

purpose, namely as an indicator for precise measurement. A compound lever with a magnification of about 600 was arranged to be moved by a micrometer at the one end, and carried the contact at the other end. Using this to measure the thickness of a parallel slip-gauge with the lower face resting on three steel balls and the contact on the upper face, it was found that repetition of observations could easily be obtained with variations not exceeding 0.5×10^{-6} inch. These experiments indicate that the advantages obtained by making contact in the grid circuit of a valve may eliminate some of the difficulties hitherto experienced in this method of measurement.

This method appears to have advantages in connexion with the reception of feeble wireless signals with the aid of a relay ; and it is also proposed to try it, on account of its freedom from sparking at the grid-circuit contact, in connexion with the location of the height of the mercury surface in the vacuum space of a standard barometer.

July 1922.

LXXIV. *Radiative Equilibrium : the Insolation of an Atmosphere.* By E. A. MILNE, M.A., Fellow of Trinity College, Cambridge*.

§ 1. *INTRODUCTION.*—The generally accepted theory of the existence of the earth's stratosphere was put forward in 1908 by Gold†. Gold showed that when radiation processes were taken into account the continued existence of an adiabatic gradient to indefinitely great heights was impossible ; for the upper portions of such an atmosphere, being very cold, would radiate very little, but on the other hand, being backed by an extensive cushion of warmer air besides the warm surface of the earth, would be subjected to low-temperature radiation of considerable intensity, and the consequent excess of absorption over emission would raise their temperature and so disturb the adiabatic gradient. Such upper portions, however, could not exchange heat with the rest of the atmosphere by convection, for they would tend to rise, not fall. Consequently

* Communicated by the Author.

† "The Isothermal Layer of the Atmosphere and Atmospheric Radiation," Proc. Roy. Soc. 82 A. p. 43 (1909). A preliminary announcement was made at the British Association meeting in 1908 ; see 'Nature,' vol. lxxviii. p. 551 (1908). See also Geophysical Memoirs, No. 5 (Met. Office), vol. i. p. 65 (1913).

their temperature would continue to increase until the extra emission due to increased temperature balanced the absorption and a new steady state set in—a state of radiative equilibrium. The direct absorption of solar radiation is small and, though important, does not affect the argument. (It is of interest to mention that exactly the same course of argument shows that even in the absence of convection a strictly isothermal atmosphere is impossible; for the outer portions would not be able to absorb as much as they emitted, and so would cool, causing convection.)

Gold embodied these ideas in analysis, in order to determine the temperature and the height of the tropopause, and he showed that the theory generally was adequate to account for the observed values. His procedure, however, was in part empirical. In the light of Schwarzschild's * theory of radiative equilibrium in a stellar atmosphere, an immediate rough evaluation of the boundary temperature is possible; if T_0 is this temperature, then $T_0^4 = \frac{1}{2}T_1^4$, where T_1 is the effective temperature of the system (earth plus atmosphere) as determined by the amount of energy radiated away into space. This energy is equal to the mean value of the absorption of solar radiation, assuming that the earth is on the average neither losing nor accumulating energy. The value of T_1 , deduced by Abbot † from the solar constant and the earth's albedo, is about 254° , giving $T_0 = 214^\circ$. The observed mean value of the temperature of the stratosphere over the British Isles is about 219° . Schwarzschild's formula, $T_0^4 = \frac{1}{2}T_1^4$, was indeed obtained independently by Humphreys ‡ in this connexion, and applied to the stratosphere. Gold, however, did not proceed in this way. Accepting the observed division of the atmosphere into two shells—an inner one in convective equilibrium with a known temperature gradient, and an outer one at a uniform temperature,—he determined the height at which the convective gradient should terminate, in order that the atmosphere above this height should, as a whole, gain as much heat by absorption as it lost by radiation; the temperature of the convective region at this height then gave the temperature of the isothermal region. It appeared that a satisfactory balance was obtained if the point of division was taken at a height given by $p = \frac{1}{4}p_1$, where p is the pressure at any height, p_1 the ground-pressure. It appeared further that there was very nearly a balance of radiation in the upper

* *Gött. Nach.* 1906, p. 41.

† *Annals Astrophys. Obs. Smithson. Inst.* ii. p. 174 (1908). 'The Sun' (Appleton, New York, 1912), p. 323.

‡ *Astrophys. Journ.* vol. xxix. p. 26 (1909).

layer of the convective region extending from $p = \frac{1}{2}p_1$ to $p = \frac{1}{4}p_1$, from which Gold deduced that in this layer the convection would be small.

It is the object of this paper to point out a certain difficulty in Gold's work, and to consider an idealized problem which is suggested by it in the absorption of radiation by an atmosphere subject to insolation.

Since the paper was first written, the author has become aware of a paper by Emden* which anticipates portions of it. Emden criticized Gold's theory on certain points, and investigated the general theory of the radiative equilibrium of an atmosphere by a method similar to that of the present paper. Where necessary, the paper has been recast to take account of Emden's work.

§ 2. *Criticism of Gold's solution.*—One of the most interesting points in Gold's discussion is his isolation and explicit formulation of the condition for a convective atmosphere. In such an atmosphere, transfers of energy are being effected both by radiation and by convection, and across any plane there will be a net radiative flux and a convective flux. Now convection can only transfer heat upwards, not downwards. But assuming a steady state, the upward convective flux plus the net radiative flux must be equal to the downward solar flux. Hence *the net radiative flux (as due to the earth and atmosphere together) must be less than the downward solar flux.* But the downward solar flux at any point cannot exceed its value at the boundary; and at the boundary the downward solar flux must be equal to the outward flux due to the earth and atmosphere. Hence another form of the condition is : *the net outward flux across any plane must be less than its value at the boundary.* Again, the upper layers must be gaining more heat by convection from below than they are losing to layers above. Hence, for a steady state, *emission of radiation must exceed absorption in the upper layers* (for emission must equal absorption plus net gain by convection). Whenever these inequalities become equalities, radiative equilibrium holds; if they become reversed the state cannot be a steady one, for it would involve convection downwards †.

* "Über Strahlungsgleichgewicht und atmosphärische Strahlung: ein Beitrag zur Theorie der oberen Inversion," *Sitz. d. K. Akad. Wiss. zu München*, 1913, pp. 55-142.

† Gold's conditions have been applied by the writer to stellar atmospheres in a paper recently communicated to the Royal Society.

Now Gold applied these conditions in various ways to show under what circumstances a convective atmosphere can or cannot exist: *e. g.*, he showed that a convective atmosphere cannot extend indefinitely, yet must extend above $p = \frac{1}{2}p_1$. But he did not point out that his final solution was inconsistent with these conditions. We shall show that although on the assumptions made the layer $(\frac{1}{4}p_1, 0)$ is neither gaining nor losing heat as a whole, yet its upper portions are emitting more than they are absorbing, and its lower portions absorbing more than they are emitting; consequently the upper layers must cool and sink, the lower ones warm and rise, convection will occur, and the state of isothermal equilibrium must be destroyed. Further, although the layer $(\frac{1}{2}p_1, \frac{1}{4}p_1)$ satisfies the conditions for convective equilibrium as a whole, emission exceeding absorption, in the upper portions absorption exceeds emission, so that a steady convective state in this region is not possible; the smallness of the excess of emission over absorption for the whole layer, attributed by Gold to the slowness of convection required, is merely the result of the excess in the lower portions being balanced by the deficiency in the upper ones.

Actually we can prove a more precise result than this, under very general conditions. We shall show that the excess of absorption over emission at the base of Gold's isothermal layer, per unit optical mass, is numerically equal to the excess of emission over absorption at the top, whatever the temperature distribution in the convective layer and whatever the law connecting the coefficient of absorption with height. To do this we shall employ the approximate form of the equations of transfer of radiant energy. It may be mentioned here that though Gold uses the exact formulæ (involving Ei functions) which take full account of the spherical divergence of the radiation, his results can be obtained more simply to the same degree of precision by using the approximate formulæ and by making free use of the optical thickness and the net flux of radiation. The quite small errors of the approximate formulæ are swallowed up in the uncertainty of the numerical data that have finally to be employed. The uncertainty arises in the final translation of the optical thicknesses into actual thicknesses; but, as in other cases of radiative equilibrium, many of the results hold in a form independent of the numerical values of the absorption coefficients.

Let τ be the optical thickness measured *inwards* from the

outer limit of the atmosphere ; if ρ is the density at height h , $k(h)$ the mass-absorption coefficient, then

$$\tau(h) = \int_h^\infty k(h)\rho dh.$$

Let $I(\tau)$ be the intensity of radiation at τ in a direction θ with the outward vertical, where $0 \leq \theta \leq \frac{1}{2}\pi$; and let $I'(\tau)$ be the intensity at ψ with the inward vertical, where $0 \leq \psi \leq \frac{1}{2}\pi$. Assume the material is *grey* (*i.e.* has an absorption coefficient the same for all the wave-lengths that are important—in this case the wave-lengths that are predominant in the low-temperature radiation considered). Let $B(\tau)$ be the intensity of black body radiation for the temperature ruling at the point τ ; and let $\pi F(\tau)$ be the net upward flux of energy per unit area across a horizontal plane at τ . Then

$$\cos \theta \frac{dI}{d\tau} = I - B, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\cos \psi \frac{dI'}{d\tau} = B - I', \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{1}{2}F(\tau) = \int_0^{\frac{1}{2}\pi} I(\tau) \sin \theta \cos \theta d\theta - \int_0^{\frac{1}{2}\pi} I'(\tau) \sin \psi \cos \psi d\psi. \quad . \quad . \quad . \quad (3)$$

Consider the expression

$$\pi F(\tau') - \pi F(\tau''), \quad (\tau' > \tau'').$$

Here $\pi F(\tau')$ is the net amount of radiant energy entering the lower boundary of the layer (τ', τ'') , $\pi F(\tau'')$ the net amount leaving the upper boundary. Hence *the difference is the excess of absorption over emission for the whole layer* (τ', τ'') . Thus $F(\tau)$ behaves as an integral, whether or no radiative equilibrium holds; this is interesting, for in certain forms of radiative equilibrium it appears naturally as an integral* of (1) and (2) in the form $F = \text{const.}$

Let τ_1 be the value of τ at the earth's surface. Now suppose with Gold that the complete atmosphere $\tau=0$ to $\tau=\tau_1$ consists of two shells—an outer one at a uniform temperature from $\tau=0$ to $\tau=\tau_2$ (say), and an inner one in convective equilibrium from $\tau=\tau_2$ to $\tau=\tau_1$. Then the outer one will be in radiative equilibrium as a whole, provided

$$F(\tau_2) - F(0) = 0, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and this is the equation which determines τ_2 .

* Monthly Notices, lxxxi. p. 362 (1921).

We now approximate. Setting $t=2\tau$, $t_1=2\tau_1$, etc., and using t as the variable specifying position, equations (1), (2), (3) can be written approximately *

$$\frac{dI}{dt} = I - B, \quad \frac{dI'}{dt} = B - I', \quad . . . (5), (6)$$

$$F(t) = I(t) - I'(t), \quad . . . (7)$$

and the equation for t_2 is

$$I(t_2) - I'(t_2) - I(0) = 0, \quad . . . (8)$$

since the incident radiation $I'(0)$ is zero †. Solving (5) and (6) with the assumption that the air near the ground has the same temperature as the ground and that the earth radiates like a black body, we find

$$I(t) = e^t \int_t^{t_1} B(t) e^{-t} dt + B(t_1) e^{-(t_1-t)} dt, \quad . . . (9)$$

$$I'(t) = e^{-t} \int_0^t B(t) e^t dt. \quad . . . (10)$$

These can be inserted in (8), and t_2 determined as soon as $B(t)$ is known as a function of t in the convective region.

Now the excess of absorption over emission in a small element of volume dv is

$$\begin{aligned} k\rho dv \left[\int I d\omega + \int I' d\omega' - 4\pi B \right] \\ = 2\pi k\rho dv \left[\int_0^{\frac{1}{2}\pi} I \sin \theta d\theta + \int_0^{\frac{1}{2}\pi} I' \sin \psi d\psi - 2B \right] \\ = 2\pi k\rho dv [I(t) + I'(t) - 2B(t)] \end{aligned}$$

approximately. Denote the expression in square brackets by $E(t)$. Then for the values of the excess of absorption over emission at the top and bottom of the isothermal region we have respectively

$$E(0) = I(0) - 2B(0),$$

$$E(t_2) = I(t_2) + I'(t_2) - 2B(t_2), \quad . . . (11)$$

or, using (8),

$$E(0) = I(t_2) - I'(t_2) - 2B(0). \quad . . . (12)$$

* For details, see *e. g.* Monthly Notices, lxxxi. p. 363 (1921).

† Ignoring solar radiation. See below.

We shall now prove that $I(t_2) = 2B(t_2)$. From (9), remembering that B is constant in $(t_2, 0)$, we have

$$I(t_2) = e^{t_2} \int_{t_2}^{t_1} B(t) e^{-t} dt + B(t_1) e^{-(t_1-t_2)},$$

$$I(0) = \int_{t_2}^{t_1} B(t) e^{-t} dt + B(t_2)(1 - e^{-t_2}) + B(t_1) e^{-t_1},$$

whence

$$I(0) - e^{-t_2} I(t_2) = B(t_2)(1 - e^{-t_2}).$$

Further, from (10),

$$I'(t_2) = B(t_2)(1 - e^{-t_2}). \quad . \quad . \quad . \quad (13)$$

Inserting in the equation for t_2 , namely (8), we find

$$I(t_2)(1 - e^{-t_2}) = 2B(t_2)(1 - e^{-t_2}),$$

which is the equality required. Making use of this, we have from (11) and (12)

$$E(t_2) = I'(t_2) = -E(0). \quad . \quad . \quad . \quad (14)$$

Now I' is essentially positive. Hence there is an excess of absorption over emission at the base and a numerically equal excess of emission over absorption at the top. This is the result stated. The excess can only be zero if $I'(t_2)$ is zero, *i. e.* if t_2 is zero.

It should be noticed that the departure from radiative equilibrium at the base and at the top is very appreciable. The ratio of the excess, $2\pi k\rho dv E(t_2)$, to the emission, $4\pi k\rho dv B(t_2)$, has the value

$$\frac{1}{2}(1 - e^{-t_2}); \quad . \quad . \quad . \quad . \quad (15)$$

if $t_2 = 1.0$ this is 0.32, and if $t_2 = 0.56$ it is 0.22; and it can be shown from Gold's data that these limits for t_2 correspond to widely separated values of t_1 , the total absorbing power of the atmosphere. Again, $E(t)$ is a continuous function of t ; and hence, since it is positive when $t = t_2$, it will be positive in the upper parts of the convective atmosphere, violating the condition for convection. As we approach the earth it decreases, soon becoming negative, showing that in the lower portions the condition is satisfied.

We have assumed the atmosphere "grey" as regards the low-temperature radiation, and we have ignored the direct absorption of solar radiation. But a variation of the coefficient of absorption with wave-length does not affect the gist of the argument; a strictly isothermal upper atmosphere would still be an impossibility unless its optical

thickness were zero. As regards the solar radiation, Gold made an allowance for this by choosing τ_2 so that the left-hand side of (4) was slightly negative; but again the argument is unaffected. It appears then that Gold's analysis, though doubtless giving the broad outlines of the phenomenon, is inadequate in its details.

§ 3. Now the complete phenomenon must be very complex. Complications arise from the rotation of the earth, the change of insolation with latitude, cloud-structure, scattering, and the light from the sky, besides probably the world-wide circulation of the air; and the suddenness of the upper inversion has always been to some extent a difficulty. Instead of attempting to take account of the various influencing causes simultaneously, it would appear to be more in accordance with scientific method to construct a number of idealized models, to work out the theoretical solution for each separately, and then to examine the extent to which the earth's atmosphere partakes of their several characteristics.

§ 4. *The problem in principle.*—As a contribution towards this, it is proposed in this paper to consider the theory of the radiative equilibrium of a mass of absorbing and radiating material subject to insolation. The material is supposed to be stratified in parallel planes, and to be subject at its outer boundary to a parallel beam of incident radiation. The latter will be supposed in the first instance to be normal to the surface; later we shall examine the effect of oblique incidence. The material will be taken in the first instance to be grey; but later we shall suppose that there may be one coefficient of absorption for the incident radiation, another coefficient for the low-temperature radiation emitted by the material itself. Further, we shall assume the material to be infinitely thick, and to be in radiative equilibrium throughout its mass. The assumption of infinite thickness involves little or no loss of generality; we could, if we liked, consider a mass of finite thickness with an inner boundary consisting of a black radiating surface, but since our results will only involve the optical thickness, we need only suppose the absorption coefficient or the density to become suddenly very large at an assigned depth in order to deduce the case of an inner boundary from the solution for an infinitely thick slab of material.

The material being in a steady state must emit energy at its outer boundary equal to the incident radiation. Across

any plane parallel to the surface there will be a net outward flux of radiation derived from the material just balancing the inward flux of the residual solar radiation. In the far interior the latter will be greatly attenuated, and consequently the outward flux there must be small too. We should expect, therefore, that the temperature gradient in the far interior would be small; and this proves to be the case. In fact, *not only is there a definite limiting temperature at the outer boundary, as in the Schwarzschild case, but there is also a definite limiting temperature in the far interior.* This is one of the most interesting characteristics of the model we are discussing.

Let τ be the optical thickness measured from the outer boundary to any point; I , I' the outward and inward intensities at any point at angles θ and ψ with the normal; $B(\tau)$ the intensity of black radiation for the temperature at the point τ ; πS the intensity of the parallel beam of incident solar radiation *defined as the energy incident per second per unit area normal to the beam.* Here I and I' are to refer only to radiation derived from the material. It must be noted that since we have assumed the solar radiation to constitute a parallel beam, the definition of its intensity is necessarily different from the standard definition for conically spreading pencils*.

The residual solar intensity at any depth τ is $\pi S e^{-\tau}$. The equations of transfer are

$$\cos \theta \frac{dI}{d\tau} = I - B, \quad . \quad . \quad . \quad (16)$$

$$\cos \psi \frac{dI'}{d\tau} = B - I'. \quad . \quad . \quad . \quad (17)$$

The amount of energy emitted by an element dv per second is $4\pi k p B dv$. That absorbed is

$$k p dv \left[\int I d\omega + \int I' d\omega' \right],$$

together with

$$\pi S e^{-\tau} k p dv.$$

Hence the equation of radiative equilibrium is

$$\int_0^{\frac{1}{2}\pi} I(\tau) \sin \theta d\theta + \int_0^{\frac{1}{2}\pi} I'(\tau) \sin \psi d\psi + \frac{1}{2} S e^{-\tau} = 2B(\tau). \quad (18)$$

The flux relation follows from (16), (17), (18), namely,

$$\int_0^{\frac{1}{2}\pi} I(\tau) \sin \theta \cos \theta d\theta - \int_0^{\frac{1}{2}\pi} I'(\tau) \sin \psi \cos \psi d\psi = \frac{1}{2} S e^{-\tau}. \quad (19)$$

* See Planck, *Wärmestrahlung*, p. 15 (3rd edition).

From (16) and (17), with the appropriate boundary conditions,

$$I(\tau) = e^{\tau \sec \theta} \int_{\tau}^{\infty} B(t) e^{-t \sec \theta} \sec \theta dt, \quad . \quad . \quad (20)$$

$$I'(\tau) = e^{-\tau \sec \psi} \int_0^{\tau} B(t) e^{t \sec \psi} \sec \psi dt. \quad . \quad . \quad (21)$$

Substitute these in (18); write $t = \tau + y \cos \theta$ in the first integral, $t = \tau - y \sec \psi$ in the second, and replace $\cos \theta$ and $\cos \psi$ by μ . We find

$$\begin{aligned} & \int_0^1 d\mu \int_0^{\infty} B(\tau + y\mu) e^{-y} dy \\ & + \int_0^1 d\mu \int_0^{\tau/\mu} B(\tau - y\mu) e^{-y} dy + \frac{1}{2} S e^{-\tau} = 2B(\tau). \end{aligned}$$

We can now reverse the order of integration in the repeated integrals*. Setting

$$C(\tau) = \int_0^{\tau} B(\tau) d\tau, \quad C'(\tau) = B(\tau),$$

we find finally for the integral equation for the temperature distribution,

$$\begin{aligned} C'(\tau) = \frac{1}{4} S e^{-\tau} + \int_0^{\tau} \frac{C(\tau + y) - C(\tau - y)}{2y} e^{-y} dy \\ + \int_{\tau}^{\infty} \frac{C(\tau + y)}{2y} e^{-y} dy. \quad . \quad . \quad . \quad (22) \end{aligned}$$

If we invert the orders of integration before making the substitutions for t , we obtain another form,

$$B(\tau) = \frac{1}{2} \int_0^{\infty} B(t) Ei(|t - \tau|) + \frac{1}{4} S e^{-\tau}, \quad . \quad . \quad (23)$$

which is the standard form for integral equations†.

Solutions of these may be sought directly. For an

* For details, cf. Monthly Notices, lxxxi, p. 365 (1921).

† In equation (23) Ei denotes the exponential-integral function. The integral equation in the form (23) is substantially equivalent to the integral equation obtained by L. V. King in the analogous problem for scattering (Phil. Trans. 212 A. p. 375, 1912); it bears the same relation to King's equation that the author's integral equation for the atmosphere of a star in radiative equilibrium (M. N. lxxxi, p. 373, 1921) bears to Schwarzschild's integral equation for scattering in a stellar atmosphere (*Berlin Sitz.* 1914, p. 1183). But the form (22) is more convenient when solutions are being sought by successive approximation, and for other purposes.

approximate solution, however, it is quicker to employ the approximate forms of equations (16) to (19), obtained in the usual way. These are

$$\frac{1}{2} \frac{dI}{d\tau} = I - B, \quad \frac{1}{2} \frac{dI'}{d\tau} = B - I', \quad . \quad . \quad (24), (25)$$

$$I + I' + \frac{1}{4} S e^{-\tau} = 2B, \quad . \quad . \quad . \quad . \quad (26)$$

$$I - I' = S e^{-\tau}. \quad . \quad . \quad . \quad . \quad (27)$$

From the two latter,

$$I = B + \frac{1}{4} S e^{-\tau},$$

$$I' = B - \frac{3}{4} S e^{-\tau},$$

But $I'(0) = 0$. Consequently $B_0 = \frac{3}{4} S$. Inserting this approximate value of I in (24), we find

$$\frac{dB}{d\tau} = \frac{3}{4} S e^{-\tau},$$

whence, using the value of B_0 already found,

$$B(\tau) = \frac{3}{2} S (1 - \frac{1}{2} e^{-\tau}). \quad . \quad . \quad . \quad . \quad (28)$$

It follows that there is a limiting temperature in the far interior, given by $B_\infty = \frac{3}{2} S$. If now T_0 is the boundary temperature, T_1 the effective temperature of the whole mass viewed from the outside, T_∞ the temperature in the far interior, and σ Stefan's constant, we have

$$\sigma T_0^4 = \pi B_0 = \frac{3}{4} \pi S,$$

$$\sigma T_\infty^4 = \pi B_\infty = \frac{3}{2} \pi S,$$

$$\sigma T_1^4 = \pi S,$$

and thus

$$T_\infty^4 : T_1^4 : T_0^4 = \frac{3}{2} : 1 : \frac{3}{4}. \quad . \quad . \quad . \quad . \quad (29)$$

It is important to notice that T_∞ is different from T_1 , contrary to what might have been anticipated; also that the relation between T_1 and T_0 is different from that in the Schwarzschild case, where the net flux is the same at all depths. Notice also $T_\infty^4 = 2T_0^4$.

These values and the general distribution of temperature given by (28) are only approximations. To test them, let us re-employ (28) in (20) to obtain I and so check the radiative equilibrium at the boundary and the net flux

there. We find

$$I(\tau, \theta) = \frac{3}{2}S \left(1 - \frac{\frac{1}{2}e^{-\tau}}{1 + \cos \theta} \right),$$

whence

$$I(0, \theta) = \frac{3}{2}S \frac{\frac{1}{2} + \cos \theta}{1 + \cos \theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

This gives the distribution of the emergent radiation—the law of bolometric darkening. Inserting in (18), the total absorption near the boundary is found to be proportional to $(2 - \frac{3}{4} \log 2)S$, the emission to $\frac{3}{2}S$ —*i. e.* 0.987 instead of unity, an error of only 1.3 per cent. Again, from (19), the net flux at the boundary is given as $(\frac{3}{2} \log 2)\pi S$ instead of πS —*i. e.* 1.040 instead of unity, an error of 4.1 per cent. The smallness of these discrepancies shows that (28) and the values (29) are satisfactory approximations.

To obtain a better approximation, knowing now something of the form of B from (28), we can assume

$$B(\tau) = a - be^{-\tau}$$

and choose a and b so that the correct net flux is given at the boundary and the condition of radiative equilibrium is satisfied there. It is found that the condition of radiative equilibrium in the far interior is then automatically satisfied, save for terms which tend to zero. We find

$$I(0, \theta) = a - \frac{b}{1 + \cos \theta},$$

whence from (18) and (19)

$$a - b(2 - \log 2) = \frac{1}{2}S,$$

$$\frac{1}{2}a - b(1 - \log 2) = \frac{1}{2}S.$$

These give

$$a = S/\log 2 = 1.4427 S,$$

$$b = \frac{1}{2}S/\log 2 = 0.7213 S;$$

whence $B_0 = 0.7213 S$, $B_\infty = 1.4427 S$, and

$$T_\infty^4 : T_1^4 : T_0^4 = 1.443 : 1 : 0.721.$$

Thus the values of T_∞ and T_0 in terms of T_1 come out about 1 per cent. smaller than on the previous approximation. The relation $T_\infty^4 = 2T_0^4$ still holds. The change is so trifling that we shall not attempt to obtain further approximations, which can be sought by using the integral equation. We shall

content ourselves with observing that in the exact solution the differential coefficient $B'(\tau)$ has a singularity at $\tau=0$, becoming infinite * like $\log \tau$. This is easily proved.

§ 5. *Extension to non-grey absorption*.—Let us now suppose that the material has a different coefficient of absorption for the incident radiation, say equal to n times that for its own low temperature radiation; n will usually be a small fraction. The inward solar intensity at τ is now $S e^{-n\tau}$. Hence in the flux equation, (19), $e^{-\tau}$ must be replaced by $e^{-n\tau}$, and in the equation of radiative equilibrium, (18), $S e^{-\tau}$ must be replaced by $n S e^{-n\tau}$. Proceeding as before, we find that

$$B(\tau) = S \frac{1 + \frac{1}{2}n}{n} [1 - (1 - \frac{1}{2}n)e^{-n\tau}], \quad . \quad . \quad (28')$$

$$B_{\infty} = S \frac{1 + \frac{1}{2}n}{n}, \quad B_0 = \frac{1}{2}S(1 + \frac{1}{2}n),$$

$$T_{\infty}^4 : T_1^4 : T_0^4 = \frac{1}{2} + n^{-1} : 1 : \frac{1}{2}(1 + \frac{1}{2}n). \quad . \quad (29')$$

As $n \rightarrow 0$, $T_{\infty} \rightarrow \infty$, $T_0^4 \rightarrow \frac{1}{2}T_1^4$, and the temperature distribution tends to

$$B(\tau) = S(\frac{1}{2} + \tau).$$

The limiting case is, in fact, the Schwarzschild case for a constant net flux πF . Notice that $T_{\infty}^4 = 2T_0^4/n$.

§ 6. *Extension to oblique incident radiation*.—Next suppose that the external radiation is incident at an angle α with the normal. If we preserve the same intrinsic intensity, the amount incident per unit area is now $S \cos \alpha$ and the amount crossing unit area at depth τ is $S \cos \alpha e^{-n\tau \sec \alpha}$. We can obtain the solution by putting $S \cos \alpha$ for S and $n \sec \alpha$ for n in the foregoing formulæ. We find

$$B(\tau) = S \frac{\cos \alpha + \frac{1}{2}n}{n} [\cos \alpha - (\cos \alpha - \frac{1}{2}n)e^{-n\tau \sec \alpha}], \quad (28'')$$

$$B_{\infty} = S \cos \alpha (\cos \alpha + \frac{1}{2}n)/n, \quad B_0 = \frac{1}{2}S(\cos \alpha + \frac{1}{2}n),$$

$$\begin{aligned} T_{\infty}^4 : T_1^4 : T_0^4 &= \cos \alpha (\frac{1}{2} + n^{-1} \cos \alpha) : \cos \alpha : \frac{1}{2}(\cos \alpha + \frac{1}{2}n) \\ &= \frac{1}{2} + n^{-1} \cos \alpha : 1 : \frac{1}{2}(1 + \frac{1}{2}n \sec \alpha). \quad . \quad (29'') \end{aligned}$$

Notice that $T_{\infty}^4 = 2 \cos \alpha T_0^4/n$.

* Cf. Monthly Notices, lxxxi. p. 367 (1921).

§ 7. These formulæ offer several points of interest. As α increases from 0 to $\frac{1}{2}\pi$ and $\cos \alpha \rightarrow 0$, T_0 tends to a definite non-zero limit, although T_1 tends to zero; T_0 steadily decreases as α increases, the limit being given by $\sigma T_0^4 = \frac{1}{4}\pi nS$; T_∞ tends to zero. It appears, then, that for sufficiently oblique incidence the boundary is warmer than the interior. Consider now the temperature distribution given by (28''). When $\cos \alpha = \frac{1}{2}n$, $B(\tau)$ is constant and equal to $\frac{1}{2}nS$ or $S \cos \alpha$, and the state is isothermal everywhere; and when $\cos \alpha < \frac{1}{2}n$, the temperature steadily decreases inwards in the interior. In spite of this there is at each point a net flux in the outward direction; so that here we have a case where the net flux is in the opposite direction to the temperature gradient. This would seem to be a novelty in the theory of radiative equilibrium. (It is easy to assure one's self that no contradiction with the second law is involved.) These results are based only on the approximate formulæ (28'') and (29''), but further investigation confirms them. It is easy to see in a general way how these curious temperature distributions arise. When the solar radiation is nearly tangential, its effective intensity is very weak, but owing to its obliquity it is entirely absorbed in a thin layer close to the surface (provided n is not zero). This layer is enabled to assume a definite temperature, but no residual radiation penetrates to the interior, which remains near the absolute zero. The outward net flux is maintained at any point in virtue of the outward radiation from the large amount of cold material inside the point overpowering the inward radiation from the small amount of warm material outside it. In the limit when $\alpha = \frac{1}{2}\pi$, the distribution of temperature is discontinuous; the temperature is zero everywhere, except at points in the surface.

§ 8. *Effect of rotation.*—These results can only be applied to a thick spherical atmosphere on the assumption that the solar energy incident on any one place is all re-radiated from that same place. Making this assumption, let us tentatively take into account the effects of rotation. We will calculate the time mean of the temperatures in any given latitude λ on the assumption that the axis of rotation is perpendicular to the ecliptic. If ϕ is the hour-angle of the sun, its zenith distance α is given by $\cos \alpha = \cos \phi \cos \lambda$. Taking (29'') as giving the "instantaneous" temperature during the day and taking the latter as zero during

the night, and using bars to denote mean values, we have

$$\begin{aligned}\frac{\sigma}{\pi} \overline{T_0^4} &= \overline{B_0} = \frac{1}{\pi} \int_0^{\frac{1}{2}\pi} B_0 d\phi = S \frac{\frac{1}{2}(\cos \lambda + \frac{1}{4}n\pi)}{\pi}, \\ \frac{\sigma}{\pi} \overline{T_1^4} &= \overline{B_1} = \frac{1}{\pi} \int_0^{\frac{1}{2}\pi} S \cos \lambda \cos \phi d\phi = S \frac{\cos \lambda}{\pi}, \\ \frac{\sigma}{\pi} \overline{T_\infty^4} &= \overline{B_\infty} = \frac{1}{\pi} \int_0^{\frac{1}{2}\pi} B_\infty d\phi = S \frac{\frac{1}{2} \cos \lambda (1 + \frac{1}{2}n^{-1} \pi \cos \lambda)}{\pi}.\end{aligned}\quad . . . (31)$$

When this averaging is taken into account, the approximately isothermal state ($\overline{T_\infty} = \overline{T_0}$) is found to occur for $\cos \lambda = n/\sqrt{2}$; for this value of λ ,

$$\overline{T_0^4}/\overline{T_1^4} = \frac{1}{2}(1 + \frac{1}{4}\pi\sqrt{2}) = 1.055,$$

which is sufficiently near unity. The general run of the change of temperature distribution with latitude has the same features as before.

§ 9. *Comparison with Emden*.—The formal problems discussed by Emden in the paper already mentioned and his method of solution are very similar to those discussed above, except that he takes the material to be bounded below by a black surface. Emden considers the radiative equilibrium of an atmosphere subject to external solar radiation in two cases: (1) the case of “grey radiation,” by which he means the case in which the mean coefficient of absorption for the solar radiation is equal to that for the atmospheric radiation; this is the case $n=1$ above; (2) the case in which the radiation spectrum can be divided into two ranges which have different mean coefficients of absorption, the solar radiation being entirely confined to one of them; this is practically our general case in which n is not unity. In each case he considers the solar radiation to be “gleichmässig verteilt,” *i. e.* not as being confined to a parallel beam, but as uniformly distributed over the solid angle 2π ; consequently he does not consider the variation of the state of equilibrium with latitude. The two main results to which he draws attention are: in case (1) the whole atmosphere must be isothermal, at a temperature equal to the “effective” temperature T_1 calculated from the incident radiation with allowance for the albedo (see p. 873 above); in case (2) the state is not isothermal and the boundary temperature T_0 is connected with the effective temperature

T_1 by the relation

$$T_0 = T_1 \left[\frac{1}{2} (1 + k_1/k_2) \right]^{\frac{1}{2}}, \quad . \quad . \quad . \quad (32)$$

where k_1 and k_2 are the coefficients of absorption for the solar and terrestrial radiations.

Both these results are in apparent contradiction with those obtained in this paper. The source of the discrepancies is in each case Emden's assumption that the incident radiation may be taken to be diffuse. The way this occurs is as follows:—The mean coefficient of absorption for diffuse radiation incident on a thin layer of material is approximately twice the coefficient of absorption for a parallel beam incident normally, *i. e.* twice the coefficient of absorption as ordinarily defined. This fact allows us to approximate to the equations of transfer (equations (1) and (2) above) by replacing them by the "equations of linear flow"; in equations (5) and (6) we have explicitly adopted a new optical thickness t equal to twice the optical thickness τ obtained directly from the ordinary coefficient of absorption; in equations (24) and (25) we have retained the optical thickness τ and simply replaced the factors $\cos \theta$ and $\cos \psi$ by the value $\frac{1}{2}$; the result is the same as if all the diffuse radiation were supposed to be confined to beams at an angle of incidence of 60° with the planes of stratification. Emden approximates in the same way as we have done, but since he takes the solar radiation to be diffuse he is adopting for this also a coefficient of absorption twice the value for a permanent beam. His results may therefore be expected to agree with ours if in ours we put $\cos \alpha = \frac{1}{2}$, $\alpha = 60^\circ$; and this in fact they do. But they lose part of their significance. His result for case (1) is of course true for diffuse radiation; indeed it is obvious thermodynamically, without proof, that material exposed to *isotropic* incident radiation will, if in radiative equilibrium, take up a temperature equal to that of the radiation: the case is practically that of a black body enclosure. But our results show that if the incident radiation occurs *as a parallel beam*—as, in fact, solar radiation does—then the isothermal state is merely the particular distribution of temperature that happens to correspond to an angle of incidence of 60° . Further, Emden's result does not suggest another of our results—that when $n \neq 1$ there also exists an isothermal state of equilibrium: namely, for $\cos \alpha = \frac{1}{2}n$ for a fixed parallel beam, and for $\cos \lambda = n/\sqrt{2}$ when rotation is taken into account. Emden's formula (32)

above should be compared with our formula for a fixed beam incident at α (from (29'')),

$$T_0 = T_1 \left[\frac{1}{2} \left(1 + \frac{1}{2} n \sec \alpha \right) \right]^{\frac{1}{2}}, \quad . \quad . \quad . \quad (33)$$

where $n = k_1/k_2$; and with the rotational mean formula (from (31)),

$$T_0 = T_1 \left[\frac{1}{2} \left(1 + \frac{1}{4} n \pi \sec \lambda \right) \right]^{\frac{1}{2}}. \quad . \quad . \quad . \quad (34)$$

Emden's formula differs but little from the latter when $\lambda = 0$, as is to be expected.

Emden does not obtain the integral equation for the temperature distribution. For the sake of completeness it seems worth putting on record the integral equation for the general case involving n and α . It is deduced in the same way as (22) :—

$$\begin{aligned} C'(\tau) = \frac{1}{4} n S e^{-n\tau \sec \alpha} + \int_0^\tau \frac{C(\tau+y) - C(\tau-y)}{2y} e^{-y} dy \\ + \int_\tau^\infty \frac{C(\tau+y)}{2y} e^{-y} dy. \quad . \quad . \quad (35) \end{aligned}$$

§ 10. *Effect of an internal boundary.*—We shall next consider the case in which the material is bounded internally by a black surface at $\tau = \tau_1$ instead of extending to infinity. It has already been mentioned that as the formulæ only involve the optical thickness τ , we may deduce the results for this case by supposing that immediately beneath $\tau = \tau_1$ the density suddenly increases indefinitely. The temperature distribution above τ_1 is unaltered. It might at first be supposed that the black surface would assume a temperature equal to T_∞ , but this is not so. For the infinite density gradient we have postulated at the level τ_1 implies an infinite radiation gradient there, and (unless we are prepared to accept the existence of an infinite temperature gradient at the black surface) the surface will take up a temperature intermediate between $T(\tau_1)$, the temperature of the material in contact with the surface, and T_∞ . This temperature, say T_s , is easily calculated. For since the surface must re-radiate all the radiation falling on it, we shall have $\sigma T_s^4 = \pi B_s$, where B_s is given by

$$B_s = I'(\tau_1) + S \cos \alpha e^{-n\tau_1 \sec \alpha} \quad . \quad . \quad (36)$$

From the equations

$$I + I' + \frac{1}{2}nS e^{-n\tau \sec \alpha} = 2B, \quad . \quad . \quad . \quad (26'')$$

$$I - I' = S \cos \alpha e^{-n\tau \sec \alpha}, \quad . \quad . \quad . \quad (27'')$$

we find

$$I' = B - \frac{1}{2}S e^{-n\tau \sec \alpha} (\cos \alpha + \frac{1}{2}n).$$

Hence

$$B_s = B(\tau_1) + \frac{1}{2}S (\cos \alpha - \frac{1}{2}n) e^{-n\tau_1 \sec \alpha}. \quad . \quad . \quad (37)$$

If $\cos \alpha > \frac{1}{2}n$, which will usually be the case in applications, B_s is greater than $B(\tau_1)$. Thus the temperature of the surface exceeds that of the material (say air) in contact with it. Hence convection currents would be set up, and the state of radiative equilibrium would be destroyed. This is a simple way of demonstrating the impossibility of the existence of a state of radiative equilibrium throughout the entire atmosphere.

§ 11. *The "greenhouse" effect.*—Inserting in (27) the value of $B(\tau_1)$ from (28''), we have

$$B_s = S \cos \alpha [(\cos \alpha + \frac{1}{2}n) - (\cos \alpha - \frac{1}{2}n) e^{-n\tau_1 \sec \alpha}] / n. \quad (38)$$

Now if the black surface were exposed to the direct insolation $\pi S \cos \alpha$, without the intervention of an atmosphere, it would take up a temperature T_s' given by

$$\sigma T_s'^4 / \pi = B_s' = S \cos \alpha.$$

Hence

$$\frac{T_s^4}{T_s'^4} = 1 + \frac{(\cos \alpha - \frac{1}{2}n)(1 - e^{-n\tau_1 \sec \alpha})}{n}. \quad . \quad . \quad (39)$$

Thus, when $\cos \alpha > \frac{1}{2}n$, the surface is maintained at a temperature higher than it would be in the absence of an atmosphere. The ratio $T_s^4/T_s'^4$ increases as n decreases, the limit as $n \rightarrow 0$ being $1 + \tau_1$. The case of diffuse incident radiation is roughly given by putting $\cos \alpha = \frac{1}{2}$, and then the condition is $n < 1$, i. e. that the atmosphere or "protecting layer" must be more transparent to the incident radiation than to the radiation returned. This is the radiation part of the "greenhouse" or "heat-trap" effect, which is sometimes the subject of fallacious statements; it must of course be distinguished from that part of the effect which is due to the prevention of convection.

§ 12. *Extension to a partially convective atmosphere.*—We will now generalize the problem a little further. Suppose that we have a state of affairs in which the material above a

given level $\tau = \tau_2$ is in strict radiative equilibrium, that below the given level merely in radiative equilibrium *as a whole*; below $\tau = \tau_2$ the temperature distribution may be of any form (with or without a lower bounding surface) subject only to the condition that the whole system below τ_2 radiates outwards as much as it absorbs; in general, convection of heat will be required in the region below $\tau = \tau_2$ in order to maintain a steady state. Then it is easily seen that the temperature distribution above τ_2 is exactly the same as if the lower region were in radiative equilibrium in the strict sense; for the upward intensity at τ_2 , namely $I(\tau_2)$, is the same in the two cases. Hence the temperature distribution we have already found applies to the region above τ_2 . The importance of this point from the point of view of applications to the earth's troposphere and stratosphere is evident*.

§ 13. *The boundary between troposphere and stratosphere.*—It is convenient to denote the regions below and above the level τ_2 in our ideal problem by the words “troposphere” and “stratosphere” respectively, without implying any reference to these actual regions in the earth's atmosphere. Then § 12 shows that under the conditions there stated a stratosphere cannot be isothermal unless its optical thickness is zero or $\cos \alpha = \frac{1}{2}n$. If the optical thickness is not zero and $\cos \alpha > \frac{1}{2}n$, the lower parts of the stratosphere must be warmer than the upper. This agrees with § 2, where it was found that Gold's stratosphere is warming up at the base.

We are now in a position to frame in a precise manner the problem of where the division between troposphere and stratosphere should occur, in the ideal case. Let us suppose that there is a certain distribution of temperature which the processes of convection tend to set up throughout the whole atmosphere. Let the corresponding black body radiation-function be expressed as a function of optical depth, say $B_c(\tau)$. This temperature distribution together with the lower boundary surface implies a definite upward intensity of radiation at any point τ , say $I_c(\tau)$, which is determinate and calculable when $B_c(\tau)$ is given. Let τ_2 denote the optical depth of the surface of separation between troposphere and stratosphere which it is required to determine. Then

* The points which are the subject of §§ 10, 11, 12 are substantially made by Emden, in the form appropriate to diffuse radiation. But Emden's analysis is in parts a little complicated by his introducing unnecessarily early into the investigation an empirical expression for the water-vapour in the earth's atmosphere as a function of height.

below τ_2 the temperature is given by $B_c(\tau)$; above τ_2 it is given by the function $B(\tau)$ given by formula (28''). At τ_2 the upward intensity of radiation is that appropriate to the state of radiative equilibrium; it is the value $I(\tau_2)$ deducible from (26'') and (27''),

$$I(\tau_2) = B(\tau_2) + \frac{1}{2}S(\cos \alpha - \frac{1}{2}n)e^{-n\tau_2 \sec \alpha}.$$

Hence τ_2 is the root of the equation

$$I_c(\tau_2) = I(\tau_2). \quad . \quad . \quad . \quad . \quad (40)$$

Suppose this equation is solved. It by no means follows that

$$B_c(\tau_2) = B(\tau_2);$$

i. e. it by no means follows that the temperature immediately below the junction is continuous with that immediately above it. Further, even if it happens that these temperatures are equal, it does not follow that the condition for a convective atmosphere is satisfied in the region immediately below τ_2 . For a physically possible distribution both these conditions must be satisfied. Hence, in general, it is not possible to determine a level τ_2 such that a *prescribed* temperature distribution exists up to τ_2 and a radiative one above it.

The question must therefore be studied in the reverse order: what conditions does the existence of a stratosphere of non-zero optical thickness impose on the temperature distribution in the upper troposphere? It would make the present paper too long to take up the investigation here. But it appears to be possible to show that if the temperature is continuous at τ_2 , then *in general* (but not necessarily) the temperature gradient is discontinuous there. This is, of course, what is observed.

In the earth's stratosphere, on the other hand, the observed absence of vertical gradients strongly suggests that if it is in strict radiative equilibrium its optical thickness is practically zero. For the particular relation ($\cos \alpha = \frac{1}{2}n$ or $\cos \lambda = n/\sqrt{2}$) which is necessary for an isothermal stratosphere of non-zero optical thickness cannot be satisfied save in very high latitudes; and even here (as we shall see) this would be prevented by the additional radiation due to world-wide convection. Further, we have seen in § 2 that if the absorption of solar radiation is neglected, an isothermal stratosphere would soon cease to be isothermal and would be disturbed by convection currents. If now the optical thickness of the stratosphere is practically zero, a state of radiative equilibrium will probably extend a little way below the tropopause, and the observed suddenness of the demarcation must

be due to a sudden diminution of absorbing power. This, again, would indicate the tropopause as the boundary of the water-vapour atmosphere. The contrary has, however, been urged by Gold * on different grounds, and it is difficult to deny the force of his arguments. The matter is obviously one of considerable difficulty.

14. *Applications.*—This concludes the discussion of the idealized problem of which it is the main business of the paper to give an account. The theory is capable of a number of applications to the earth's atmosphere, but the principal of these, at least in the case when the incident radiation may on the average be taken to be diffuse, have already been made by Emden. Perhaps the result most directly useful is the correction to the Schwarzschild boundary temperature due to the absorption of the incident radiation there, given by formulæ (33) or (34), or Emden's form (32). (The Schwarzschild temperature is given by putting $n=0$.) From a discussion of the observational material, Emden finds that n may be taken to be $\frac{1}{23}$. With $\cos \alpha = \frac{1}{2}$ this gives an increase of 1 per cent., making the calculated value of T_0 (see § 1) about 216° . If n is taken equal to $\frac{1}{10}$, T_0 becomes 219° , the observed value.

Another application made by Emden is to show that an atmosphere entirely in radiative equilibrium would be an impossibility, even in the absence of the warming effect due to the earth's surface (§ 10 above). For, taking into account the water-vapour distribution, radiative equilibrium implies at a sufficient depth temperature gradients in excess of the critical gradient for stability; so that convection currents would be set up. Emden finds that this would occur at a height of about 3 km.

When Emden wrote, the variation of the temperature of the stratosphere and height of the tropopause with latitude was not fully appreciated. And in the light of this variation, the small improvement in agreement, due to the introduction of n , between the Schwarzschild temperature and the observed mean temperature for S.E. England becomes largely meaningless. What is astonishing, *a priori*, is that the two temperatures should agree as well as they do. The agreement can only mean that the *actual* temperature of the stratosphere over S.E. England must be very close to the *mean* temperature over the earth. The agreement is partly helped by the circumstance that the latitude of England is

* Geophysical Memoirs, No. 5, p. 129 (1913).

close to 60° ; and we have seen that the value $\alpha = 60^\circ$ plays a special part in the theory.

It is therefore interesting to inquire whether the theory developed in the present paper has any bearing on the question of the origin of the variation with latitude. The facts to explain are that the temperature of the stratosphere increases as the latitude increases, and that the height of the tropopause decreases. Reference to recent books on meteorology and the physics of the air shows that there is no accepted detailed explanation.

Assuming that a stratosphere is optically very thin, it will be very nearly isothermal and its temperature will be equal to T_0 . Formula (34) shows at once that the ratio of T_0 to T_1 (the effective temperature of the insolation) increases as λ increases, provided n is not zero, and that the increase becomes relatively large for high latitudes. As seen in § 7, this is due to the increased absorption of the solar radiation in the more superficial layers. This does not seem to have been suggested before, and it may in fact be one of the contributory causes of the increased temperature in higher latitudes. But the first of formulæ (31) shows that the actual value of T_0 decreases as λ increases. All the theory indicates is that T_0 decreases much more slowly than T_1 as λ increases, and so the absolute increase of T_0 as observed is not accounted for. The tendency to increase would be helped if it could be shown that n increased with λ , *i. e.* if the ratio of absorption of solar radiation to that of terrestrial radiation increased with latitude. Some effect of this kind there must be; for carbon dioxide is more important as an absorber of solar radiation than of terrestrial radiation, and owing to the decreased humidity in high latitudes the ratio of carbon dioxide to water vapour is there greater. But this, again, would not appear to be sufficient.

It must now be recalled that we have assumed throughout that any portion of the surface radiates away an amount of energy equal to that incident on it. But we know that this is not true for the earth's surface. Heat is convected from the equatorial regions toward the polar regions: otherwise the change of surface temperature with latitude would be much more severe than it is. Hence the equatorial regions must radiate less than they receive, the polar regions more. Now if πF is the additional net outward flux (positive or negative), the radiative distribution of temperature is obtained by adding the term $F(\frac{1}{2} + \tau)$ to the right-hand of (28''), and the boundary temperature is given by

$$\sigma T_0^4 / \pi = \frac{1}{2} F + \frac{1}{2} S (\cos \alpha + \frac{1}{2} n). \quad . \quad . \quad . \quad (41)$$

A similar formula holds when rotation is taken into account. Since F is positive in high latitudes and negative in low, T_0 should be greater in high latitudes and smaller in low than it would be in the absence of convection. Moreover, for some particular latitude F will be zero, and here the value of T_0 will be the same as in the absence of world-wide convection. The agreement between calculation and observation for S.E. England thus implies that in this latitude the amount radiated is about equal to the solar radiation incident.

But T_0 for high latitudes *exceeds* that for low. Now $\pi(F + S \cos \alpha)$ is the total outward radiation to space. Hence (on the assumptions made) the total radiation to space in high latitudes must exceed that in low, unless the change of n is very considerable. This is a surprising result, but not necessarily impossible; if the stratospheric temperatures are really maintained to great heights under the influence of radiation there seems little escape from it. The difficulty is, of course, not new. Gold dealt with it as follows*. Gold showed that if the absorbing power of the atmosphere increases, then the theoretical height of the tropopause increases. In the notation of § 2, it can be deduced from (4) that τ_2 though increasing with τ_1 is fairly insensitive to it. Roughly speaking, then, on Gold's theory the isothermal state sets in at a fixed optical depth below the outer boundary; hence the more absorbing the atmosphere the smaller is τ_2/τ_1 , the smaller is p_2/p_1 , and the greater is the height of the tropopause. The known increased humidity over the equator, with consequent increased absorbing power, would thus account for the observed increased height of the tropopause; and the lower temperature of the stratosphere follows from the increased height through which a convective gradient holds, even allowing for the higher ground temperature. But the above difficulty still remains, for the increased absorbing power implies a decreased outward radiation.

Gold argued from the improbability of this that "the atmosphere is not a 'grey' body, but must have nearly perfect transparency for some spectral region." It is well known that the coefficient of absorption varies considerably from place to place in the spectrum, whereas we have assumed it to possess but two values—one for solar radiation and one for terrestrial. But it is very doubtful whether this removes the difficulty. For there still has to be an equilibrium of radiation. One might reason generally that the transparency of the air in certain spectral regions

* Geophysical Memoirs, i. p. 128 (1913).

would permit the escape of extra radiation which would have no effect in controlling the temperature. But the boundary temperature is not necessarily lower: it may be higher or lower, according to the spectral regions in which the air is not transparent. It has recently been shown by the author* that if a thin layer of material is exposed on one side to black radiation of effective temperature T_1 and is in radiative equilibrium, then it will take up a temperature T_0 which is equal to $2^{-\frac{1}{2}}T_1$ (the Schwarzschild value) when the material is grey, but which, whatever the optical properties of the material, will satisfy the inequalities

$$T_1 > T_0 > \frac{1}{2}T_1;$$

and that T_0 will approximate to T_1 if the material is transparent save in the extreme ultra-violet, to $\frac{1}{2}T_1$ if transparent save in the extreme infra-red. (Here "ultra-violet" and "infra-red" must be interpreted as relative to the value of λ_{\max} . corresponding to T_1 .) If, for simplicity, we neglect the absorption of solar radiation, the theorem can be applied as it stands to the stratosphere. It shows that the temperature will be less than for uniform absorption only if the absorption occurs principally on the long wave-length side of λ_{\max} .—in this case about $10\ \mu$. But water-vapour is least opaque† to long wave-length radiation in the region $7\ \mu$ to $20\ \mu$, more particularly in the region $8\ \mu$ to $12\ \mu$. There is, indeed, important carbon-dioxide absorption‡ in the region $13\ \mu$ to $16\ \mu$, but this is usually considered to be not large compared with the water-vapour absorption. Without a detailed numerical investigation it would be difficult to estimate the resultant effect; but if carbon-dioxide absorption is important near the equator, it should be more important in higher latitudes, and this would go against the argument. If a relation $T_0 = \epsilon T_1$ holds above the equator in virtue of selective radiative equilibrium, it ought to hold too in higher latitudes; for it is difficult to see why the atmosphere at 20 km. above the equator should be optically different from that in higher latitudes in the direction of being relatively more absorptive above the equator on the red side of λ_{\max} . Moreover, if we do modify the selective optical properties of the atmosphere from equator to pole, then Gold's explanation

* "The Temperature in the Outer Atmosphere of a Star," Monthly Notices R. A.S. lxxii. p. 368 (1922). See also Fabry, *Astrophys. Journ.* xlv. p. 269 (1917).

† Abbot, *Annals Astrophys. Obs. Smithson. Inst.* ii. p. 167 (1908).

‡ See, for example, Humphreys, 'Physics of the Air,' p. 88, 1920.

of the change of height of the tropopause no longer holds, at least without further examination.

In view of these considerations, it seems on the whole tenable that the outward radiation from the equator is less than from higher latitudes, and that the variation of stratospheric temperature must be principally due to the general circulation of the air in the convective region. (The connexion between the variation of temperature of the stratosphere and the observed uniformity of pressure at 20 km. over the whole earth has been pointed out by W. H. Dines*.) The higher upper-air temperatures of high latitudes may still be helped by the increased direct absorption there, in the way we have seen.

§ 15. *Summary.*—It is shown that if the atmosphere is divided into two shells—a lower one (the troposphere) in convective equilibrium, and an isothermal one (the stratosphere),—then the stratosphere cannot be in strict radiative equilibrium unless its optical thickness for low-temperature radiation is zero, even if it is in radiative equilibrium as a whole. The only exception is when the lower region is also in radiative equilibrium as a whole and when, in addition, a special relation ($\cos \alpha = \frac{1}{2}n$, or $\cos \lambda = n/\sqrt{2}$ when rotation is allowed for) exists between the angle of incidence of the solar radiation and the ratio of the coefficients of absorption of solar and terrestrial radiation. The theory of atmospheres in radiative equilibrium subject to insolation is discussed in detail for various cases, including the dependence of the temperature distribution on the angle of incidence of the solar radiation. An integral equation for the temperature is obtained. Comparison is made with Emden's work. From an application of the results to the earth's atmosphere it is inferred that the variation of the temperature of the stratosphere with latitude cannot be accounted for on radiation principles unless the total radiation of the earth to space is greater in high latitudes than in low latitudes. This is probably the case, and the observed distribution of stratospheric temperature is probably connected with the general circulation of the air; however, the increased direct absorption of solar energy in the upper levels in high latitudes must have some effect.

It is intended to insist principally on the general theory, and the applications are only made tentatively.

July 17, 1922.

* *Geophysical Memoirs*, No. 13, p. 71 (1919).