

# Error Rate Analysis and Optimal Power Allocation in Multiple Access Relay Channels with Analog Network Coding

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**Abstract**—In this paper, we examine multi-source multi-relay systems that employ Analog Network Coding for which we provide a two-fold contribution: i) We derive a closed-form upper bound of the average symbol error rate (SER) per source of the system, which is shown to be tight in the high-Signal-to-Noise Ratio (SNR) region, especially for an adequate number of relays, and ii) based on this bound, for a given total power budget to be distributed among the source and relay nodes we formulate the power allocation optimization problem with the aim of minimizing the SER per source. Results show that for a common target SER among the sources, an increase in their number results in an increase in the expected energy gains over the equal power allocation for all the nodes policy.

## I. INTRODUCTION

Network Coding was introduced as a method to achieve the maximum information flow in a network with multiple nodes by enabling the intermediate nodes to perform coding operations at the incoming packets, such as the Exclusive-OR operation [1]. Although initially aimed for wired networks, Network Coding can be readily extended to wireless networks that consist of intermediate nodes between the sources and the destination, such as relays [2].

Among the several Network Coding techniques proposed [3]-[7], it is generally agreed that the amplify-and-forward approach, which is also dubbed as Analog Network Coding [8], has the lowest complexity since it is the only technique in which the relays do not need to perform coding operations at the incoming signals other than just forwarding them to the destination after amplification. Hence, it is considered as an attractive option for future relay-based systems that employ Network Coding, especially when the relays are battery-powered terminals in which the relaying operations should be kept as simple as possible.

Since its introduction, there have been several works in the literature exploiting Analog Network Coding from an information-theoretic point of view [9]-[12]. However, designers of practical systems are usually interested in the

expected system diversity and error probabilities when employing a particular modulation scheme. Towards this end, [13] presented a diversity analysis of Analog Network Coding schemes that employ relay selection and distributed space-time coding [14]. However, since [13] is solely focused on the diversity analysis of Analog Network Coding, a loose upper bound is derived for the end-to-end error probability of the system. A tight upper bound is important though to analytically estimate the transmission power of the sources and the relays needed to achieve a target SER, which is of interest regarding the energy efficient design of the system since the sources and the relays can be mobile terminals with limited energy resources. More specifically, for a given total power budget to be distributed among the source and relay nodes, the optimal power distribution of power for minimizing the SER per source can be extracted from it. The work in [15] is based on these aims, however only the single-user case is investigated, and, moreover, the variable-gain relay type is considered, whereas the fixed-gain type is a better alternative for low-capability relay terminals since the estimation of the channel coefficients of the source-relay links is not required at the relays.

*Contribution:* Motivated by the practical need of having an analytical framework for evaluating the SER per source of Analog Network Coding systems, which can enable their energy-efficient design, in this work we: i) first present such a framework for fixed-gain relays, which is tight in the high-SNR region and, as we show in the numerical results of Section IV, and ii) based on it, for a given total power budget to be distributed among the source and relay nodes, we formulate the power allocation optimization problem with the aim of minimizing the SER per source of the system. Results show that for a common target SER among the sources, an increase in their number results in higher total energy savings compared to the equal power allocation policy.

*Organization:* The rest of the paper is organized as follows: In Section II, the system model is given. In Section III,

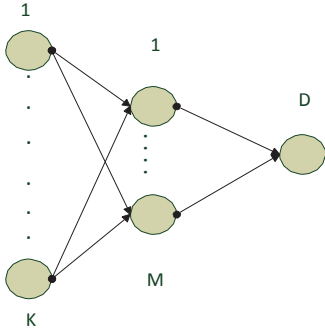


Fig. 1. The Multiple Access Relay Channel.

the theoretical analysis of the end-to-end SER per source is presented together with the problem formulation of the optimal power distribution among the source and relay nodes for minimizing the SER per source. In Section IV, numerical results are provided which substantiate the tightness of the analytical framework for the end-to-end SER per source in the high-SNR region. Moreover, numerical values are given for the energy gains that the optimal power allocation achieves over the equal power allocation for all the nodes approach. Finally, Section V concludes this paper.

*Notation:* The following notation is used throughout this paper: i)  $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp(-u^2/2) du$  denotes the Q-function; ii)  $E\{x\}$  denotes the mean value of the stochastic process  $x$ ; iii)  $|\cdot|$  and  $\|\cdot\|_F$  are the absolute value of a complex number and the Frobenius norm of a vector, respectively; iv)  $K_\nu(\cdot)$  denotes the modified Bessel function of the second kind and of the  $\nu$ th order, v)  $E_1(\cdot)$  is the exponential integral function, vi)  $\log(\cdot)$  is the natural logarithm, and vii) matrices and vectors are denoted in boldface.

## II. SYSTEM MODEL

We consider the multiple access relay channel (MARC) depicted in Fig. 1, which is a typical Uplink scenario for next generation cellular networks.  $K$  sources want to communicate with a common destination through the use of  $M$  fixed-gain half-duplex relays. We assume single-antenna nodes and that no direct link exists between the sources and the destination, which can be attributed to a heavy-shadowing environment, for instance. Let  $h_{km} \sim \sigma_{km}^2 CN(0, 1)$  be the channel coefficient from the  $k$ th source to the  $m$ th relay and  $f_m \sim \sigma_{mD}^2 CN(0, 1)$  the channel coefficient from the  $m$ th relay to the destination,  $k = 1, 2, \dots, K$ ,  $m = 1, 2, \dots, M$ , where  $\sigma_{km}^2$  and  $\sigma_{mD}^2$  are the path-loss coefficients (including shadowing effects) of the source-relay and relay-destination links, respectively. To simplify the notations and without loss of generality, we assume that  $\sigma_{km}^2 = \sigma_{SR}^2$  and  $\sigma_{mD}^2 = \sigma_{RD}^2$ . Physically, this means that

$$m_D = \sigma_{RD}$$

the sources and the relays are clustered and, hence, the path-loss coefficients of the source-relay and relay-destination links can be considered equal, respectively. Moreover, we assume that each source has the packet  $s_k$  to be dispatched to the

destination, which perfectly knows the instantaneous values of  $h_{km}$  and  $f_m$ . In addition, the relays have statistical knowledge of the source-relay links, which means that they are aware of  $\sigma_{SR}^2$ . Finally, without loss of generality we consider that the sources and the relays have equal transmission powers, which are denoted as  $P_S$  and  $P_R$ , respectively.

We distinguish two transmission phases in the system:

*1st Phase-Transmission from the sources to the relays:* During the first phase, which has a duration of 1 time slot, the  $K$  sources simultaneously transmit their modulated packets  $s_k$  to the relays. Hence, the received signal  $y_m$  at the  $m$ th relay is given by

$$y_m = \sum_{k=1}^K \sqrt{P_S} h_{km} s_k + n_m \quad (1)$$

where  $n_m \sim CN(0, 1)$  is the Additive White Gaussian Noise realization at  $m$ th relay.

*2nd Phase-Transmission from the relays to the destination:* During the second phase, the fixed-gain relays amplify their received signal and forward it to the destination. As a case-study, we consider the time-orthogonal transmission protocol from the relays, which means that they forward their signal to the destination one after the other in non-overlapping time slots. Based on this protocol,  $M + 1$  time slots are required for the end-to-end communication between the sources and the destination.

The gain  $r_m$  of the relays, which normalizes their average transmission power with respect to the average received power of the sources, is given by [16]

$$r_m = \frac{P_R \sigma_{RD}^2}{P_S \sigma_{SR}^2 + 1} \quad (2)$$

Consequently, the received signal at the destination from each of the relays is

$$\begin{aligned} y_{Dm} &= \sqrt{P_R} \sigma_{RD}^2 r_m y_m f_m + n_{Dm} \\ &= \frac{P_R \sigma_{RD}^2}{P_S \sigma_{SR}^2 + 1} \sum_{k=1}^K h_{km} s_k + \tilde{n}_{Dm} \end{aligned} \quad (3)$$

where  $n_{Dm} \sim CN(0, 1)$  is the Additive White Gaussian Noise Realization at the destination and  $\tilde{n}_{Dm} \sim CN(0, \frac{P_R \sigma_{RD}^2}{P_S \sigma_{SR}^2 + 1})$ .

By having the channel coefficients that correspond to the source-relay and relay-destination links, the destination can employ the optimal maximum-likelihood detection, and jointly obtain the transmitted symbols from the sources as

$$\begin{aligned} \mathbf{s}_{det} &= \arg \min_{\mathbf{s}_k} \sum_{m=1}^M \left( \frac{P_S \sigma_{SR}^2 P_R \sigma_{RD}^2}{P_S \sigma_{SR}^2 + 1} \sum_{k=1}^K |h_{km} s_k - \tilde{y}_{Dm}|^2 \right) \\ &= \left( s_{1det}, s_{2det}, \dots, s_{Kdet} \right)^T \end{aligned} \quad (4)$$

where  $\mathbf{s}_{det} = (s_{1det}, s_{2det}, \dots, s_{Kdet})^T$  is the detected symbol vector.

### III. PERFORMANCE ANALYSIS AND OPTIMAL POWER ALLOCATION

#### A. Derivation of the SER Bound

In this section, we aim to derive an analytical closed-form framework for the Union Bound of the SER per source. Based on [17], with  $\{s_q\}$  we denote the set of all possible  $Q$  symbols transmitted from a particular source, which we assume to be equally probable. Furthermore, with  $\{s\}$  we denote the set of

all  $Q^K$  symbol vectors to be transmitted from the  $K$  sources and with  $\{s_i\}$  we define a subset of  $\{s\}$  in which the symbol

vectors have  $s_q$  transmitted from a particular source and with  $\{s_j\}$  we define the set of all symbol vectors in which the symbol transmitted from that source is different than  $s_q$ .

Assuming that all the sources employ the same modulation order  $Q$ , the Union Bound of the SER is given as [17]

$$SER \leq \frac{1}{Q^K} \sum_{q, i, j} PEP_{s_q, ij} \quad (5)$$

where  $PEP_{s_q, ij}$  denotes the pairwise error probability (PEP) of detecting the symbol vector  $s_j$  when the vector  $s_i$  is transmitted and it is given by

$$PEP_{s_q, ij} = \frac{1}{Q} \sum_{m=1}^M E \left\{ \frac{1}{2} \left( 1 + \frac{P_S \sigma^2 + P_R \sigma^2}{P_R \sigma^2} \frac{|\mathbf{h}_m \Delta \mathbf{s}_{i,j}|^2}{|f_m|^2} \right)^{-1} \right\} \quad (6)$$

where  $\mathbf{h}_m = (h_{1m} \ h_{2m} \ \dots \ h_{Km})$  and  $\Delta \mathbf{s}_{i,j} = \mathbf{s}_i - \mathbf{s}_j$ . Due to the difficulty of analytically solving (6) in its exact form, we consider the exponential approximation of the  $Q$ -function [18]:

$$Q(x) \approx \frac{1}{12} e^{-x^2} + \frac{1}{6} e^{-3x^2} \quad (7)$$

By plugging (7) into (6) and by taking into account that all the channel links are independent and identically distributed, we obtain

$$PEP_{s_q, ij} = \frac{1}{12} \prod_{m=1}^M E \left\{ e^{-u_m} \right\} + \frac{1}{6} \prod_{m=1}^M E \left\{ e^{-3u_m} \right\} \quad (8)$$

where  $u_m = \frac{4 P_S \sigma^2 + P_R \sigma^2}{P_R \sigma^2} \frac{|\mathbf{h}_m \Delta \mathbf{s}_{i,j}|^2}{|f_m|^2}$  and  $f_m(u)$  is the probability density function (pdf) of  $u_m$ . According to [16, Eq. (10)],

$$f_m(u) = \frac{2}{b} e^{-\frac{cu}{ab}} K_1 \left( 2 \sqrt{\frac{cu}{ab}} \right) + \frac{c}{a} K_0 \left( 2 \sqrt{\frac{cu}{ab}} \right) \quad (9)$$

where  $a = P_R \sigma_{RD}^2$ ,  $b = \frac{1}{4} P_S \sigma_{SR}^2 \frac{|\Delta \mathbf{s}_{i,j}|^2}{F}$ , and  $c = KP_S \sigma_{SR}^2 + 1$ . By plugging (9) into (8) and by using [19, Eq.

(6.643.3)], [20, Eq. (13.1.33) and Eq. (13.2.5)], we obtain (10) at the top of the next page. Consequently, by plugging (10) into (5) we obtain the Union Bound of the SER per source.

*Coding and Diversity gain:* For  $(P_S, P_R) \rightarrow \infty$  and by using the inequality  $E_1(z) < e^{-z} \log \left( 1 + \frac{1}{z} \right)$  [16, Eq. (5.1.20)], (12) is upper bounded as

$$PEP_{s_q, ij} < \frac{K^M}{2} \frac{4^M}{12} + \frac{3^M}{6} \left( \frac{1}{\log \left( \log \left( \frac{P_R \sigma^2}{P_S \sigma^2} \right) \right)} \right)^M \times \left( \frac{1}{P_R \sigma_{RD}^2} \right)^M \frac{1}{\log \left( \frac{P_R \sigma^2}{P_S \sigma^2} \right)} \quad (11)$$

Hence, the pair-wise coding gain  $G_{ML}^{cod}$  and diversity gain  $G_{ML}^{div}$  of the ML detection are asymptotically  $\log \left( \frac{P_S \sigma_{RD}^2}{P_R \sigma_{RD}^2} \right) \ll \log \left( \frac{P_S \sigma_{RD}^2}{P_R \sigma_{RD}^2} \right)$  when  $P_R \rightarrow \infty$  given by

$$G_{ML}^{cod} = \left[ \frac{K^M}{\sigma_{RD}^2 |\Delta \mathbf{s}_{i,j}|^2} \frac{4^M}{12} + \frac{3^M}{6} \right]^{-\frac{1}{M}} \quad (12)$$

and

$$G_{ML}^{div} = M \quad (12)$$

Consequently, a full diversity, equal to the number of relays, is achieved with ML detection.

#### B. Optimal Power Allocation

Let us assume that there is a total power budget in the network, denoted as  $P_{budget}$ , to be allocated to the sources and relay nodes. Hence,

$$P_{budget} = KP_S + MP_R \quad (13)$$

The goal of the optimal power allocation is to distribute  $P_{budget}$  among the sources and relay nodes in a way that minimizes the SER of the system. Mathematically, the problem formulation is as follows:

$$\begin{aligned} & \text{Given } P_{budget} \\ & \text{find } P_S, P_R \\ & \text{such that SER is minimized} \\ & \text{subject to } P_{budget} = KP_S + MP_R \end{aligned} \quad (14)$$

To make the optimal power allocation problem analytically more tractable, we consider the optimal power allocation in the high-SNR region, which means that  $P_S, P_R \gg 1$ . Then, (10) can be upper bounded as

$$PEP_{s_q, ij} < \frac{1}{|\Delta \mathbf{s}_{i,j}|^2} \frac{(4K)^M}{12} + \frac{(3K)^M}{6} \frac{1}{\log \left( 1 + \frac{P_R \sigma^2}{P_S \sigma^2} \right)^M} \times \frac{1}{\sigma_{SR}^2 (P_{budget} - MP_R)} + \frac{1}{P_R \sigma_{RD}^2} \quad (15)$$

$$PEP_{s_q,ij} \approx \frac{1}{12} \frac{1}{b+1} + \frac{bc}{a(b+1)^2} e^{\frac{\epsilon}{a(b+1)}} E_1 \left( \frac{c}{a(b+1)} \right)^M + \frac{1}{6} \frac{3}{4b+3} + \frac{12bc}{a(4b+3)^2} e^{\frac{3c}{a(4b+3)}} E_1 \left( \frac{3c}{a(4b+3)} \right)^M \quad (10)$$

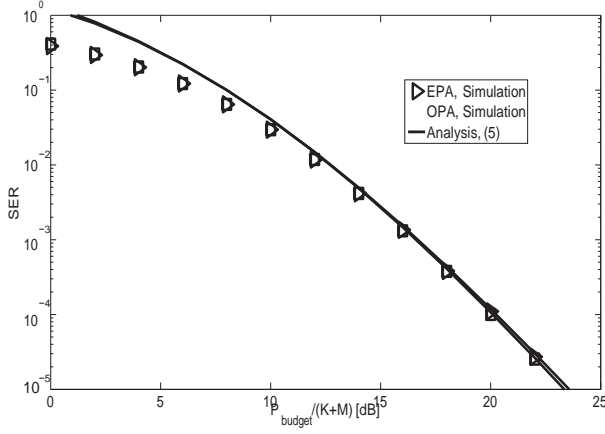


Fig. 2. SER vs.  $P_{budget}/(K+M)$  for  $K=2$ ,  $M=4$ ,  $Q=4$ , and  $\sigma_{SR} = \sigma_{RD} = 1$ .

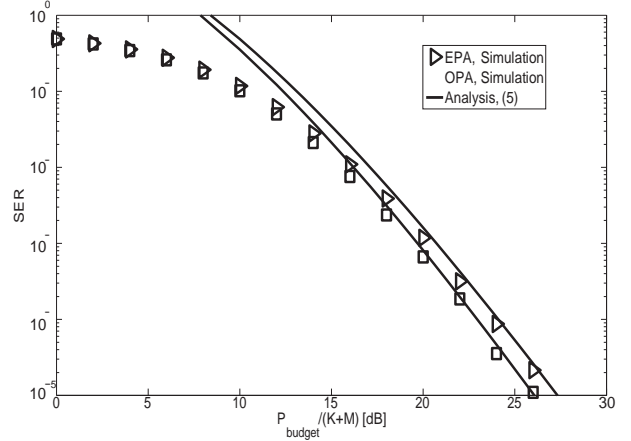


Fig. 3. SER vs.  $P_{budget}/(K+M)$  for  $K=4$ ,  $M=4$ ,  $Q=4$ , and  $\sigma_{SR} = \sigma_{RD} = 1$ .

where we have used  $P_S = \frac{P_{budget} - MP_R}{K}$  and the inequality  $E_1(z) < e^{-z} \log(1 + \frac{1}{z})$ . As we observe from (15), the factor that is included in all the pairwise error probabilities and depends on the power allocation is  $\frac{P_R \sigma_{RD}^2}{\sigma^2} + \log(1 + P_R \sigma_{RD}^2)$ . Hence, the objective

function of the SER bound to be included in the power allocation problem so as to minimize the SER is  $D(P_R) = \frac{P_R \sigma_{RD}^2}{\sigma^2} + \log(1 + P_R \sigma_{RD}^2)$ . Based on this, (14) is formulated as follows:

$$\begin{aligned} & \text{Given } P_{budget} \\ & \text{find } P_R \in [0, \frac{P_{budget}}{M}] \\ & \text{such that} \\ & D(P_R) = \frac{P_R \sigma_{RD}^2}{\sigma^2} + \log(1 + P_R \sigma_{RD}^2) \\ & \text{is minimized} \end{aligned} \quad (16)$$

It can be easily proven that (16) is a convex optimization problem and, hence, its solution can be found through a line search in MATLAB, for instance. After obtaining the optimal value of  $P_R$ , the optimal value of  $P_S$  can be found from (13).

#### IV. NUMERICAL RESULTS

In this section, our aim is twofold: i) To substantiate the tight behavior in the high-SNR region of our derived analytical framework for the SER per source with respect to Monte Carlo simulations and ii) To show how the energy gains achieved by

employing the optimum power allocation algorithm (OPA) of (18), with respect to the equal power allocation (EPA) policy, are affected by the number of sources.

Fig. 2 and Fig. 3 show the SER vs.  $P_{budget}/(K+M)$  curves of the EPA and OPA cases for  $K=2$  and  $K=4$ , respectively. In addition, in both cases we used  $\sigma_{SR} = \sigma_{RD} = 1$ ,  $M=4$

and  $Q=4$ . As we observe from Fig. 2 and Fig. 3, i) The derived closed-form union bound of the SER is tight with respect to Monte Carlo simulations in the high-SNR region and ii) The OPA benefits over the EPA become more pronounced as the number of sources increases. In particular, for a target SER per source, the energy gain achieved by the implementation of the OPA over the EPA is defined as

$$\text{Energy Gain} = \frac{P_{budget}^{EPA} - P_{budget}^{OPA}}{P_{budget}^{EPA}} [\%] \quad (17)$$

where  $P_{budget}^{EPA}$  and  $P_{budget}^{OPA}$  refer to the  $P_{budget}$  value needed to achieve the target SER for the EPA and OPA policy, respectively. For a target SER of  $10^{-4}$ , Fig. 4 illustrates the energy gain achieved for different number of users, which clearly indicates that the higher the number of users in the network is, the higher the achieved energy gains are by employing OPA. Intuitively thinking, this trend is attributed to the fact that with increasing number of sources there is a higher sub-optimality of EPA compared to OPA, since there are more nodes in the network for which the same power level is allocated in the former case.

#### V. CONCLUSIONS

In this paper, we tried to bridge the gap that exists in literature regarding the information-theoretic analysis with the

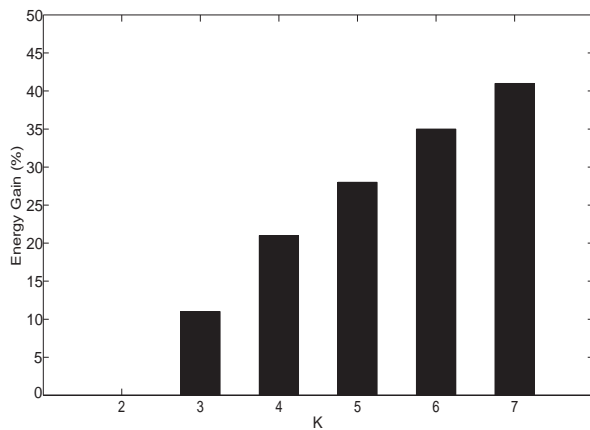


Fig. 4. Energy Gain (%) vs.  $K$  for  $M = 4$ ,  $Q = 2$ , and  $\alpha_{SR}^2 = \sigma_{RD}^2 = 1$ .

practical system design of multi-user multi-relay systems that employ Analog Network Coding. In particular, our contributions can be summarized as: i) We presented an analytical framework for the calculation of the SER per source that results in a close match with Monte Carlo simulations in the high-SNR region, especially for an adequate number of relays, and ii) based on this framework, we formulated the optimal power distribution criterion among the source and relay nodes.

Results show that the the energy gains of the optimal power allocation policy, with respect to the equal one, increase for increasing number of sources.

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