ON A TYPE OF MODULAR RELATION

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1. *Definitions and Notation.*—In these *Proceedings*, Ser. 2, Vol. 18, p. xx, a statement is made by Ramanujan with reference to the functions $G(x)$, $H(x)$, defined as basic series or as infinite products.* From the basic series forms, it follows that

$$H(x)/G(x) = \frac{1}{1+}\,\frac{x}{1+}\,\frac{x^2}{1+}\,\frac{x^3}{1+}\cdots;$$

from the product forms it follows that

$$G(x) = \frac{1-x^2-x^3+x^9+x^{11}-\dots}{1-x-x^2+x^5+x^7-\dots},$$

$$H(x) = \frac{1-x-x^4+x^7+x^{13}-\dots}{1-x-x^2+x^5+x^7}.$$

It will be convenient to make a slight alteration in the use of the argument-symbol x, by writing q^2 for x, to bring the series into line with the usual elliptic function notation. Moreover, it will be better to adopt what we may call a *standard* form of ϑ-function, in which the numerators of all indices are perfect squares. This is easily done by multiplying by a suitable power of x in each case, which we may call the *standardizing* power with standardizing index. Thus the numerators of $G(x)$ and $H(x)$ require indices $\frac{1}{40}$, $\frac{9}{40}$ respectively, and the common denominator $\frac{1}{24}$.

I write then

$$g = \frac{q^{\frac{1}{20}}(1-q^4-q^6+\dots)}{q^{\frac{1}{12}}(1-q^2-q^4+\dots)}, \qquad h = \frac{q^{\frac{9}{20}}(1-q^2-q^8+\dots)}{q^{\frac{1}{12}}(1-q^2-q^4+\dots)}, \qquad (1.1$$

* In the numerator of the second term of the series for $G(x)$, *for* 1, *read* x.

so that Ramanujan's identity now takes the form

$$gh(g^{10}-11g^5h^5-h^{10}) = 1. \tag{1.2}$$

The quotient h/g will be written μ (1.3), and the results of replacing q by q^p in g, h, μ will be written g_p, h_p, μ_p (1.31). The continued fraction form for μ is now

$$\mu = \frac{q^{\frac{2}{5}}}{1+}\,\frac{q^2}{1+}\,\frac{q^4}{1+}\,\frac{q^6}{1+}\,\cdots, \tag{1.4}$$

which has the advantage of conciseness in form, but is otherwise, as the basic series are, irrelevant to the present investigation, which is based purely on ϑ-function identities.

I may point out that Ramanujan's other identity (*loc. cit.*)

$$H(x)\,G(x^{11})-x^2G(x)\,H(x^{11}) = 1,$$

now becomes

$$hg_{11}-gh_{11} = 1, \tag{1.5}$$

which, with (1.2), has the advantage of containing no extraneous powers of the argument.

The denominators of g and h are

$$\frac{1}{\sqrt{3}}\,\vartheta_1\left(\frac{\pi}{3},\ q^{\frac{1}{3}}\right) = q^{\frac{1}{12}}\prod_1^{\infty}(1-q^{2n}),$$

and will be written P,* with P_p for

$$\frac{1}{\sqrt{3}}\,\vartheta_1\left(\frac{\pi}{3},\ q^{\frac{1}{3}p}\right).$$

Dashes attached to g, h, μ, will denote like functions of the complementary modulus q' (1.6).

2. *Proof of identity* (1.2).—Writing c_1 for $\cos\frac{1}{10}\pi$, c_3 for $\cos\frac{3}{10}\pi$, and

$$a = c_3/c_1 = \tfrac{1}{2}(\sqrt{5}-1), \tag{2.1}$$

we have

$$\vartheta_2\left(\tfrac{1}{10}\pi,\ q^{\frac{1}{5}}\right) = 2c_1\left(q^{\frac{1}{20}}+aq^{\frac{9}{20}}-aq^{\frac{49}{20}}-q^{\frac{81}{20}}-\ldots\right),$$

or, replacing the ϑ_2-function by its equivalent product,

$$P(g+ah) = q^{\frac{1}{20}}\prod_1^{\infty}\left(1+q^{\frac{2}{5}n}e^{\frac{1}{5}\pi i}\right)\left(1+q^{\frac{2}{5}n}e^{-\frac{1}{5}\pi i}\right)q^{-\frac{1}{60}}P_{\frac{1}{5}}, \tag{2.2}$$

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* The use of G as in Whittaker and Watson's *Modern Analysis*, p. 465, is at present untenable, and is also unstandardised.

while

$$\vartheta_2(\tfrac{3}{10}\pi,\ q^{\frac{1}{5}}) = 2c_3 P\left(g - \frac{h}{a}\right)$$

$$= 2c_3 q^{\frac{1}{20}} \prod_1^\infty (1+q^{\frac{2}{5}n} e^{\frac{3}{5}\pi i})(1+q^{\frac{2}{5}n} e^{-\frac{3}{5}\pi i})\, q^{-\frac{1}{60}} P_{\frac{1}{5}}. \tag{2.3}$$

Hence

$$P^2(g+ah)\left(g - \frac{1}{a}\,h\right) = P^2(g^2-gh-h^2) = q^{\frac{1}{10}} \prod_1^\infty \left(\frac{1-q^{2n}}{1-q^{\frac{2}{5}n}}\right) q^{-\frac{1}{30}} P^2_{\frac{1}{5}}$$

$$= P P_{\frac{1}{5}}. \tag{2.4}$$

Moreover $$P^2 gh = q^{\frac{1}{2}}(1-q^4)(1-q^6)\ldots(1-q^{10})(1-q^{20})\ldots$$

$$\times(1-q^2)(1-q^8)\ldots(1-q^{10})(1-q^{20})\ldots$$

$$= PP_5. \tag{2.5}$$

But, by changing q into $q\omega$, $q\omega^2$, ..., where $\omega = e^{\frac{2}{5}\pi i}$, and multiplying the results (2.2), we have on the left-hand side a factor $g^5+a^5h^5$, since h/g is altered to $\omega^2 h/g$, &c., while on the right-hand side the product arising from any factor $1+q^{\frac{2}{5}n}e^{\frac{1}{5}\pi i}$, *i.e.* $1-q^{\frac{2}{5}n}\omega^2$, is $1-q^{2n}$, if n is not a multiple of 5, but is otherwise $(1-q^{2m}\omega^2)^5$, where $n = 5m$.

Hence

$$P^5(g^5+a^5h^5)$$

$$= q^{\frac{1}{3}} \Pi(1-q^{2n})^2 \prod_1^\infty (1-q^{2m}\omega^2)^5 (1-q^{2m}\omega^3)^5\, \Pi(1-q^{2n}) \prod_1^\infty (1-q^{2m})^5,$$

where n has all positive integral values except multiples of 5, *i.e.*

$$P^5(g^5+a^5h^5) = q^{\frac{1}{3}} \frac{\prod_1^\infty (1-q^{2n})^8}{\prod_1^\infty (1-q^{10n})^3} \prod_1^\infty (1+q^{2n}e^{\frac{1}{5}\pi i})^5 (1+q^{2n}e^{-\frac{1}{5}\pi i})^5$$

$$= \frac{P^3}{P_5^3}\, \frac{\vartheta_2(\frac{1}{10}\pi,\ q)^5}{(2c_1)^5} = \frac{P^3}{P_5^3}(g_5+ah_5)^5 P_5^5, \tag{2.6}$$

i.e. $$g^5+a^5h^5 = P_5^2 (g_5+ah_5)^5/P^2. \tag{2.7}$$

Similarly, from (2.3), or by changing $\sqrt{5}$ to $-\sqrt{5}$ in a, we have

$$g^5-a^{-5}h^5 = P_5^2(g_5-a^{-1}h_5)^5/P^2. \tag{2.8}$$

Changing q to q^5, (2 . 4) becomes

$$g_5^2 - g_5 h_5 - h_5^2 = P/P_5, \tag{2.9}$$

while (2 . 5) becomes $$gh = P_5/P: \tag{2.10}$$

whence, seeing that $a^5 = \frac{1}{2}(5\sqrt{5}-11)$, we have

$$gh\,(g^{10} - 11g^5h^5 - h^{10}) = P_5^5\,(g_5^2 - g_5 h_5 - h_5^2)^5/P^5 = 1.$$

It may be here observed that a similar relation exists between the ϑ-series derived from $\vartheta_2(x, q^{1/p})$, where

$$x = \frac{\pi}{2p},\ \frac{3\pi}{2p},\ \ldots,\ \frac{p-2}{p}\pi,$$

each being divided by P. The relation asserts the equality to unity of a homogeneous algebraic function of degree $\frac{1}{2}(p^2-1)$ in the ϑ-quotients corresponding to g and h when $p = 5$.

3. The main object of the present memoir is to establish algebraic relations connecting μ and μ_p when $p = 2, 3, 5, 11$.

It is easy to see that such algebraic relations exist. For, by (2 . 4), (2 . 5), and (1 . 3), (1 . 7),

$$1 - \mu - \mu^2 = \mu\, P_{\frac{1}{5}}/P_5.$$

But $P_{\frac{1}{5}}/P$ and P_5/P are known each to be connected algebraically with the moduli and multipliers in the quintic transformation of elliptic functions, and hence with the modulus k, so that μ is connected algebraically with k; and hence the modular equation of any order p implies an algebraic relation between μ and μ_p.

4. *Complementary relation.*—From the formula

$$\sqrt{w}\,\vartheta_2(x, q) = \sum_{n=-\infty}^{\infty} (-1)^n e^{-\pi/w\,(n + x/\pi)^2},$$

where $e^{-\pi w} = q$, we have

$$\frac{\vartheta_2(\frac{3}{10}\pi,\ q^{\frac{1}{5}})}{\vartheta_2(\frac{1}{10}\pi,\ q^{\frac{1}{5}})} = \frac{\Sigma(-1)^n e^{-5\pi/w\,(n+\frac{3}{10})^2}}{\Sigma(-1)^n e^{-5\pi/w\,(n+\frac{1}{10})^2}}.$$

which, with the notation of (1 . 3), (1 . 6), and (2 . 1), gives

$$\frac{a-\mu}{1+a\mu} = \mu'.^* \tag{4.1}$$

5. *Quadratic relation.*—Writing u_n for $\vartheta_2(\frac{1}{10}n, q)$, where

$$n = 0,\ 1,\ 2,\ 3,\ 4,$$

we have from the formula

$$\vartheta_2(x+y+z)\,\vartheta_2(x)\,\vartheta_2(y)\,\vartheta_2(z)+\vartheta_1(x+y+z)\,\vartheta_1(x)\,\vartheta_1(y)\,\vartheta_1(z)$$
$$= \vartheta_2(0)\,\vartheta_2(y+z)\,\vartheta_2(z+x)\,\vartheta_2(x+y).$$

when $x = y = \frac{1}{10}\pi,\ z = \frac{1}{5}\pi$,

$$u_1 u_4 (u_1 u_2 + u_3 u_4) = u_0 u_2 u_3^2, \tag{5.1}$$

and when $x = y = \frac{3}{10}\pi,\ z = \frac{3}{5}\pi$,

$$u_2 u_3 (u_1 u_2 - u_3 u_4) = u_0 u_1^2 u_4. \tag{5.2}$$

But
$$\frac{\vartheta_1(\frac{1}{5}\pi,\ q^2)}{\vartheta_1(\frac{3}{5}\pi,\ q^2)} = \frac{\vartheta_1(\frac{1}{10}\pi)\,\vartheta_2(\frac{1}{10}\pi)}{\vartheta_1(\frac{3}{10}\pi)\,\vartheta_2(\frac{3}{10}\pi)},$$

from a known formula for $\vartheta_1(2x, q^2)$, *i.e.*

$$\frac{\vartheta_2(\frac{3}{10}\pi,\ q^2)}{\vartheta_2(\frac{1}{10}\pi,\ q^2)} = \frac{u_1 u_4}{u_3 u_2}. \tag{5.3}$$

By § 4, $u_3/u_1 = \mu'$, so that (5 . 3) gives $u_4/u_2 = \mu'\mu_2'$, and (5 . 1), (5 . 2) give by division

$$\frac{u_4^2}{u_2^2}\,\frac{1+\mu' u_4/u_2}{1-\mu' u_4/u_2} = \mu'^2.$$

Suppressing dashes and changing μ, μ_2 to μ_2, μ we get, finally,

$$\mu^3\mu_2^2+\mu^2+\mu\mu_2^3-\mu_2 = 0. \tag{5.4}$$

6. *Cubic relation.*—Since

$$Q\vartheta_1(3x,\ q^3) = \vartheta_1(x)\,\{\vartheta_2^2(x)\,\vartheta_1^2(\tfrac{1}{3}\pi)-\vartheta_1^2(x)\,\vartheta_2^2(\tfrac{1}{3}\pi)\}/\vartheta_2^2(0),$$

where
$$Q = \vartheta_1^2(\tfrac{1}{3}\pi)\,\vartheta_1'(0)/3\vartheta_1'(0,\ q^3),$$

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* This gives a simple numerical value for μ when $q = e^{-\pi}$, for $a/(1-a^2) = 1$, so that $a = \tan(\frac{1}{2}\tan^{-1}2)$, and $\mu = \tan(\frac{1}{4}\tan^{-1}2)$.

we easily deduce that

$$\vartheta_1(3x, q^3)\,\vartheta_1^3(2x)-\vartheta_1(6x, q^3)\,\vartheta_1(x)$$

$$= 3\,\frac{\vartheta_1'(0, q^3)}{\vartheta_1'(0)\,\vartheta_2^2(0)}\,\vartheta_1(x)\,\vartheta_1(2x)\,\{\vartheta_2^2(x)\,\vartheta_1^2(2x)-\vartheta_1^2(x)\,\vartheta_2^2(2x)\}$$

$$= 3\,\frac{\vartheta_1'(0, q^3)}{\vartheta_1'(0)}\,\vartheta_1^2(x)\,\vartheta_1(2x)\,\vartheta_1(3x). \qquad (6\,.\,1)$$

By changing x into xi/w, we have a relation connecting ϑ_1-functions for moduli q'^3 and q', which after suppressing dashes and changing q to q^3, gives

$$\vartheta_1(x)\,\vartheta_1^3(2x, q^3)-\vartheta_1(2x)\,\vartheta_1^3(x, q^3) = \frac{\vartheta_1'(0)}{\vartheta_1'(0, q^3)}\,\vartheta_1^2(x, q^3)\,\vartheta_1(2x, q^3)\,\vartheta_1(3x, q^3). \quad (6\,.\,2)$$

Multiplying (6 . 1) and (6 . 2), the factors independent of x cancel out. If $x = \frac{1}{5}\pi$, $\vartheta_1(3x) = \vartheta_1(2x)$, and

$$\frac{\vartheta_1(x)}{\vartheta_1(2x)} = \frac{u_3}{u_1} = \mu'.$$

Also $\vartheta_1(6x, q^3) = -\vartheta_1(x, q^3)$, so that in the resultant equation we connect $\vartheta_1(x)/\vartheta_1(2x)$ with $\vartheta_1(x, q^3)/\vartheta_2(2x, q^3)$, *i.e.* μ' with μ_3'. Writing then μ_3 for the former and μ for the latter, we get

$$(1+\mu\mu_3^3)(\mu_3-\mu^3) = 3\mu^2\mu_3^2,$$

or

$$\mu^4\mu_3^3+\mu^3+3\mu^2\mu_3^2-\mu\mu_3^4-\mu_3 = 0. \qquad (6\,.\,2)$$

7. *Quintic relation.*—From (2 . 7), (2 . 8), we have

$$\frac{a^5-\mu^5}{1+a^5\mu^5} = \left(\frac{a-\mu_5}{1+a\mu_5}\right)^5;$$

i.e.

$$\mu^5 = \mu_5\,\frac{1-2\mu_5+4\mu_5^2-3\mu_5^3+\mu_5^4}{1+3\mu_5+4\mu_5^2+2\mu_5^3+\mu_5^4}. \qquad (7\,.\,1)$$

8. *Relation when* $p = 11$.—This is immediately deduced from Ramanujan's formulæ (1 . 2) and (1 . 5), viz.

$$\mu\mu_{11}(1-11\mu^5-\mu^{10})(1-11\mu_{11}^5-\mu_{11}^{10}) = (\mu-\mu_{11})^{12}. \qquad (8\,.\,1)$$

9. The identity (1 . 5), together with others of the same type,* may be

* These were communicated privately to me in February 1919, but, as I understand that Ramanujan has left no proof, I suggest the proof given in this section.

proved by Schröter's formulæ,* connected with the multiplication of two ϑ_3-series of different orders. As his formulæ require some modification and specialisation for the present problem, it will be simpler to give what is virtually his method in detail, and to employ summational forms instead of ϑ-function notation.

Let p be a prime number, and α, β any integers, such that

$$\alpha m^2+\beta = \lambda p, \tag{9.1}$$

m being odd. The indicial letters r, s, t, σ following the symbol Σ will denote summation extending through all values $0, \pm 1, \pm 2, \dots, \pm \infty$. Let

$$\Sigma(-1)^r q^{p\alpha(r+mv)^2}.\Sigma(-1)^s q^{p\beta(s+v)^2} = \Sigma\Sigma(-1)^{r+s} q^l, \tag{9.2}$$

so that

$$l = p\alpha(r+mv)^2+p\beta(s+v)^2.$$

Put $r = ms+t$, so that for any given value of s, t is equally general with r, and

$$l = p\alpha\{m(s+v)+t\}^2+p\beta(s+v)^2$$

$$= \lambda p^2(s+v)^2+2p\alpha mt(s+v)+p\alpha t^2 \quad [\text{by } (9.1)]$$

$$= \lambda\left\{p(s+v)+\frac{\alpha mt}{\lambda}\right\}^2+\frac{\alpha\beta}{\lambda}t^2,$$

while

$$(-1)^{r+s} = (-1)^{(m+1)s+t} = (-1)^t.$$

Now let v have the p values $\frac{1}{2p}, \frac{3}{2p}, \dots, \frac{2p-1}{2p}$, and add together all the equations (9.2) so obtained. The series on the left-hand side will be equal in pairs, while their values for $v = p/2p$ will be zero. On the right-hand side we have $\Sigma\Sigma(-1)^t q^l$, where now

$$l = \lambda\left\{p\left(s+\frac{2n-1}{2p}\right)+\frac{\alpha mt}{\lambda}\right\}^2+\frac{\alpha\beta}{\lambda}t^2 \quad \left(n = 1, 2, \dots, \frac{p-1}{2}\right)$$

$$= \lambda\left\{\frac{2\sigma+1}{2}+\frac{\alpha mt}{\lambda}\right\}^2+\frac{\alpha\beta}{\lambda}t^2. \tag{9.3}$$

Let $p = 5$, $m = 1$, so that $\alpha+\beta = 5\lambda$. If $v = \frac{1}{10}$,

$$\Sigma(-1)^r q^{5\alpha(r+\frac{1}{10})^2} = q^{\alpha/20}(1-q^{4\alpha}-q^{6\alpha}+\dots) = g_\alpha P_\alpha,$$

according to the notation of § 1. If $v = \frac{3}{10}$,

$$\Sigma(-1)^r q^{5\alpha(r+\frac{3}{10})^2} = h_\alpha P_\alpha.$$

* Enneper, *Elliptische Functionen*, Zweite Auflage, p. 474.

Hence $$2\,(g_\alpha g_\beta + h_\alpha h_\beta)\,P_\alpha P_\beta = \Sigma\Sigma\,(-1)\;q^{l}, \tag{9.4}$$

as in (9.3). Again, if $p = 5$, $m = 3$, so that $9\alpha+\beta = 5\lambda$, then when $v = \frac{1}{10}$,

$$\Sigma\,(-1)^r\,q^{5\alpha(r+\frac{1}{10})^2} = h_\alpha,$$

and when $v = \frac{3}{10}$, $$\Sigma\,(-1)^r\,q^{5\alpha(r+\frac{3}{10})^2} = -g_\alpha :$$

so that $$2\,(h_\alpha g_\beta - h_\beta g_\alpha) = \Sigma\Sigma\,q^{I}\,(-1)^{t}, \tag{9.41}$$

as in (9.3).

Now suppose $p = 3$, $m = 1$, and use letters a, b, l instead of α, β, λ. We shall get the same value of I as in (9.3), provided $l = \lambda$, $ab = \alpha\beta$, and

$$\frac{\alpha m}{\lambda} \pm \frac{a}{l} \text{ is an integer.} \tag{9.5}$$

When $v = \frac{1}{6}$ or $\frac{5}{6}$,

$$\Sigma\,(-1)^r\,q^{3a(r+v)^2} = q^{\frac{1}{12}a}\,(1-q^{2a}-q^{4a}+\dots) = P_a,$$

so that the left-hand side is

$$2P_aP_b. \tag{9.6}$$

Hence, by (9.4), $$g_\alpha g_\beta + h_\alpha h_\beta = P_aP_b\;P_\alpha P_\beta. \tag{9.7}$$

where $$\alpha+\beta = 5\lambda, \quad a+b = 3l = 3\lambda :$$

while, by (9.5), $$h_\alpha g_\beta - h_\beta g_\alpha = P_aP_b/P_\alpha P_\beta. \tag{9.71}$$

where $$\alpha+\beta = 5\lambda, \quad a+b = 3l = 3\lambda.$$

Thus, if $\alpha = 1$, $\beta = 11$, $m = 3$, $\lambda = 4$, then $a = 1$, $b = 11$, $l = 4$, so that (9.5) is satisfied; and, by (9.71),

$$hg_{11} - h_{11}g = PP_{11}/PP_{11} = 1,$$

as in (1.5).

If $\alpha = 1$, $\beta = 9$, $m = 1$, $\lambda = 2$, $a = 3$, $b = 3$, $l = 2$, we have

$$gg_9 + hh_9 = P_3^2/PP_9. \tag{9.8}$$

When $\alpha = 1$, $\beta = 14$, $m = 1$, $\lambda = 3$, $a = 2$, $b = 7$, $l = 3$, and

$$gg_{14} + hh_{14} = P_2P_7/PP_{14}. \tag{9.81}$$

When $\alpha = 2$, $\beta = 7$, $m = 3$, $\lambda = 5$, $a = 1$, $b = 14$, $l = 5$, and $3\alpha m/\lambda - a/l = 1$, and

$$h_2g_7 - h_7g_2 = PP_{14}/P_2P_7. \tag{9.82}$$

10. Certain cases of (9 . 4) and (9 . 41) may be treated without the help of the results for $p = 3$. For instance, if $\lambda = 1$, when of course $m = 1$, then I in (9 . 3) is equivalent to $\left(\frac{2\sigma+1}{2}\right)^2+\alpha\beta t^2$: for amt is an integer, and may be merged in the general symbol σ. In this case

$$2(g_\alpha g_\beta+h_\alpha h_\beta) = \Sigma q^{(\sigma+\frac{1}{2})^2}\Sigma(-1)^t q^{\alpha\beta t^2}/P_\alpha P_\beta$$
$$= \vartheta_2(0)\,\vartheta(0,\, q^{\alpha\beta})/P_\alpha P_\beta.$$

Thus
$$gg_4+hh_4 = \tfrac{1}{2}\vartheta_2(0)\,\vartheta(0,\, q^4)/PP_4,$$
$$g_2 g_3+h_2 h_3 = \tfrac{1}{2}\vartheta_2(0)\,\vartheta(0,\, q^5)/PP_6. \qquad (10\,.\,1)$$

Again, when $\alpha = 1$, $\beta = 6$, $m = 3$, $\lambda = 3$,

$$I = 3(\sigma+\tfrac{1}{2}+t)^2+It^2 \equiv 3(\sigma+\tfrac{1}{2})^2+It^2.$$

Hence
$$hg_6-h_6 g = \tfrac{1}{2}\Sigma q^{3(\sigma+\frac{1}{2})^2}\Sigma q^{2t^2}(-1)^t/PP_6$$
$$= \tfrac{1}{2}\vartheta_2(0,\, q^3)\,\vartheta(0,\, q^2)/PP_6. \qquad (10\,.\,2)$$

Again, when $p = 2$, $m = 1$, the left-hand series in (9 . 2) have only one form, derived from $v = \frac{1}{4}$ or $v = \frac{3}{4}$, viz.

$$\Sigma(-1)^r q^{2(r+\frac{1}{4})^2} = q^{\frac{1}{8}}(1-q-q^3+\ldots)$$
$$= q^{\frac{1}{8}}(1-q)(1-q^3)(1-q^5)(1-q^7)\ldots(1-q^4)(1-q^8)$$
$$= P_{\frac{1}{2}}P_2/P. \qquad (10\,.\,3)$$

Thus, if $\alpha = 3$, $\beta = 8$, $m = 3$, $\lambda = 7$, $a = 2$, $b = 12$, $p = 2$, $l = 7$, so that

$$\frac{ma}{\lambda}-\frac{a}{l} = 1,$$

we have
$$h_3 g_8-h_8 g_3 = \frac{PP_4}{P_2 P_8}\,\frac{P_6 P_{24}}{P_3 P_{12}}. \qquad (10\,.\,4)$$

When $\alpha = 1$, $\beta = 16$, $m = 3$, $\lambda = 5$, $a = 2$, $b = 12$, $p = 2$, $l = 7$, so that

$$\frac{ma}{\lambda}+\frac{a}{l} = 1,$$

$$hg_{16}-h_{16}g = \frac{PP_4}{P_2}\,\frac{P_4 P_{16}}{P_8}\,\frac{1}{PP_{16}} = \frac{P_4^2}{P_2 P_8}. \qquad (10\,.\,5)$$

These results, as well as many others, have all been given by Ramanujan.

11. To resume the theory of the modular connection between μ and μ_p, there still remains the case of $p = 7$, which presents great difficulties when treated by the methods of § 5 and § 6. The relation

$$(gg_{14}+hh_{14})(h_2 g_7 - h_7 g_2) = 1,$$

derived from (9 . 81) and (9 . 82), combined with (1 . 2), with its extension to suffixes 2, 7, 14, would, in connection with (5 . 4), give a relation between μ and μ_7, but the method is impracticable.

It is to be observed, however, that in the cases of $p = 2, 3, 11$ the relations are of degree $p+1$, just as the Jacobian modular equation in $\sqrt[4]{k}$ and $\sqrt[4]{l}$ is of degree $p+1$, except in the quadratic cases. Though it is not obvious how we may set forth a general hypothesis, we may at least see what the roots of μ are when $\mu_p (p = 2, 3, 11)$ is supposed given. Writing $\mu(q^{2p})$ for μ_p, we see that p of the roots of the equation are $\mu(q^2)$, $\mu(q^2\omega)$, $\mu(q^2\omega^2)$, ..., where $\omega = e^{2\pi i/p}$. Now

$$\mu(q^2) = q^{\frac{2}{5}}\Pi(1-q^{2n})^{\pm 1},$$

according as $n \equiv \pm 1$, or ± 2 (mod 5). The product of these p roots is therefore

$$q^{\frac{2}{5}p}\Pi(1-q^{2np})^{\pm 1}\,\Pi(1-q^{2m})^{\pm p},$$

where $n \not\equiv 0 \pmod{p}$, but $m \equiv 0 \pmod{p}$; except that, in the case of $p = 2$, $\mu(-q^2)$ is negative, and a negative sign must be placed before the expression.

Now $\Pi(1-q^{2np})^{\pm 1}$ includes all the binomial factors of μ_p, except those for which $n \equiv 0 \pmod{p}$. These can all be supplied by μ_{p^2}, either by multiplication or division.

Thus, when $p = 3$, we have $(1-q^6)(1-q^{24})\ldots/(1-q^{12})\ldots$, where $1-q^{36}$ fails in the numerator and $1-q^{18}$ in the denominator. Hence the required product of the p roots is $\mu_3\mu_9$; and in general, when $p = \pm 2$ (mod 5), it is $\mu_p\mu_{p^2}$. If, however, $p = \pm 1$ (mod 5), as when $p = 11$, we have $(1-q^{22})(1-q^{88})\ldots/(1-q^{44})\ldots$, where $1-q^{242}$ fails in the numerator and $1-q^{484}$ in the denominator, so that the product is μ_{11}/μ_{121}, or in general μ_p/μ_{p^2}.

Similarly in $\Pi(1-q^{2m})^{\pm p}$, when $m \equiv 0 \pmod{p}$, we have all the binomial factors of μ_p^{-p}, if $p \equiv \pm 2$ (mod 5), but all the binomial factors of μ_p^p, if $p \equiv \pm 1$ (mod 5). Hence the product of the p roots is $\mu_p^{-p+1}\mu_{p^2}$, when $p \equiv \pm 2$ (mod 5), and μ_p^{p+1}/μ_{p^2} when $p \equiv \pm 1$ (mod 5). Thus in the quadratic case, where, by (5 . 4), the product of all the roots in μ is $1/\mu_2$, it follows, since $\mu(-q^2)$ is negative, that the third root is $-1\ \mu_4$.

In the cubic case, by (6 . 2), the product of all the roots in μ is $-1/\mu_3^2$, so that the fourth root is $-1/\mu_9$.

In the quintic case (see § 7), the above considerations do not apply, and μ^5 is explicit in μ_5.

In the 11-ic case, the product of all the roots is μ_{11}^{12}, so that the 12th root is μ_{121}.

In conclusion we may notice that if $\mu_p = a$ then $\mu_{1/p} = 0$, by § 4; *i.e.* $q'_{1/p} = 0$, which by the modular theory of Jacobi and Sohncke implies that p roots of q' are zero, *i.e.* p roots of μ are a. Thus, when $\mu_2 = a$, (5 . 4) reduces to

$$(\mu - a)^2 \left(\mu - \frac{1}{a}\right) = 0:$$

when $\mu_3 = a$, (6 . 2) reduces to

$$(\mu - a)^3 \left(\mu + \frac{1}{a}\right) = 0:$$

when $\mu_{11} = a$, (8 . 1) reduces to

$$(\mu - a)^{12} = 0.$$