + 4800000^2 &= 1440000000004^2 &:= 4147200000023040000000032 \\ 143999999999996^2 + 48000000^2 &= 144000000000004^2 &:= 41472000000002304000000000032 \end{aligned} \tag{3.255}$$

▶ **Division by 8**

$$\begin{aligned} 7198^2 + 240^2 &= 7202^2 \\ 719998^2 + 2400^2 &= 720002^2 &:= 1036805760008 \\ 71999998^2 + 24000^2 &= 72000002^2 &:= 10368000576000008 \\ 7199999998^2 + 240000^2 &= 7200000002^2 &:= 103680000057600000008 \\ 719999999998^2 + 2400000^2 &= 720000000002^2 &:= 1036800000005760000000008 \\ 71999999999998^2 + 24000000^2 &= 72000000000002^2 &:= 10368000000000576000000000008 \end{aligned} \tag{3.256}$$

▶ **Division by 16**

$$
\begin{aligned}
3599^2 + 120^2 &= 3601^2 \\
359999^2 + 1200^2 &= 360001^2 && := 259201440002 \\
35999999^2 + 12000^2 &= 36000001^2 && := 2592000144000002 \\
3599999999^2 + 120000^2 &= 3600000001^2 && := 25920000014400000002 \\
359999999999^2 + 1200000^2 &= 360000000001^2 && := 259200000001440000000002 \\
35999999999999^2 + 12000000^2 &= 36000000000001^2 && := 2592000000000144000000000002
\end{aligned} \tag{3.257}
$$

- **For $n = 5;\ m = 60, 600, 6000, 60000, ...$ in (3.245):**

$$
\begin{aligned}
57575^2 + 2400^2 &= 57625^2 \\
5759975^2 + 24000^2 &= 5760025^2 \\
575999975^2 + 240000^2 &= 576000025^2 && := 331776028800000625 \\
57599999975^2 + 2400000^2 &= 57600000025^2 && := 3317760002880000000625 \\
5759999999975^2 + 24000000^2 &= 5760000000025^2 && := 33177600000288000000000625
\end{aligned} \tag{3.258}
$$

The first triple $(551, 240, 601)$ for $n = 5;\ m = 6$ is not written above as it doesn't give good pattern.

(ii) For $n = 1, 2, 3, 4, 5;\ m = 7, 70, 700, 7000, 70000, ...$:

- **For $n = 1;\ m = 70, 700, 7000, 70000, ...$ in (3.245):**

$$
\begin{aligned}
78399^2 + 560^2 &= 78401^2 \\
7839999^2 + 5600^2 &= 7840001^2 && := 61465615680001 \\
783999999^2 + 56000^2 &= 784000001^2 && := 614656001568000001 \\
78399999999^2 + 560000^2 &= 78400000001^2 && := 6146560000156800000001 \\
7839999999999^2 + 5600000^2 &= 7840000000001^2 && := 61465600000015680000000001
\end{aligned} \tag{3.259}
$$

The first triple $(783, 56, 7852)$ for $n = 1;\ m = 7$ is not written above as it doesn't give good pattern.

- **For $n = 2;\ m = 70, 700, 7000, 70000, ...$ in (3.245):**

$$
\begin{aligned}
78396^2 + 1120^2 &= 78404^2 \\
7839996^2 + 11200^2 &= 7840004^2 && := 61465662720016 \\
783999996^2 + 112000^2 &= 784000004^2 && := 614656006272000016 \\
78399999996^2 + 1120000^2 &= 78400000004^2 && := 6146560000627200000016 \\
7839999999996^2 + 11200000^2 &= 7840000000004^2 && := 61465600000062720000000016
\end{aligned} \tag{3.260}
$$

The first triple $(780, 112, 788)$ for $n = 2;\ m = 7$ is not written above as it doesn't give good pattern.

► **Division by 2**

$$\begin{aligned}
39198^2 + 560^2 &= 39202^2 \\
3919998^2 + 5600^2 &= 3920002^2 &&:= 30732831360008 \\
391999998^2 + 56000^2 &= 392000002^2 &&:= 3073280031360000008 \\
39199999998^2 + 560000^2 &= 39200000002^2 &&:= 307328000313600000008 \\
3919999999998^2 + 5600000^2 &= 3920000000002^2 &&:= 30732800000031360000000008 \\
391999999999998^2 + 56000000^2 &= 392000000000002^2 &&:= 3073280000000313600000000008 \qquad (3.261)
\end{aligned}$$

▶ **Division by 4**

$$\begin{aligned}
19599^2 + 280^2 &= 19601^2 \\
1959999^2 + 2800^2 &= 1960001^2 &&:= 7683207840002 \\
195999999^2 + 28000^2 &= 196000001^2 &&:= 76832000784000002 \\
19599999999^2 + 280000^2 &= 19600000001^2 &&:= 768320000078400000002 \\
1959999999999^2 + 2800000^2 &= 1960000000001^2 &&:= 7683200000007840000000002 \\
195999999999999^2 + 28000000^2 &= 196000000000001^2 &&:= 76832000000000784000000000002 \qquad (3.262)
\end{aligned}$$

- **For $n = 3;\ m = 70, 700, 7000, 70000, ...$ in (3.245):**

$$\begin{aligned}
78391^2 + 1680^2 &= 78409^2 \\
7839991^2 + 16800^2 &= 7840009^2 \\
783999991^2 + 168000^2 &= 784000009^2 &&:= 614656014112000081 \\
78399999991^2 + 1680000^2 &= 78400000009^2 &&:= 6146560001411200000081 \\
7839999999991^2 + 16800000^2 &= 7840000000009^2 &&:= 61465600000141120000000081 \qquad (3.263)
\end{aligned}$$

The first triple $(775, 168, 793)$ for $n = 3;\ m = 6$ is not written above as it doesn't give good pattern.

- **For $n = 4;\ m = 70, 700, 7000, 70000, ...$ in (3.245):**

$$\begin{aligned}
78384^2 + 2240^2 &= 78416^2 \\
7839984^2 + 22400^2 &= 7840016^2 \\
783999984^2 + 224000^2 &= 784000016^2 &&:= 614656025088000256 \\
78399999984^2 + 2240000^2 &= 78400000016^2 &&:= 6146560002508800000256 \\
7839999999984^2 + 22400000^2 &= 7840000000016^2 &&:= 61465600000250880000000256 \qquad (3.264)
\end{aligned}$$

The first triple $(768, 224, 800)$ for $n = 4;\ m = 7$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
39192^2 + 1120^2 &= 39208^2 \\
3919992^2 + 11200^2 &= 3920008^2 &&:= 15366462720064 \\
391999992^2 + 112000^2 &= 392000008^2 &&:= 153664006272000064 \\
39199999992^2 + 1120000^2 &= 39200000008^2 &&:= 1536640000627200000064 \\
3919999999992^2 + 11200000^2 &= 3920000000008^2 &&:= 15366400000062720000000064 \qquad (3.265)
\end{aligned}$$

▶ **Division by 4**

$$\begin{aligned}
19596^2 + 560^2 &= 19604^2 \\
1959996^2 + 5600^2 &= 1960004^2 &&:= 3841615680016 \\
195999996^2 + 56000^2 &= 196000004^2 &&:= 38416001568000016 \\
19599999996^2 + 560000^2 &= 19600000004^2 &&:= 384160000156800000016 \\
1959999999996^2 + 5600000^2 &= 1960000000004^2 &&:= 3841600000015680000000016 \qquad (3.266)
\end{aligned}$$

▶ **Division by 8**

$$\begin{aligned}
9798^2 + 280^2 &= 9802^2 \\
979998^2 + 2800^2 &= 980002^2 &&:= 960403920004 \\
97999998^2 + 28000^2 &= 98000002^2 &&:= 9604000392000004 \\
9799999998^2 + 280000^2 &= 9800000002^2 &&:= 96040000039200000004 \\
979999999998^2 + 2800000^2 &= 980000000002^2 &&:= 960400000003920000000004 \qquad (3.267)
\end{aligned}$$

▶ **Division by 16**

$$\begin{aligned}
4899^2 + 140^2 &= 4901^2 &&:= 24019801 \\
489999^2 + 1400^2 &= 490001^2 &&:= 240100980001 \\
48999999^2 + 14000^2 &= 49000001^2 &&:= 2401000098000001 \\
4899999999^2 + 140000^2 &= 4900000001^2 &&:= 24010000009800000001 \\
489999999999^2 + 1400000^2 &= 490000000001^2 &&:= 240100000000980000000001 \qquad (3.268)
\end{aligned}$$

- **For $n = 5;\ m = 70, 700, 7000, 70000, \ldots$ in (3.245):**

$$\begin{aligned}
78375^2 + 2800^2 &= 78425^2 \\
7839975^2 + 28000^2 &= 7840025^2 \\
783999975^2 + 280000^2 &= 784000025^2 &&:= 614656039200000625 \\
78399999975^2 + 2800000^2 &= 78400000025^2 &&:= 6146560003920000000625 \\
7839999999975^2 + 28000000^2 &= 7840000000025^2 &&:= 61465600000392000000000625 \qquad (3.269)
\end{aligned}$$

The first triple $(759, 280, 809)$ for $n = 5;\ m = 7$ is not written above as it doesn't give good pattern.

(iii) For $n = 1, 2, 3, 4, 5;\ m = 6, 66, 666, 6666, 66666, \ldots$:

- **For $n = 1;\ m = 666, 6666, 66666, \ldots$ in (3.245):**

$$\begin{aligned}
7096895^2 + 5328^2 &= 7096897^2\\
710968895^2 + 53328^2 &= 710968897^2\\
71109688895^2 + 533328^2 &= 71109688897^2\\
7111096888895^2 + 5333328^2 &= 7111096888897^2
\end{aligned} \tag{3.270}$$

The first two triple $(575, 48, 577)$ and $(69695, 528, 69697)$ for $n = 1;\ m = 6$ and 66 are not written above as they don't give good pattern.

- **For $n = 2;\ m = 666, 6666, 66666, \ldots$ in (3.245):**

$$\begin{aligned}
7096892^2 + 10656^2 &= 7096900^2\\
710968892^2 + 106656^2 &= 710968900^2\\
71109688892^2 + 1066656^2 &= 71109688900^2\\
7111096888892^2 + 10666656^2 &= 7111096888900^2\\
711110968888892^2 + 106666656^2 &= 711110968888900^2
\end{aligned} \tag{3.271}$$

The first two triple $(572, 96.580)$ and $(69692, 1056, 69700)$ for $n = 2;\ m = 6$ and 66 are not written above as they don't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
34846^2 + 528^2 &= 34850^2\\
3548446^2 + 5328^2 &= 3548450^2\\
355484446^2 + 53328^2 &= 355484450^2\\
35554844446^2 + 533328^2 &= 35554844450^2\\
3555548444446^2 + 5333328^2 &= 3555548444450^2\\
355555484444446^2 + 53333328^2 &= 355555484444450^2
\end{aligned} \tag{3.272}$$

▶ **Division by 4**

$$\begin{aligned}
143^2 + 24^2 &= 145^2\\
17423^2 + 264^2 &= 17425^2\\
1774223^2 + 2664^2 &= 1774225^2\\
177742223^2 + 26664^2 &= 177742225^2\\
17777422223^2 + 266664^2 &= 17777422225^2\\
1777774222223^2 + 2666664^2 &= 1777774222225^2\\
177777742222223^2 + 26666664^2 &= 177777742222225^2
\end{aligned} \tag{3.273}$$

- **For $n = 3$; $m = 666, 6666, 66666, ...$ in (3.245):**

$$\begin{aligned}
7096887^2 + 15984^2 &= 7096905^2 \\
710968887^2 + 159984^2 &= 710968905^2 \\
71109688887^2 + 1599984^2 &= 71109688905^2 \\
7111096888887^2 + 15999984^2 &= 7111096888905^2
\end{aligned} \tag{3.274}$$

The first two triple (567, 144, 585) and (69687, 1584, 69705) for $n = 3$; $m = 6$ and 66 are not written above as they don't give good pattern.

- **For $n = 4$; $m = 666, 6666, 66666, ...$ in (3.245):**

$$\begin{aligned}
70968802^2 + 21312^2 &= 7096912^2 \\
7109688802^2 + 213312^2 &= 710968912^2 \\
711096888802^2 + 2133312^2 &= 71109688912^2 \\
71110968888802^2 + 21333312^2 &= 7111096888912^2 \\
711110968888880^2 + 213333312^2 &= 711110968888912^2
\end{aligned} \tag{3.275}$$

The first two triple (5602, 192, 592) and (696802, 2112, 69712) for $n = 4$; $m = 6$ and 66 are not written above as they don't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
34840^2 + 1056^2 &= 34856^2 \\
3548440^2 + 10656^2 &= 3548456^2 \\
355484440^2 + 106656^2 &= 355484456^2 \\
35554844440^2 + 1066656^2 &= 35554844456^2 \\
3555548444440^2 + 10666656^2 &= 3555548444456^2 \\
355555484444440^2 + 106666656^2 &= 355555484444456^2
\end{aligned} \tag{3.276}$$

▶ **Division by 4**

$$\begin{aligned}
17420^2 + 528^2 &= 17428^2 \\
1774220^2 + 5328^2 &= 1774228^2 \\
177742220^2 + 53328^2 &= 177742228^2 \\
17777422220^2 + 533328^2 &= 17777422228^2 \\
1777774222220^2 + 5333328^2 &= 1777774222228^2 \\
177777742222220^2 + 53333328^2 &= 177777742222228^2
\end{aligned} \tag{3.277}$$

▶ **Division by 8**

$$\begin{aligned}
70^2 + 24^2 &= 74^2\\
8710^2 + 264^2 &= 8714^2\\
887110^2 + 2664^2 &= 887114^2\\
88871110^2 + 26664^2 &= 88871114^2\\
8888711110^2 + 266664^2 &= 8888711114^2\\
888887111110^2 + 2666664^2 &= 888887111114^2\\
88888871111110^2 + 26666664^2 &= 88888871111114^2
\end{aligned} \qquad (3.278)$$

▶ **Division by 16**

$$\begin{aligned}
35^2 + 12^2 &= 37^2\\
4355^2 + 132^2 &= 4357^2\\
443555^2 + 1332^2 &= 443557^2\\
44435555^2 + 13332^2 &= 44435557^2\\
4444355555^2 + 133332^2 &= 4444355557^2\\
444443555555^2 + 1333332^2 &= 444443555557^2\\
44444435555555^2 + 13333332^2 &= 44444435555557^2
\end{aligned} \qquad (3.279)$$

- **For $n = 5;\ m = 666, 6666, 66666, ...$ in (3.245):**

$$\begin{aligned}
70968712^2 + 26640^2 &= 7096921^2\\
7109688712^2 + 266640^2 &= 710968921^2\\
711096888712^2 + 2666640^2 &= 71109688921^2\\
71110968888712^2 + 26666640^2 &= 7111096888921^2
\end{aligned} \qquad (3.280)$$

The first two triple (551, 240, 601) and (69671, 2640, 69721) for $n = 5;\ m = 6$ and 66 are not written above as they don't give good pattern.

(iv) For $n = 1, 2, 3, 4, 5;\ m = 9, 99, 999, 9999, 99999, ...$:

- **For $n = 1;\ m = 99, 999, 9999, 99999, ...$ in (3.245):**

$$\begin{aligned}
156815^2 + 792^2 &= 156817^2\\
15968015^2 + 7992^2 &= 15968017^2\\
1599680015^2 + 79992^2 &= 1599680017^2 &&:= 2558976156789120289\\
159996800015^2 + 799992^2 &= 159996800017^2 &&:= 25598976015679891200289\\
15999968000015^2 + 7999992^2 &= 15999968000017^2 &&:= 255998976001567998912000289
\end{aligned} \qquad (3.281)$$

The first triple (1295, 72, 1297) for $n = 1;\ m = 9$ is not written above as it doesn't give good pattern.

- **For $n = 2$; $m = 99, 999, 9999, 99999, ...$ in (3.245):**

$$\begin{aligned}
156812^2 + 1584^2 &= 15682^2 \\
15968012^2 + 15984^2 &= 15968020^2 \\
1599680012^2 + 159984^2 &= 1599680020^2 &&:= 2558976166387200400 \\
159996800012^2 + 1599984^2 &= 159996800020^2 &&:= 25598976016639872000400 \\
15999968000012^2 + 15999984^2 &= 15999968000020^2 &&:= 255998976001663998720000400
\end{aligned} \tag{3.282}$$

The first triple $(1292, 144, 1300)$ for $n = 2$; $m = 9$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
78406^2 + 792^2 &= 78410^2 \\
7984006^2 + 7992^2 &= 7984010^2 \\
799840006^2 + 79992^2 &= 799840010^2 &&:= 639744041596800100 \\
79998400006^2 + 799992^2 &= 79998400010^2 &&:= 6399744004159968000100 \\
7999984000006^2 + 7999992^2 &= 7999984000010,^2 &&:= 63999744000415999680000100
\end{aligned} \tag{3.283}$$

▶ **Division by 4**

$$\begin{aligned}
323^2 + 36^2 &= 325^2 \\
39203^2 + 396^2 &= 39205^2 \\
3992003^2 + 3996^2 &= 3992005^2 &&:= 15936103920025 \\
399920003^2 + 39996^2 &= 399920005,^2 &&:= 159936010399200025 \\
39999200003^2 + 399996^2 &= 39999200005^2 &&:= 1599936001039992000025 \\
3999992000003^2 + 3999996^2 &= 3999992000005^2 &&:= 15999936000103999920000025
\end{aligned} \tag{3.284}$$

▶ **Division by 8**

$$\begin{aligned}
156807^2 + 2376^2 &= 156825^2 \\
15968007^2 + 23976^2 &= 15968025^2 \\
1599680007^2 + 239976^2 &= 1599680025^2 &&:= 2558976182384000625 \\
159996800007^2 + 2399976^2 &= 159996800025^2 &&:= 25598976018239840000625 \\
15999968000007^2 + 23999976^2 &= 15999968000025^2 &&:= 255998976001823998400000625
\end{aligned} \tag{3.285}$$

- **For $n = 3;\ m = 99, 999, 9999, 99999, \ldots$ in (3.245):**

$$\begin{aligned}
156807^2 + 2376^2 &= 156825^2 \\
15968007^2 + 23976^2 &= 15968025^2 \\
1599680007^2 + 239976^2 &= 1599680025^2 &&:= 2558976182384000625 \\
159996800007^2 + 2399976^2 &= 159996800025^2 &&:= 25598976018239840000625 \\
15999968000007^2 + 23999976^2 &= 15999968000025^2 &&:= 255998976001823998400000625
\end{aligned} \tag{3.286}$$

The first triple $(1287, 216, 1305)$ for $n = 3;\ m = 9$ is not written above as it doesn't give good pattern.

- **For $n = 4;\ m = 99, 999, 9999, 99999, \ldots$ in (3.245):**

$$\begin{aligned}
156800^2 + 3168^2 &= 156832^2 \\
15968000^2 + 31968^2 &= 15968032^2 \\
1599680000^2 + 319968^2 &= 1599680032^2 &&:= 2558976204779521024 \\
159996800000^2 + 3199968^2 &= 159996800032^2 &&:= 25598976020479795201024 \\
15999968000000^2 + 31999968^2 &= 15999968000032^2 &&:= 255998976002047997952001024
\end{aligned} \tag{3.287}$$

The first triple $(1280, 288, 1312)$ for $n = 4;\ m = 9$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
78400^2 + 1584^2 &= 78416^2 \\
7984000^2 + 15984^2 &= 7984016^2 \\
799840000^2 + 159984^2 &= 799840016^2 &&:= 639744051194880256 \\
79998400000^2 + 1599984^2 &= 79998400016^2 &&:= 6399744005119948800256 \\
7999984000000^2 + 15999984^2 &= 7999984000016^2 &&:= 63999744000511999488000256
\end{aligned} \tag{3.288}$$

▶ **Division by 4**

$$\begin{aligned}
320^2 + 72^2 &= 328^2 \\
39200^2 + 792^2 &= 39208^2 \\
3992000^2 + 7992^2 &= 3992008^2 &&:= 15936127872064 \\
399920000^2 + 79992^2 &= 399920008^2 &&:= 159936012798720064 \\
39999200000^2 + 799992^2 &= 39999200008^2 &&:= 1599936001279987200064 \\
3999992000000^2 + 7999992^2 &= 3999992000008^2 &&:= 15999936000127999872000064
\end{aligned} \tag{3.289}$$

▶ **Division by 8**

$$\begin{aligned}
160^2 + 36^2 &= 164^2 \\
19600^2 + 396^2 &= 19604^2 \\
1996000^2 + 3996^2 &= 1996004^2 &:= 3984031968016 \\
199960000^2 + 39996^2 &= 199960004^2 &:= 39984003199680016 \\
19999600000^2 + 399996^2 &= 19999600004^2 &:= 399984000319996800016 \\
1999996000000^2 + 3999996^2 &= 1999996000004^2 &:= 3999984000031999968000016
\end{aligned} \quad (3.290)$$

▶ **Division by 16**

$$\begin{aligned}
80^2 + 18^2 &= 82^2 \\
9800^2 + 198^2 &= 9802^2 &:= 96079204 \\
998000^2 + 1998^2 &= 998002^2 &:= 996007992004 \\
99980000^2 + 19998^2 &= 99980002^2 &:= 9996000799920004 \\
9999800000^2 + 199998^2 &= 9999800002^2 &:= 99996000079999200004 \\
999998000000^2 + 1999998^2 &= 999998000002^2 &:= 999996000007999992000004
\end{aligned} \quad (3.291)$$

▶ **Division by 32**

$$\begin{aligned}
40^2 + 9^2 &= 41^2 \\
4900^2 + 99^2 &= 4901^2 &:= 24019801 \\
499000^2 + 999^2 &= 499001^2 &:= 249001998001 \\
49990000^2 + 9999^2 &= 49990001^2 &:= 2499000199980001 \\
4999900000^2 + 99999^2 &= 4999900001^2 &:= 24999000019999800001 \\
499999000000^2 + 999999^2 &= 499999000001^2 &:= 249999000001999998000001
\end{aligned} \quad (3.292)$$

- **For $n = 5;\ m = 999, 9999, 99999, \ldots$ in (3.245):**

$$\begin{aligned}
156791^2 + 3960^2 &= 156841^2 \\
15967991^2 + 39960^2 &= 15968041^2 \\
1599679991^2 + 399960^2 &= 1599680041^2 &:= 2558976233573761681 \\
159996799991^2 + 3999960^2 &= 159996800041^2 &:= 25598976023359737601681 \\
15999967999991^2 + 39999960^2 &= 15999968000041^2 &:= 255998976002335997376001681
\end{aligned} \quad (3.293)$$

The first triple $(1271, 360, 1321)$ for $n = 5;\ m = 9$ is not written above as it doesn't give good pattern.

The subsection below give **patterned Pythagorean triples** for all the four Procedures. In case of patterns with even numbers, division by 2, 4, 8, etc. are considered to get more patterns utill we reach patterns with primitive values.

4 Pandigital Palindromic-Type Patterns

4.1 Part I

Below are some examples of pattern in Pythagorean triples with palindromic type numbers having all the digits from 1 to 9.

- **For $n = 1$; $m = 55, 555, 5555, 55555, \ldots$ in (3.170):**

$$\begin{aligned}
099^2 + 20^2 &= 1\,01^2 \\
12\,099^2 + 220^2 &= 121\,01^2 \\
1232\,099^2 + 2220^2 &= 12321\,01^2 \\
123432\,099^2 + 22220^2 &= 1234321\,01^2 \\
12345432\,099^2 + 222220^2 &= 123454321\,01^2 \\
1234565432\,099^2 + 2222220^2 &= 12345654321\,01^2 \\
123456765432\,099^2 + 22222220^2 &= 1234567654321\,01^2 \\
12345678765432\,099^2 + 222222220^2 &= 123456787654321\,01^2 \\
1234567898765432\,099^2 + 2222222220^2 &= 12345678987654321\,01^2 \qquad (4.1)
\end{aligned}$$

Here we have total 9 examples os similar kind. See below more 8 examples:

- **For $n = 2$; $m = 55, 555, 5555, 55555, \ldots$ in (3.170) :**

$$\begin{aligned}
096^2 + 40^2 &= 1\,04^2 \\
12\,096^2 + 440^2 &= 121\,04^2 \\
1232\,096^2 + 4440^2 &= 12321\,04^2 \\
123432\,096^2 + 44440^2 &= 1234321\,04^2 \\
12345432\,096^2 + 444440^2 &= 123454321\,04^2 \\
1234565432\,096^2 + 4444440^2 &= 12345654321\,04^2 \\
123456765432\,096^2 + 44444440^2 &= 1234567654321\,04^2 \\
12345678765432\,096^2 + 444444440^2 &= 123456787654321\,04^2 \\
1234567898765432\,096^2 + 4444444440^2 &= 12345678987654321\,04^2 \qquad (4.2)
\end{aligned}$$

- **For $n = 3$; $m = 55, 555, 5555, 55555, \ldots$ in (3.170) :**

$$\begin{aligned}
091^2 + 60^2 &= 1\,09^2\\
12\,091^2 + 660^2 &= 121\,09^2\\
1232\,091^2 + 6660^2 &= 12321\,09^2\\
123432\,091^2 + 66660^2 &= 1234321\,09^2\\
12345432\,091^2 + 666660^2 &= 123454321\,09^2\\
1234565432\,091^2 + 6666660^2 &= 12345654321\,09^2\\
123456765432\,091^2 + 66666660^2 &= 1234567654321\,09^2\\
12345678765432\,091^2 + 666666660^2 &= 123456787654321\,09^2\\
1234567898765432\,091^2 + 6666666660^2 &= 12345678987654321\,09^2
\end{aligned} \tag{4.3}$$

- **For $n = 4$; $m = 55, 555, 5555, 55555, ...$ in (3.170) :**

$$\begin{aligned}
084^2 + 80^2 &= 1\,16^2\\
12\,084^2 + 880^2 &= 121\,16^2\\
1232\,084^2 + 8880^2 &= 12321\,16^2\\
123432\,084^2 + 88880^2 &= 1234321\,16^2\\
12345432\,084^2 + 888880^2 &= 123454321\,16^2\\
1234565432\,084^2 + 8888880^2 &= 12345654321\,16^2\\
123456765432\,084^2 + 88888880^2 &= 1234567654321\,16^2\\
12345678765432\,084^2 + 888888880^2 &= 123456787654321\,16^2\\
1234567898765432\,084^2 + 8888888880^2 &= 12345678987654321\,16^2
\end{aligned} \tag{4.4}$$

- **For $n = 5$; $m = 55, 555, 5555, 55555, ...$ in (3.170) :**

$$\begin{aligned}
075^2 + 100^2 &= 1\,25^2\\
12\,075^2 + 1100^2 &= 121\,25^2\\
1232\,075^2 + 11100^2 &= 12321\,25^2\\
123432\,075^2 + 111100^2 &= 1234321\,25^2\\
12345432\,075^2 + 1111100^2 &= 123454321\,25^2\\
1234565432\,075^2 + 11111100^2 &= 12345654321\,25^2\\
123456765432\,075^2 + 111111100^2 &= 1234567654321\,25^2\\
12345678765432\,075^2 + 1111111100^2 &= 123456787654321\,25^2\\
1234567898765432\,075^2 + 11111111100^2 &= 12345678987654321\,25^2
\end{aligned} \tag{4.5}$$

- **For $n = 6$; $m = 55, 555, 5555, 55555, ...$ in (3.170) :**

$$
\begin{aligned}
064^2 + 120^2 &= 1\,36^2 \\
12\,064^2 + 1320^2 &= 121\,36^2 \\
1232\,064^2 + 13320^2 &= 12321\,36^2 \\
123432\,064^2 + 133320^2 &= 1234321\,36^2 \\
12345432\,064^2 + 1333320^2 &= 123454321\,36^2 \\
1234565432\,064^2 + 13333320^2 &= 12345654321\,36^2 \\
123456765432\,064^2 + 133333320^2 &= 1234567654321\,36^2 \\
12345678765432\,064^2 + 1333333320^2 &= 123456787654321\,36^2 \\
1234567898765432\,064^2 + 13333333320^2 &= 12345678987654321\,36^2
\end{aligned} \tag{4.6}
$$

- **For $n = 7;\ m = 55, 555, 5555, 55555, ...$ in (3.170) :**

$$
\begin{aligned}
051^2 + 140^2 &= 1\,49^2 \\
12\,051^2 + 1540^2 &= 121\,49^2 \\
1232\,051^2 + 15540^2 &= 12321\,49^2 \\
123432\,051^2 + 155540^2 &= 1234321\,49^2 \\
12345432\,051^2 + 1555540^2 &= 123454321\,49^2 \\
1234565432\,051^2 + 15555540^2 &= 12345654321\,49^2 \\
123456765432\,051^2 + 155555540^2 &= 1234567654321\,49^2 \\
12345678765432\,051^2 + 1555555540^2 &= 123456787654321\,49^2 \\
1234567898765432\,051^2 + 15555555540^2 &= 12345678987654321\,49^2
\end{aligned} \tag{4.7}
$$

- **For $n = 8;\ m = 55, 555, 5555, 55555, ...$ in (3.170) :**

$$
\begin{aligned}
036^2 + 160^2 &= 1\,64^2 \\
12\,036^2 + 1760^2 &= 121\,64^2 \\
1232\,036^2 + 17760^2 &= 12321\,64^2 \\
123432\,036^2 + 177760^2 &= 1234321\,64^2 \\
12345432\,036^2 + 1777760^2 &= 123454321\,64^2 \\
1234565432\,036^2 + 17777760^2 &= 12345654321\,64^2 \\
123456765432\,036^2 + 177777760^2 &= 1234567654321\,64^2 \\
12345678765432\,036^2 + 1777777760^2 &= 123456787654321\,64^2 \\
1234567898765432\,036^2 + 17777777760^2 &= 12345678987654321\,64^2
\end{aligned} \tag{4.8}
$$

- **For $n = 9;\ m = 55, 555, 5555, 55555, ...$ in (3.170) :**

$$\begin{aligned}
019^2 + 180^2 &= 1\,81^2\\
12\,019^2 + 1980^2 &= 121\,81^2\\
1232\,019^2 + 19980^2 &= 12321\,81^2\\
123432\,019^2 + 199980^2 &= 1234321\,81^2\\
12345432\,019^2 + 1999980^2 &= 123454321\,81^2\\
1234565432\,019^2 + 19999980^2 &= 12345654321\,81^2\\
123456765432\,019^2 + 199999980^2 &= 1234567654321\,81^2\\
12345678765432\,019^2 + 1999999980^2 &= 123456787654321\,81^2\\
1234567898765432\,019^2 + 19999999980^2 &= 12345678987654321\,81^2
\end{aligned} \tag{4.9}$$

For extended study of above type of patterns refer to author's wrok [6].

Remark 8. *We observe the last number in each row is a perfect square, for example, i.e.,*

$$01 = 1^2,\ 04 = 2^2,\ 09 = 3^2,\ \ldots,\ 81 = 9^2.$$

Moreover, the sum of last two digits of first and third rows are always 100. See below:

▶ ***Sum of Last Two Digits: First and Third Rows***

$$\begin{aligned}
n = 1 \quad &\Rightarrow \quad 99 + 01 = 100\\
n = 2 \quad &\Rightarrow \quad 96 + 04 = 100\\
n = 3 \quad &\Rightarrow \quad 91 + 09 = 100\\
n = 4 \quad &\Rightarrow \quad 84 + 16 = 100\\
n = 5 \quad &\Rightarrow \quad 75 + 25 = 100\\
n = 6 \quad &\Rightarrow \quad 64 + 36 = 100\\
n = 7 \quad &\Rightarrow \quad 51 + 49 = 100\\
n = 8 \quad &\Rightarrow \quad 19 + 64 = 100\\
n = 9 \quad &\Rightarrow \quad 19 + 81 = 100
\end{aligned}$$

4.2 Part II

Below are some examples of pattern in Pythagorean triples with palindromic type numbers having all the digits from 1 to 9 with 0 in between.

- **For $n = 1;\ m = 10, 1010, 101010, 10101010, 1010101010, ...$ in (3.47):**

$$\begin{aligned}
0099^2 + 20^2 &= 1\,01^2 \\
102\,0099^2 + 2020^2 &= 10201\,01^2 \\
1020302\,0099^2 + 202020^2 &= 102030201\,01^2 \\
10203040302\,0099^2 + 20202020^2 &= 1020304030201\,01^2 \\
102030405040302\,0099^2 + 2020202020^2 &= 10203040504030201\,01^2 \\
1020304050605040302\,0099^2 + 202020202020^2 &= 102030405060504030201\,01^2 \\
10203040506070605040302\,0099^2 + 20202020202020^2 &= 1020304050607060504030201\,01^2 \\
102030405060708070605040302\,0099^2 + 2020202020202020^2 &= 10203040506070807060504030201\,01^2 \\
1020304050607080908070605040302\,0099^2 + 202020202020202020^2 &= 102030405060708090807060504030201\,01^2
\end{aligned} \tag{4.10}$$

- **For $n = 2$; $m = 10, 1010, 101010, 10101010, 1010101010, \ldots$ in (3.47):**

$$\begin{aligned}
0096^2 + 40^2 &= 1\,04^2 \\
102\,0096^2 + 4040^2 &= 10201\,04^2 \\
1020302\,0096^2 + 404040^2 &= 102030201\,04^2 \\
10203040302\,0096^2 + 40404040^2 &= 1020304030201\,04^2 \\
102030405040302\,0096^2 + 4040404040^2 &= 10203040504030201\,04^2 \\
1020304050605040302\,0096^2 + 404040404040^2 &= 102030405060504030201\,04^2 \\
10203040506070605040302\,0096^2 + 40404040404040^2 &= 1020304050607060504030201\,04^2 \\
102030405060708070605040302\,0096^2 + 4040404040404040^2 &= 10203040506070807060504030201\,04^2 \\
1020304050607080908070605040302\,0096^2 + 404040404040404040^2 &= 102030405060708090807060504030201\,04^2
\end{aligned} \tag{4.11}$$

- **For $n = 3$; $m = 10, 1010, 101010, 10101010, 1010101010, \ldots$ in (3.47):**

$$\begin{aligned}
0091^2 + 60^2 &= 1\,09^2 \\
102\,0091^2 + 6060^2 &= 10201\,09^2 \\
1020302\,0091^2 + 606060^2 &= 102030201\,09^2 \\
10203040302\,0091^2 + 60606060^2 &= 1020304030201\,09^2 \\
102030405040302\,0091^2 + 6060606060^2 &= 10203040504030201\,09^2 \\
1020304050605040302\,0091^2 + 606060606060^2 &= 102030405060504030201\,09^2 \\
10203040506070605040302\,0091^2 + 60606060606060^2 &= 1020304050607060504030201\,09^2 \\
102030405060708070605040302\,0091^2 + 6060606060606060^2 &= 10203040506070807060504030201\,09^2 \\
1020304050607080908070605040302\,0091^2 + 606060606060606060^2 &= 102030405060708090807060504030201\,09^2
\end{aligned} \tag{4.12}$$

- **For $n = 4$; $m = 10, 1010, 101010, 10101010, 1010101010, \ldots$ in (3.47):**

$$\begin{aligned}
0084^2 + 80^2 &= 1\,16^2\\
102\,0084^2 + 8080^2 &= 10201\,16^2\\
1020302\,0084^2 + 808080^2 &= 102030201\,16^2\\
10203040302\,0084^2 + 80808080^2 &= 1020304030201\,16^2\\
102030405040302\,0084^2 + 8080808080^2 &= 10203040504030201\,16^2\\
1020304050605040302\,0084^2 + 808080808080^2 &= 102030405060504030201\,16^2\\
10203040506070605040302\,0084^2 + 80808080808080^2 &= 1020304050607060504030201\,16^2\\
102030405060708070605040302\,0084^2 + 8080808080808080^2 &= 10203040506070807060504030201\,16^2\\
1020304050607080908070605040302\,0084^2 + 808080808080808080^2 &= 102030405060708090807060504030201\,16^2
\end{aligned} \tag{4.13}$$

- **For $n = 5$; $m = 10, 1010, 101010, 10101010, 1010101010, ...$ in (3.47):**

$$\begin{aligned}
0075^2 + 100^2 &= 1\,25^2\\
102\,0075^2 + 10100^2 &= 10201\,25^2\\
1020302\,0075^2 + 1010100^2 &= 102030201\,25^2\\
10203040302\,0075^2 + 101010100^2 &= 1020304030201\,25^2\\
102030405040302\,0075^2 + 10101010100^2 &= 10203040504030201\,25^2\\
1020304050605040302\,0075^2 + 1010101010100^2 &= 102030405060504030201\,25^2\\
10203040506070605040302\,0075^2 + 101010101010100^2 &= 1020304050607060504030201\,25^2\\
102030405060708070605040302\,0075^2 + 10101010101010100^2 &= 10203040506070807060504030201\,25^2\\
1020304050607080908070605040302\,0075^2 + 1010101010101010100^2 &= 102030405060708090807060504030201\,25^2
\end{aligned} \tag{4.14}$$

- **For $n = 6$; $m = 10, 1010, 101010, 10101010, 1010101010, ...$ in (3.47):**

$$\begin{aligned}
0064^2 + 120^2 &= 1\,36^2\\
102\,0064^2 + 12120^2 &= 10201\,36^2\\
1020302\,0064^2 + 1212120^2 &= 102030201\,36^2\\
10203040302\,0064^2 + 121212120^2 &= 1020304030201\,36^2\\
102030405040302\,0064^2 + 12121212120^2 &= 10203040504030201\,36^2\\
1020304050605040302\,0064^2 + 1212121212120^2 &= 102030405060504030201\,36^2\\
10203040506070605040302\,0064^2 + 121212121212120^2 &= 1020304050607060504030201\,36^2\\
102030405060708070605040302\,0064^2 + 12121212121212120^2 &= 10203040506070807060504030201\,36^2\\
1020304050607080908070605040302\,0064^2 + 1212121212121212120^2 &= 102030405060708090807060504030201\,36^2
\end{aligned} \tag{4.15}$$

- **For $n = 7$; $m = 10, 1010, 101010, 10101010, 1010101010, ...$ in (3.47):**

$$\begin{aligned}
0051^2 + 140^2 &= 1\,49^2\\
102\,0051^2 + 14140^2 &= 10201\,49^2\\
1020302\,0051^2 + 1414140^2 &= 102030201\,49^2\\
10203040302\,0051^2 + 141414140^2 &= 1020304030201\,49^2\\
102030405040302\,0051^2 + 14141414140^2 &= 10203040504030201\,49^2\\
1020304050605040302\,0051^2 + 1414141414140^2 &= 102030405060504030201\,49^2\\
10203040506070605040302\,0051^2 + 141414141414140^2 &= 1020304050607060504030201\,49^2\\
102030405060708070605040302\,0051^2 + 14141414141414140^2 &= 10203040506070807060504030201\,49^2\\
1020304050607080908070605040302\,0051^2 + 1414141414141414140^2 &= 102030405060708090807060504030201\,49^2
\end{aligned} \tag{4.16}$$

- **For $n=8$; $m = 10, 1010, 101010, 10101010, 1010101010, \ldots$ in (3.47):**

$$\begin{aligned}
0036^2 + 160^2 &= 1\,64^2\\
102\,0036^2 + 16160^2 &= 10201\,64^2\\
1020302\,0036^2 + 1616160^2 &= 102030201\,64^2\\
10203040302\,0036^2 + 161616160^2 &= 1020304030201\,64^2\\
102030405040302\,0036^2 + 16161616160^2 &= 10203040504030201\,64^2\\
1020304050605040302\,0036^2 + 1616161616160^2 &= 102030405060504030201\,64^2\\
10203040506070605040302\,0036^2 + 161616161616160^2 &= 1020304050607060504030201\,64^2\\
102030405060708070605040302\,0036^2 + 16161616161616160^2 &= 10203040506070807060504030201\,64^2\\
1020304050607080908070605040302\,0036^2 + 1616161616161616160^2 &= 102030405060708090807060504030201\,64^2
\end{aligned} \tag{4.17}$$

- **For $n=9$; $m = 10, 1010, 101010, 10101010, 1010101010, \ldots$ in (3.47):**

$$\begin{aligned}
0019^2 + 180^2 &= 1\,81^2\\
102\,0019^2 + 18180^2 &= 10201\,81^2\\
1020302\,0019^2 + 1818180^2 &= 102030201\,81^2\\
10203040302\,0019^2 + 181818180^2 &= 1020304030201\,81^2\\
102030405040302\,0019^2 + 18181818180^2 &= 10203040504030201\,81^2\\
1020304050605040302\,0019^2 + 1818181818180^2 &= 102030405060504030201\,81^2\\
10203040506070605040302\,0019^2 + 181818181818180^2 &= 1020304050607060504030201\,81^2\\
102030405060708070605040302\,0019^2 + 18181818181818180^2 &= 10203040506070807060504030201\,81^2\\
1020304050607080908070605040302\,0019^2 + 1818181818181818180^2 &= 102030405060708090807060504030201\,81^2
\end{aligned} \tag{4.18}$$

For extended study of above type of pattern refer to author's wrok [7].

Remark 9. *We observe the last number in each row is a perfect square, for example, i.e.,*

$$01 = 1^2,\ 04 = 2^2,\ 09 = 3^2,\ \ldots,\ 81 = 9^2.$$

Moreover, the sum of last two digits of first and third rows are always 100. See below:

▶ ***Sum of Last Two Digits: First and Third Rows***

$$
\begin{array}{lcl}
n=1 & \Rightarrow & 99+01=100\\
n=2 & \Rightarrow & 96+04=100\\
n=3 & \Rightarrow & 91+09=100\\
n=4 & \Rightarrow & 84+16=100\\
n=5 & \Rightarrow & 75+25=100\\
n=6 & \Rightarrow & 64+36=100\\
n=7 & \Rightarrow & 51+49=100\\
n=8 & \Rightarrow & 19+64=100\\
n=9 & \Rightarrow & 19+81=100
\end{array}
$$

4.3 Part III

This subsection brings pandigial palindromic-type patterns similar to one given in subsection 4.2. Only difference is in fixed values. Moreover, in this case can go up to 99 patterns instead of 9.

- **For $n=1$; $m=25, 2525, 252525, 25252525, \ldots$ in (3.245):**

$$
\begin{array}{rl}
009999^2+200^2 & =1\,0001^2\\
102\,009999^2+20200^2 & =10201\,0001^2\\
1020302\,009999^2+2020200^2 & =102030201\,0001^2\\
10203040302\,009999^2+202020200^2 & =1020304030201\,0001^2\\
102030405040302\,009999^2+20202020200^2 & =10203040504030201\,0001^2\\
1020304050605040302\,009999^2+2020202020200^2 & =102030405060504030201\,0001^2\\
10203040506070605040302\,009999^2+202020202020200^2 & =1020304050607060504030201\,0001^2\\
102030405060708070605040302\,009999^2+20202020202020200^2 & =10203040506070807060504030201\,0001^2\\
1020304050607080908070605040302\,009999^2+2020202020202020200^2 & =102030405060708090807060504030201\,0001^2
\end{array}
\tag{4.19}
$$

- **For $n=2$; $m=25, 2525, 252525, 25252525, \ldots$ in (3.245):**

$$
\begin{array}{rl}
009996^2+400^2 & =1\,0004^2\\
102\,009996^2+40400^2 & =10201\,0004^2\\
1020302\,009996^2+4040400^2 & =102030201\,0004^2\\
10203040302\,009996^2+404040400^2 & =1020304030201\,0004^2\\
102030405040302\,009996^2+40404040400^2 & =10203040504030201\,0004^2\\
1020304050605040302\,009996^2+4040404040400^2 & =102030405060504030201\,0004^2\\
10203040506070605040302\,009996^2+404040404040400^2 & =1020304050607060504030201\,0004^2\\
102030405060708070605040302\,009996^2+40404040404040400^2 & =10203040506070807060504030201\,0004^2\\
1020304050607080908070605040302\,009996^2+4040404040404040400^2 & =102030405060708090807060504030201\,0004^2
\end{array}
\tag{4.20}
$$

- **For $n = 3$; $m = 25, 2525, 252525, 25252525, ...$ in (3.245):**

$$\begin{aligned}
009991^2 + 600^2 &= 1\,0009^2\\
102\,009991^2 + 60600^2 &= 10201\,0009^2\\
1020302\,009991^2 + 6060600^2 &= 102030201\,0009^2\\
10203040302\,009991^2 + 606060600^2 &= 1020304030201\,0009^2\\
102030405040302\,009991^2 + 60606060600^2 &= 10203040504030201\,0009^2\\
1020304050605040302\,009991^2 + 6060606060600^2 &= 102030405060504030201\,0009^2\\
10203040506070605040302\,009991^2 + 606060606060600^2 &= 1020304050607060504030201\,0009^2\\
102030405060708070605040302\,009991^2 + 60606060606060600^2 &= 10203040506070807060504030201\,0009^2\\
1020304050607080908070605040302\,009991^2 + 6060606060606060600^2 &= 102030405060708090807060504030201\,0009^2
\end{aligned} \tag{4.21}$$

- **For $n = 4$; $m = 25, 2525, 252525, 25252525, ...$ in (3.245):**

$$\begin{aligned}
009984^2 + 800^2 &= 1\,0016^2\\
102\,009984^2 + 80800^2 &= 10201\,0016^2\\
1020302\,009984^2 + 8080800^2 &= 102030201\,0016^2\\
10203040302\,009984^2 + 808080800^2 &= 1020304030201\,0016^2\\
102030405040302\,009984^2 + 80808080800^2 &= 10203040504030201\,0016^2\\
1020304050605040302\,009984^2 + 8080808080800^2 &= 102030405060504030201\,0016^2\\
10203040506070605040302\,009984^2 + 808080808080800^2 &= 1020304050607060504030201\,0016^2\\
102030405060708070605040302\,009984^2 + 80808080808080800^2 &= 10203040506070807060504030201\,0016^2\\
1020304050607080908070605040302\,009984^2 + 8080808080808080800^2 &= 102030405060708090807060504030201\,0016^2
\end{aligned} \tag{4.22}$$

- **For $n = 5$; $m = 25, 2525, 252525, 25252525, ...$ in (3.245):**

$$\begin{aligned}
009975^2 + 1000^2 &= 1\,0025^2\\
102\,009975^2 + 10100^2 &= 10201\,0025^2\\
1020302\,009975^2 + 1010100^2 &= 102030201\,0025^2\\
10203040302\,009975^2 + 101010100^2 &= 1020304030201\,0025^2\\
102030405040302\,009975^2 + 10101010100^2 &= 10203040504030201\,0025^2\\
1020304050605040302\,009975^2 + 1010101010100^2 &= 102030405060504030201\,0025^2\\
10203040506070605040302\,009975^2 + 101010101010100^2 &= 1020304050607060504030201\,0025^2\\
102030405060708070605040302\,009975^2 + 10101010101010100^2 &= 10203040506070807060504030201\,0025^2\\
1020304050607080908070605040302\,009975^2 + 1010101010101010100^2 &= 102030405060708090807060504030201\,0025^2
\end{aligned} \tag{4.23}$$

- **For $n = 6$; $m = 25, 2525, 252525, 25252525, ...$ in (3.245):**

$$\begin{aligned}
009964^2 + 1200^2 &= 1\,0036^2\\
102\,009964^2 + 121200^2 &= 10201\,0036^2\\
1020302\,009964^2 + 12121200^2 &= 102030201\,0036^2\\
10203040302\,009964^2 + 1212121200^2 &= 1020304030201\,0036^2\\
102030405040302\,009964^2 + 121212121200^2 &= 10203040504030201\,0036^2\\
1020304050605040302\,009964^2 + 12121212121200^2 &= 102030405060504030201\,0036^2\\
10203040506070605040302\,009964^2 + 1212121212121200^2 &= 1020304050607060504030201\,0036^2\\
102030405060708070605040302\,009964^2 + 121212121212121200^2 &= 10203040506070807060504030201\,0036^2\\
1020304050607080908070605040302\,009964^2 + 12121212121212121200^2 &= 102030405060708090807060504030201\,0036^2
\end{aligned} \tag{4.24}$$

- **For $n = 7$; $m = 25, 2525, 252525, 25252525, ...$ in (3.245):**

$$\begin{aligned}
009951^2 + 1400^2 &= 1\,0049^2\\
102\,009951^2 + 141400^2 &= 10201\,0049^2\\
1020302\,009951^2 + 14141400^2 &= 102030201\,0049^2\\
10203040302\,009951^2 + 1414141400^2 &= 1020304030201\,0049^2\\
102030405040302\,009951^2 + 141414141400^2 &= 10203040504030201\,0049^2\\
1020304050605040302\,009951^2 + 14141414141400^2 &= 102030405060504030201\,0049^2\\
10203040506070605040302\,009951^2 + 1414141414141400^2 &= 1020304050607060504030201\,0049^2\\
102030405060708070605040302\,009951^2 + 141414141414141400^2 &= 10203040506070807060504030201\,0049^2\\
1020304050607080908070605040302\,009951^2 + 14141414141414141400^2 &= 102030405060708090807060504030201\,0049^2
\end{aligned} \tag{4.25}$$

- **For $n = 8$; $m = 25, 2525, 252525, 25252525, ...$ in (3.245):**

$$\begin{aligned}
009936^2 + 1600^2 &= 1\,0064^2\\
102\,009936^2 + 161600^2 &= 10201\,0064^2\\
1020302\,009936^2 + 16161600^2 &= 102030201\,0064^2\\
10203040302\,009936^2 + 1616161600^2 &= 1020304030201\,0064^2\\
102030405040302\,009936^2 + 161616161600^2 &= 10203040504030201\,0064^2\\
1020304050605040302\,009936^2 + 16161616161600^2 &= 102030405060504030201\,0064^2\\
10203040506070605040302\,009936^2 + 1616161616161600^2 &= 1020304050607060504030201\,0064^2\\
102030405060708070605040302\,009936^2 + 161616161616161600^2 &= 10203040506070807060504030201\,0064^2\\
1020304050607080908070605040302\,009936^2 + 16161616161616161600^2 &= 102030405060708090807060504030201\,0064^2
\end{aligned} \tag{4.26}$$

- **For $n = 9$; $m = 25, 2525, 252525, 25252525, ...$ in (3.245):**

$$\begin{aligned}
009919^2 + 1800^2 &= 1\,0081^2\\
102\,009919^2 + 181800^2 &= 10201\,0081^2\\
1020302\,009919^2 + 18181800^2 &= 102030201\,0081^2\\
10203040302\,009919^2 + 1818181800^2 &= 1020304030201\,0081^2\\
102030405040302\,009919^2 + 181818181800^2 &= 10203040504030201\,0081^2\\
1020304050605040302\,009919^2 + 18181818181800^2 &= 102030405060504030201\,0081^2\\
10203040506070605040302\,009919^2 + 1818181818181800^2 &= 1020304050607060504030201\,0081^2\\
102030405060708070605040302\,009919^2 + 181818181818181800^2 &= 10203040506070807060504030201\,0081^2\\
1020304050607080908070605040302\,009919^2 + 18181818181818181800^2 &= 102030405060708090807060504030201\,0081^2
\end{aligned} \tag{4.27}$$

- **For $n = 10$; $m = 25, 2525, 252525, 25252525, ...$ in (3.245):**

$$\begin{aligned}
009900^2 + 2000^2 &= 1\,0100^2\\
102\,009900^2 + 202000^2 &= 10201\,0100^2\\
1020302\,009900^2 + 20202000^2 &= 102030201\,0100^2\\
10203040302\,009900^2 + 2020202000^2 &= 1020304030201\,0100^2\\
102030405040302\,009900^2 + 202020202000^2 &= 10203040504030201\,0100^2\\
1020304050605040302\,009900^2 + 20202020202000^2 &= 102030405060504030201\,0100^2\\
10203040506070605040302\,009900^2 + 2020202020202000^2 &= 1020304050607060504030201\,0100^2\\
102030405060708070605040302\,009900^2 + 202020202020202000^2 &= 10203040506070807060504030201\,0100^2\\
1020304050607080908070605040302\,009900^2 + 20202020202020202000^2 &= 102030405060708090807060504030201\,0100^2
\end{aligned} \tag{4.28}$$

- **For $n = 11$; $m = 25, 2525, 252525, 25252525, ...$ in (3.245):**

$$\begin{aligned}
009879^2 + 2200^2 &= 1\,0121^2\\
102\,009879^2 + 222200^2 &= 10201\,0121^2\\
1020302\,009879^2 + 22222200^2 &= 102030201\,0121^2\\
10203040302\,009879^2 + 2222222200^2 &= 1020304030201\,0121^2\\
102030405040302\,009879^2 + 222222222200^2 &= 10203040504030201\,0121^2\\
1020304050605040302\,009879^2 + 22222222222200^2 &= 102030405060504030201\,0121^2\\
10203040506070605040302\,009879^2 + 2222222222222200^2 &= 1020304050607060504030201\,0121^2\\
102030405060708070605040302\,009879^2 + 222222222222222200^2 &= 10203040506070807060504030201\,0121^2\\
1020304050607080908070605040302\,009879^2 + 22222222222222222200^2 &= 102030405060708090807060504030201\,0121^2
\end{aligned} \tag{4.29}$$

- **For $n = 12$; $m = 25, 2525, 252525, 25252525, ...$ in (3.245):**

$$
\begin{aligned}
009856^2 + 2400^2 &= 1\,0144^2 \\
102\,009856^2 + 242400^2 &= 10201\,0144^2 \\
1020302\,009856^2 + 24242400^2 &= 102030201\,0144^2 \\
10203040302\,009856^2 + 2424242400^2 &= 1020304030201\,0144^2 \\
102030405040302\,009856^2 + 242424242400^2 &= 10203040504030201\,0144^2 \\
1020304050605040302\,009856^2 + 24242424242400^2 &= 102030405060504030201\,0144^2 \\
10203040506070605040302\,009856^2 + 2424242424242400^2 &= 1020304050607060504030201\,0144^2 \\
102030405060708070605040302\,009856^2 + 242424242424242400^2 &= 10203040506070807060504030201\,0144^2 \\
1020304050607080908070605040302\,009856^2 + 24242424242424242400^2 &= 102030405060708090807060504030201\,0144^2
\end{aligned} \tag{4.30}
$$

- **For $n = 13$; $m = 25, 2525, 252525, 25252525, ...$ in (3.245):**

$$
\begin{aligned}
009831^2 + 2600^2 &= 1\,0169^2 \\
102\,009831^2 + 262600^2 &= 10201\,0169^2 \\
1020302\,009831^2 + 26262600^2 &= 102030201\,0169^2 \\
10203040302\,009831^2 + 2626262600^2 &= 1020304030201\,0169^2 \\
102030405040302\,009831^2 + 262626262600^2 &= 10203040504030201\,0169^2 \\
1020304050605040302\,009831^2 + 26262626262600^2 &= 102030405060504030201\,0169^2 \\
10203040506070605040302\,009831^2 + 2626262626262600^2 &= 1020304050607060504030201\,0169^2 \\
102030405060708070605040302\,009831^2 + 262626262626262600^2 &= 10203040506070807060504030201\,0169^2 \\
1020304050607080908070605040302\,009831^2 + 26262626262626262600^2 &= 102030405060708090807060504030201\,0169^2
\end{aligned} \tag{4.31}
$$

- **For $n = 14$; $m = 25, 2525, 252525, 25252525, ...$ in (3.245):**

$$
\begin{aligned}
009804^2 + 2800^2 &= 1\,0196^2 \\
102\,009804^2 + 282800^2 &= 10201\,0196^2 \\
1020302\,009804^2 + 28282800^2 &= 102030201\,0196^2 \\
10203040302\,009804^2 + 2828282800^2 &= 1020304030201\,0196^2 \\
102030405040302\,009804^2 + 282828282800^2 &= 10203040504030201\,0196^2 \\
1020304050605040302\,009804^2 + 28282828282800^2 &= 102030405060504030201\,0196^2 \\
10203040506070605040302\,009804^2 + 2828282828282800^2 &= 1020304050607060504030201\,0196^2 \\
102030405060708070605040302\,009804^2 + 282828282828282800^2 &= 10203040506070807060504030201\,0196^2 \\
1020304050607080908070605040302\,009804^2 + 28282828282828282800^2 &= 102030405060708090807060504030201\,0196^2
\end{aligned} \tag{4.32}
$$

- **For $n = 15$; $m = 25, 2525, 252525, 25252525, ...$ in (3.245):**

$$\begin{aligned}
009775^2 + 3000^2 &= 1\,0225^2\\
102\,009775^2 + 303000^2 &= 10201\,0225^2\\
1020302\,009775^2 + 30303000^2 &= 102030201\,0225^2\\
10203040302\,009775^2 + 3030303000^2 &= 1020304030201\,0225^2\\
102030405040302\,009775^2 + 303030303000^2 &= 10203040504030201\,0225^2\\
1020304050605040302\,009775^2 + 30303030303000^2 &= 102030405060504030201\,0225^2\\
10203040506070605040302\,009775^2 + 3030303030303000^2 &= 1020304050607060504030201\,0225^2\\
102030405060708070605040302\,009775^2 + 303030303030303000^2 &= 10203040506070807060504030201\,0225^2\\
1020304050607080908070605040302\,009775^2 + 30303030303030303000^2 &= 102030405060708090807060504030201\,0225^2
\end{aligned} \tag{4.33}$$

- **For $n = 16$; $m = 25, 2525, 252525, 25252525, \ldots$ in (3.245):**

$$\begin{aligned}
009744^2 + 3200^2 &= 1\,0256^2\\
102\,009744^2 + 323200^2 &= 10201\,0256^2\\
1020302\,009744^2 + 32323200^2 &= 102030201\,0256^2\\
10203040302\,009744^2 + 3232323200^2 &= 1020304030201\,0256^2\\
102030405040302\,009744^2 + 323232323200^2 &= 10203040504030201\,0256^2\\
1020304050605040302\,009744^2 + 32323232323200^2 &= 102030405060504030201\,0256^2\\
10203040506070605040302\,009744^2 + 3232323232323200^2 &= 1020304050607060504030201\,0256^2\\
102030405060708070605040302\,009744^2 + 323232323232323200^2 &= 10203040506070807060504030201\,0256^2\\
1020304050607080908070605040302\,009744^2 + 32323232323232323200^2 &= 102030405060708090807060504030201\,0256^2
\end{aligned} \tag{4.34}$$

- **For $n = 17$; $m = 25, 2525, 252525, 25252525, \ldots$ in (3.245):**

$$\begin{aligned}
009711^2 + 3400^2 &= 1\,0289^2\\
102\,009711^2 + 343400^2 &= 10201\,0289^2\\
1020302\,009711^2 + 34343400^2 &= 102030201\,0289^2\\
10203040302\,009711^2 + 3434343400^2 &= 1020304030201\,0289^2\\
102030405040302\,009711^2 + 343434343400^2 &= 10203040504030201\,0289^2\\
1020304050605040302\,009711^2 + 34343434343400^2 &= 102030405060504030201\,0289^2\\
10203040506070605040302\,009711^2 + 3434343434343400^2 &= 1020304050607060504030201\,0289^2\\
102030405060708070605040302\,009711^2 + 343434343434343400^2 &= 10203040506070807060504030201\,0289^2\\
1020304050607080908070605040302\,009711^2 + 34343434343434343400^2 &= 102030405060708090807060504030201\,0289^2
\end{aligned} \tag{4.35}$$

- **For $n = 18$; $m = 25, 2525, 252525, 25252525, \ldots$ in (3.245):**

$$
\begin{aligned}
009676^2 + 3600^2 &= 1\,0324^2\\
102\,009676^2 + 363600^2 &= 10201\,0324^2\\
1020302\,009676^2 + 36363600^2 &= 102030201\,0324^2\\
10203040302\,009676^2 + 3636363600^2 &= 1020304030201\,0324^2\\
102030405040302\,009676^2 + 363636363600^2 &= 10203040504030201\,0324^2\\
1020304050605040302\,009676^2 + 36363636363600^2 &= 102030405060504030201\,0324^2\\
10203040506070605040302\,009676^2 + 3636363636363600^2 &= 1020304050607060504030201\,0324^2\\
102030405060708070605040302\,009676^2 + 363636363636363600^2 &= 10203040506070807060504030201\,0324^2\\
1020304050607080908070605040302\,009676^2 + 36363636363636363600^2 &= 102030405060708090807060504030201\,0324^2
\end{aligned}
\tag{4.36}
$$

- **For $n = 19$; $m = 25, 2525, 252525, 25252525, ...$ in (3.245):**

$$
\begin{aligned}
009639^2 + 3800^2 &= 1\,0361^2\\
102\,009639^2 + 383800^2 &= 10201\,0361^2\\
1020302\,009639^2 + 38383800^2 &= 102030201\,0361^2\\
10203040302\,009639^2 + 3838383800^2 &= 1020304030201\,0361^2\\
102030405040302\,009639^2 + 383838383800^2 &= 10203040504030201\,0361^2\\
1020304050605040302\,009639^2 + 38383838383800^2 &= 102030405060504030201\,0361^2\\
10203040506070605040302\,009639^2 + 3838383838383800^2 &= 1020304050607060504030201\,0361^2\\
102030405060708070605040302\,009639^2 + 383838383838383800^2 &= 10203040506070807060504030201\,0361^2\\
1020304050607080908070605040302\,009639^2 + 38383838383838383800^2 &= 102030405060708090807060504030201\,0361^2
\end{aligned}
\tag{4.37}
$$

- **For $n = 20$; $m = 25, 2525, 252525, 25252525, ...$ in (3.245):**

$$
\begin{aligned}
009600^2 + 4000^2 &= 1\,0400^2\\
102\,009600^2 + 404000^2 &= 10201\,0400^2\\
1020302\,009600^2 + 40404000^2 &= 102030201\,0400^2\\
10203040302\,009600^2 + 4040404000^2 &= 1020304030201\,0400^2\\
102030405040302\,009600^2 + 404040404000^2 &= 10203040504030201\,0400^2\\
1020304050605040302\,009600^2 + 40404040404000^2 &= 102030405060504030201\,0400^2\\
10203040506070605040302\,009600^2 + 4040404040404000^2 &= 1020304050607060504030201\,0400^2\\
102030405060708070605040302\,009600^2 + 404040404040404000^2 &= 10203040506070807060504030201\,0400^2\\
1020304050607080908070605040302\,009600^2 + 40404040404040404000^2 &= 102030405060708090807060504030201\,0400^2
\end{aligned}
\tag{4.38}
$$

... ...
... ...

- **For $n = 99$; $m = 25, 2525, 252525, 25252525, ...$ in (3.245):**

$$
\begin{aligned}
000199^2 + 19800^2 &= 1\,9801^2 \\
102\,000199^2 + 1999800^2 &= 10201\,9801^2 \\
1020302\,000199^2 + 199999800^2 &= 102030201\,9801^2 \\
10203040302\,000199^2 + 19999999800^2 &= 1020304030201\,9801^2 \\
102030405040302\,000199^2 + 1999999999800^2 &= 10203040504030201\,9801^2 \\
1020304050605040302\,000199^2 + 199999999999800^2 &= 102030405060504030201\,9801^2 \\
10203040506070605040302\,000199^2 + 19999999999999800^2 &= 1020304050607060504030201\,9801^2 \\
102030405060708070605040302\,000199^2 + 1999999999999999800^2 &= 10203040506070807060504030201\,9801^2 \\
1020304050607080908070605040302\,000199^2 + 199999999999999999800^2 &= 102030405060708090807060504030201\,9801^2
\end{aligned}
$$

$$\ldots \quad (414)$$

Remark 10. *We observe the last four number in last column and each row is a perfect square, for example, i.e.,*

$$0001 = 1^2,\ 0004 = 2^2,\ 0009 = 3^2,\ \ldots,\ 9409 = 97^2,\ 9604 = 98^2,\ 9801 = 99^2.$$

We have written only 20 patterned triples in a sequence and last pattern in number 99. We can go further, but in many cases, middle term of the first row don't follow the same pattern. For more details refer author's work [7].

Moreover, the sum of last four digits of first and third rows are always 10000. See below:

- ***Sum of Last Four Digits: First and Third Rows***

$$
\begin{aligned}
n = 1 &\quad \Rightarrow \quad 9999 + 0001 = 10000 \\
n = 2 &\quad \Rightarrow \quad 9996 + 0004 = 10000 \\
n = 3 &\quad \Rightarrow \quad 9991 + 0009 = 10000 \\
n = 4 &\quad \Rightarrow \quad 9984 + 0016 = 10000 \\
\ldots &\quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
\ldots &\quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
n = 97 &\quad \Rightarrow \quad 0591 + 9409 = 10000 \\
n = 98 &\quad \Rightarrow \quad 0396 + 9604 = 10000 \\
n = 99 &\quad \Rightarrow \quad 0199 + 9801 = 10000
\end{aligned}
$$

5 Final Comments

The whole work is concentrated to bring pattered Pythagorean triples. Four Procedures of two variables are used to bring these results. Even though, we can write the fifth procedure but it don't bring much different results. Most of the results can be obtained just from the Procedures 1 and 2. See below the fifth procedure:

$$
\begin{aligned}
F_5(m) &:= 64m^2 - n^2 \\
G_5(m) &:= 16mn \\
H_5(m) &:= 64m^2 + n^2
\end{aligned}
$$

In case of patterns with even numbers, divisions by 2, 4, 8, etc. are considered to reach patterns with primitive values. In some cases, there are same examples in two or more procedures.

In subsections 4.1, 4.2 and 4.3 we brought the patterns connected with palindromic- type pandigital pythagorean triples. In subsections 4.2 and 4.3, the results are similar and are depending in number 0 in between. While in 4.1 the results are not depends on 0 in between. In subsections 4.1 and 4.2, we have only 9 examples in each case, while in case of 4.3 there are total 99 examples. The Procedure 2 is well known in the literature, while the Procedures 1, 3 and 4 are not very much known. Moreover, all the results given in subsections 4.1, 4.2 and 4.3 can be obtained from the Procedure 2. For more details refer author's work [6, 7]. It is interesting to observe the ending numbers in each case:

- **Ending Numbers in Pandigital Patterns:**

$$\begin{aligned}
&\text{Subsection 4.1} \;\Rightarrow\; 01 = 1^2,\ 04 = 2^2,\ 09 = 3^2,\ \ldots,\ 81 = 9^2 \\
&\text{Subsection 4.2} \;\Rightarrow\; 01 = 1^2,\ 04 = 2^2,\ 09 = 3^2,\ \ldots,\ 81 = 9^2 \\
&\text{Subsection 4.3} \;\Rightarrow\; 01 = 1^2,\ 04 = 2^2,\ 09 = 3^2,\ \ldots,\ 9801 = 99^2.
\end{aligned}$$

In case of single variable, there are five Procedures and in case of double variables there are four Procedures. The connections of Pythagorean triples with magic squares refer author's another work [3].

Detailed study of **Palindromic-Type Pandigital Patterns** are given in Taneja [7, 8]. Extension to **multiple-type** examples, refer to Taneja [9]. Applications to generating magic squares are given in Taneja [10, 11]

6 Summary of Procedures

This section revises the Procedures given above in one and two variables.

6.1 Single Variable

6.1.1 Procedure 1

Let's consider the following three functions:

$$\begin{aligned}
f_1(n) &:= 2n(n+1) \\
g_1(n) &:= 2n+1 \\
h_1(n) &:= 2n^2+2n+1
\end{aligned} \tag{6.1}$$

We can easily check that

$$\begin{aligned}
f_1(n)^2 + g_1(n)^2 &= 4n^2(n+1)^2 + (2n+1)^2 \\
&= 4n^4 + 8n^3 + 4n^2 + +4n^2 + 4n + 1 \\
&= 4n^4 + 8n^3 + 8n^2 + 4n + 1 \\
&= (2n^2+2n+1)^2 \\
&= h_1(n)^2
\end{aligned}$$

This proves that the triple (f_1, g_1, h_1) is a **Pythagorean triple**.

6.1.2 Procedure 2

Let's consider the following three functions:

$$\begin{aligned} f_2(m) &:= m(m+2) \\ g_2(m) &:= 2(m+1) \\ h_2(m) &:= m^2+2m+2 \end{aligned} \tag{6.2}$$

We can easily check that

$$\begin{aligned} f_2(m)^2 + g_2(m)^2 &= (m(m+2))^2 + (2(m+1))^2 \\ &= m^4 + 4m^3 + 4m^2 + 4m^2 + 8m + 4 \\ &= m^4 + 4m^3 + 8m^2 + 8m + 4 \\ &= (m^2+2m+2)^2 \\ &= h_2(m)^2 \end{aligned}$$

This proves that the triple (f_2, g_2, h_2) is a **Pythagorean triple**.

6.1.3 Procedure 3

Let's consider the following three functions:

$$\begin{aligned} f_3(m) &:= m^2-1 \\ g_3(m) &:= 2m \\ h_3(m) &:= m^2+1 \end{aligned} \tag{6.3}$$

Then we can easily check that

$$\begin{aligned} f_3(m)^2 + g_3(m)^2 &= \left(m^2-1\right)^2 + (2m)^2 \\ &= m^4 - 2m^2 + 4m^2 + 1 \\ &= m^4 + 2m^2 + 1 \\ &= \left(m^2+1\right)^2 = h_3(m)^2 \end{aligned}$$

This proves that the triple (f_3, g_3, h_3) is a **Pythagorean triple**.

6.1.4 Procedure 4

Let's consider the following three functions:

$$\begin{aligned} f_4(m) &:= 4m^2-1 \\ g_4(m) &:= 4m \\ h_4(m) &:= 4m^2+1 \end{aligned} \tag{6.4}$$

Then we can easily check that

$$\begin{aligned} f_4(m)^2 + g_4(m)^2 &= \left(4m^2-1\right)^2) + (4m)^2 \\ &= 16m^4 - 8m^2 + 1 + 16m^2 \\ &= 16m^4 + 8m^2 + 1 \\ &= \left(4m^2+1\right)^2 = h_4(m)^2 \end{aligned}$$

This proves that the triple (f_4, g_4, h_4) is a **Pythagorean triple**.

6.1.5 Procedure 5

Let's consider the following three functions:

$$\begin{aligned} f_5(m) &:= 16m^2 - 1 \\ g_5(m) &:= 8m \\ h_5(m) &:= 16m^2 + 1 \end{aligned} \tag{6.5}$$

Then we can easily check that

$$\begin{aligned} f_5(m)^2 + g_5(m)^2 &= \left(16m^2 - 1\right)^2 + (8m)^2 \\ &= 256m^4 - 32m^2 + 1 + 64m^2 \\ &= 256m^4 + 32m^2 + 1 \\ &= \left(16m^2 + 1\right)^2 = h_5(m)^2. \end{aligned}$$

This proves that the triple (f_5, g_5, h_5) is a **Pythagorean triple**.

6.2 Double Variables

6.2.1 Procedure 1

For all $m, n \in N_+$, let's consider the following three functions:

$$\begin{aligned} F_1(m,n) &:= m(2n+m) \\ G_1(m,n) &:= 2n(n+m) \\ H_1(m,n) &:= m^2 + 2mn + 2n^2 \end{aligned} \tag{6.6}$$

We can easily check that

$$\begin{aligned} F_1(m,n)^2 + G_1(m,n)^2 &= (m(2n+m))^2 + (2n(n+m))^2 \\ &= m^4 + 4m^3n + 4m^2n^2 + 4m^2n^2 + 8mn^3 + 4n^4 \\ &= m^4 + 4m^3n + 8m^2n^2 + 8mn^3 + 4n^4 \\ &= \left(m^2 + 2mn + 2n^2\right)^2 = H_1(m,n)^2. \end{aligned}$$

This proves that the triple (F_1, G_1, H_1) is a **Pythagorean triple** for all $m, n \in N_+$. In particular, we have

$$\begin{aligned} (F_1(1,n), G_1(1,n), H_1(1,n)) &= (f_1(n), g_1(n), h_1(n)) \\ (F_1(m,1), G_1(m,1), H_1(m,1)) &= (f_2(m), g_2(m), h_2(m)) \end{aligned}$$

where

$$\begin{aligned} f_1(n) &:= 2n + 1 \\ g_1(n) &:= 2n(n+1) \\ h_1(n) &:= 2n^2 + 2n + 1 \end{aligned} \tag{6.7}$$

and

$$\begin{aligned} f_2(m) &:= m(m+2) \\ g_2(m) &:= 2(m+1) \\ h_2(m) &:= m^2 + 2m + 2 \end{aligned} \tag{6.8}$$

The Procedures (6.7) and (6.8) are two single variable procedures given in (6.1) and (6.2) respectively

6.2.2 Procedure 2

For all $m, n \in N_+, \quad m > n \geq 1$, let's consider the following three functions:

$$\begin{aligned} F_2(m,n) &:= m^2 - n^2 \\ G_2(m,n) &:= 2mn \\ H_2(m,n) &:= m^2 + n^2 \end{aligned} \tag{6.9}$$

Then we can easily check that

$$\begin{aligned} F_2(m,n)^2 + G_2(m,n)^2 &= \left(m^2 - n^2\right)^2 + (2mn)^2 \\ &= m^4 - 2m^2n^2 + n^4 + 4m^2n^2 \\ &= m^4 + 2m^2n^2 + n^4 \\ &= \left(m^2 + n^2\right)^2 = H_2(m,n)^2. \end{aligned}$$

This proves that the triple (F_2, G_2, H_2) is a **Pythagorean triple** for all $m, n \in N_+, \quad m > n \geq 1$. In particular, when $n = 1$, we get $F_2(m,1) = f_3(m), \; G_2(m,1) = g_3(m)$ and $H_2(m,1) = h_3(m)$, i.e.,

$$(F_2(m,1), G_2(m,1), H_2(m,1)) = (f_3(m), g_3(m), h_3(m)).$$

where

$$\begin{aligned} f_3(m) &:= m^2 - 1 \\ g_3(m) &:= 2m \\ h_3(m) &:= m^2 + 1 \end{aligned} \tag{6.10}$$

The Procedure (6.10) in single variable is same as given in (6.3)

6.2.3 Procedure 3

For all $m, n \in N_+, \quad 2m > n \geq 1$, let's consider the following three functions:

$$\begin{aligned} F_3(m,n) &:= 4m^2 - n^2 \\ G_3(m,n) &:= 4mn \\ H_3(m,n) &:= 4m^2 + n^2 \end{aligned} \tag{6.11}$$

Then we can easily check that

$$\begin{aligned} F_3(m,n)^2 + G_3(m,n)^2 &= \left(4m^2 - n^2\right)^2 + (4mn)^2 \\ &= 16m^4 - 8m^2n^2 + n^4 + 16m^2n^2 \\ &= 16m^4 + 8m^2n^2 + n^4 \\ &= \left(4m^2 + n^2\right)^2 = H_3(m,n)^2. \end{aligned}$$

This proves that the triple (F_3, G_3, H_3) is a **Pythagorean triple** for all $m, n \in N_+, \quad 2m > n \geq 1$. In particular, when $n = 1$, we get $F_3(m,1) = f_4(m), \; G_3(m,1) = g_4(m)$ and $H_3(m,1) = h_4(m)$, i.e.,

$$(F_3(m,1), G_3(m,1), H_3(m,1)) = (f_4(m), g_4(m), h_4(m)),$$

where

$$\begin{aligned} f_4(m) &:= 4m^2 - 1 \\ g_4(m) &:= 4m \\ h_4(m) &:= 4m^2 + 1 \end{aligned} \tag{6.12}$$

The Procedure (6.12) is same as given in (6.4).

6.2.4 Procedure 4

For all $m, n \in N_+,\ \ 4\,m > n \geq 1$, let's consider the following three functions:

$$\begin{aligned} F_4(m,n) &:= 16m^2 - n^2 \\ G_4(m,n) &:= 8mn \\ H_4(m,n) &:= 16m^2 + n^2 \end{aligned} \tag{6.13}$$

Then we can easily check that

$$\begin{aligned} F_4(m,n)^2 + G_4(m,n)^2 &= \left(16m^2 - n^2\right)^2 + (8mn)^2 \\ &= 256m^4 - 32m^2n^2 + n^4 + 64m^2n^2 \\ &= 256m^4 + 32m^2n^2 + n^4 \\ &= \left(16m^2 + n^2\right)^2 = H_4(m,n)^2. \end{aligned}$$

This proves that the triple (F_4, G_4, H_4) is a **Pythagorean triple** for all $m, n \in N_+,\ \ 4\,m > n \geq 1$. In particular, when $n = 1$, we get $F_4(m,1) = f_5(m)$, $G_4(m,1) = g_5(m)$ and $H_4(m,1) = h_5(m)$, i.e.,

$$(F_4(m,1), G_4(m,1), H_4(m,1)) = (f_5(m), g_5(m), h_5(m)),$$

where

$$\begin{aligned} f_5(m) &:= 16m^2 - 1 \\ g_5(m) &:= 8m \\ h_5(m) &:= 16m^2 + 1 \end{aligned} \tag{6.14}$$

The Procedure (6.14) is same as given in (6.5).

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