CERTAIN SURFACES OF VOSS AND SURFACES ASSOCIATED WITH THEM.

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In a recent memoir published in these Rendiconti [1]) BIANCHI has studied at length a class of surfaces defined as follows. We take any surface S and a point O fixed in space. At each point M of S we consider the trihedral of the normal and the tangents to the lines of curvature. When the vertex M of the trihedral moves over the surface S, the point O assumes, with respect to the trihedral, a series of positions, which in general constitute a surface, called the tractrix surface for S and O. The surfaces studied by BIANCHI are those which admit as a tractrix a quadric which has the planes of the trihedral for planes of symmetry. These surfaces depend essentially on pseudospherical congruences, defined by the property that the distances between the limit points and between the focal points are constant. In fact, the normals to these surfaces are parallel to the rays of a pseudospherical congruence, and moreover the lines of curvature of the surfaces correspond to the asymptotic lines on the focal surfaces of the congruence. There are different types of these surfaces determined by the character of the quadric tractrix. Most of the types belong to a general class of surfaces discussed by the author in a former number of these Rendiconti [2]). We established in particular a transformation of surfaces of Voss into surfaces of Voss, such that the lines joining corresponding points on a surface and a transform form a congruence meeting the two surfaces in their geodesic conjugate systems. In this case we say that the two surfaces are in the relation of a transformation Ω. We showed further that the lines of intersection of corresponding tangent planes to two surfaces in this relation form a normal congruence, that the lines of this congruence

[1]) L. BIANCHI, *Sopra una classe di superficie collegate alle congruenze pseudosferiche* [Rendiconti del Circolo Matematico di Palermo, t. XL (2° semestre 1915), pp. 110-152].

[2]) L. P. EISENHART, *Conjugate systems with equal Tangential Invariants and the Transformation of* MOUTARD [Rendiconti del Circolo Matematico di Palermo, t. XXXIX (1° semestre 1915), pp. 153-176]. In what follows this memoir will be referred to as M.

are parallel to the rays of a pseudospherical congruence, and that the developables of the former congruence correspond to the asymptotic lines on the focal surfaces of the latter. In certain cases the surfaces orthogonal to these normal congruences are of the class considered by BIANCHI, and it is the purpose of this paper to investigate these surfaces from this point of view.

If Σ is a surface of the type under consideration, one knows how to draw planes through the normals to Σ such that these planes envelope a surface of Voss. In general two surfaces of Voss are obtained in this manner. They are special surfaces of Voss, characterized by the property that if W denotes the distance from the origin to the tangent planes to the surface, then W is a solution of a completely integrable system of three partial differential equations of the second order. Conversely, each solution of this system determines a special surface of Voss, and the determination of the solutions of the system is the same analytical problem as that of finding the BÄCKLUND transformations of pseudospherical surfaces. When such a surface is known, it is possible to draw in its tangent planes lines forming a normal congruence met orthogonally by surfaces Σ. Moreover, the analytical determination of these congruences is the same problem as finding the BÄCKLUND transfor mations of the pseudospherical surface whose asymptotic lines have the same spherical representation as the geodesic conjugate system on the given surface of Voss.

The tangents to the curves of either family of the geodesic conjugate system on a special surface of Voss are normal to a surface Σ with a quadric of revolution for tractrix. As thus given Σ is a surface of GUICHARD, and it is readily shown that the other focal surface of the accompanying congruence of GUICHARD is of the same type.

We establish two types of Transformations of RIBAUCOUR of these surfaces Σ, and for certain types the transforms by reciprocal radii are surfaces of the same kind. At the same time these transformations carry with them certain induced transformations of special surfaces of Voss.

1. *Equations of a surface V and preliminary formulas.* — We consider a surface V referred to the geodesic conjugate system. Since this system has the same spherical representation as the asymptotic lines on a pseudospherical surface P, the linear element of this spherical representation can be given the form

$$d\sigma^2 = du^2 + 2\cos 2\omega\, du\, dv + dv^2 \quad (1)$$ [3]),

where ω is a function of u and v satisfying the equation

$$\frac{\partial^2 \omega}{\partial u \partial v} + \sin\omega\cos\omega = 0. \quad (2)$$

If X_1, Y_1, Z_1, and X_2, Y_2, Z_2, denote respectively the direction-cosines of the bisectors of the angles between the parametric curves of the spherical representation,

3) The formulas of this section can be obtained from those of §§ 1, 3, M, by taking $\rho = \text{const.}$

and X, Y, Z, the direction-cosines of the normal to a surface V with this representation, we have

$$(3)\qquad \begin{cases} \dfrac{\partial X_1}{\partial u} = -\dfrac{\partial \omega}{\partial u} X_2 - \sin\omega X, & \dfrac{\partial X_1}{\partial v} = \dfrac{\partial \omega}{\partial v} X_2 + \sin\omega X, \\ \dfrac{\partial X_2}{\partial u} = \dfrac{\partial \omega}{\partial u} X_1 - \cos\omega X, & \dfrac{\partial X_2}{\partial v} = -\dfrac{\partial \omega}{\partial v} X_1 - \cos\omega X, \\ \dfrac{\partial X}{\partial u} = \sin\omega X_1 + \cos\omega X_2, & \dfrac{\partial X}{\partial v} = -\sin\omega X_1 + \cos\omega X_2. \end{cases}$$

The tangential coordinates of V, namely X, Y, Z and W, satisfy the equation

$$(4)\qquad \frac{\partial^2 W}{\partial u \partial v} + \cos 2\omega \cdot W = 0.$$

In terms of the tangential coordinates the rectangular coordinates, x, y, z, of V are expressible in the form

$$(5)\qquad x = WX + \frac{1}{\sin 2\omega}\left[X_1 \cos\omega\left(\frac{\partial W}{\partial u} - \frac{\partial W}{\partial v}\right) + X_2 \sin\omega\left(\frac{\partial W}{\partial u} + \frac{\partial W}{\partial v}\right)\right].$$

From these we obtain

$$(6)\qquad \begin{cases} \dfrac{\partial x}{\partial u} = -\dfrac{D}{\sin 2\omega}(\cos\omega X_1 + \sin\omega X_2), \\ \dfrac{\partial x}{\partial v} = \dfrac{D''}{\sin 2\omega}(\cos\omega X_1 - \sin\omega X_2), \end{cases}$$

where

$$(7)\qquad \begin{cases} D = -\dfrac{\partial^2 W}{\partial u^2} + 2\cot 2\omega \dfrac{\partial \omega}{\partial u}\dfrac{\partial W}{\partial u} - \dfrac{2}{\sin 2\omega}\dfrac{\partial \omega}{\partial u}\dfrac{\partial W}{\partial v} - W, \\ D'' = -\dfrac{\partial^2 W}{\partial v^2} - \dfrac{2}{\sin 2\omega}\dfrac{\partial \omega}{\partial v}\dfrac{\partial W}{\partial u} + 2\cot 2\omega \dfrac{\partial \omega}{\partial v}\dfrac{\partial W}{\partial v} - W. \end{cases}$$

The CODAZZI equations for V are

$$(8)\qquad \frac{\partial D}{\partial v} - \frac{2}{\sin 2\omega}\frac{\partial \omega}{\partial u} D'' = 0, \qquad \frac{\partial D''}{\partial u} - \frac{2}{\sin 2\omega}\frac{\partial \omega}{\partial v} D = 0.$$

When the surface P is subjected to a BÄCKLUND transformation, the linear element of the spherical representation of the asymptotic lines on the new surface P_1 is given by

$$(9)\qquad d\sigma_1^2 = du^2 + 2\cos 2\theta\, du\, dv + dv^2,$$

where θ is a solution of equation (2) satisfying also the equations

$$(10)\qquad \begin{cases} \dfrac{\partial \theta}{\partial u} - \dfrac{\partial \omega}{\partial u} = -\cot\dfrac{\sigma}{2}\sin(\theta + \omega), \\ \dfrac{\partial \theta}{\partial v} + \dfrac{\partial \omega}{\partial v} = \tan\dfrac{\sigma}{2}\sin(\theta - \omega), \end{cases}$$

σ being the constant angle between the tangent planes to P and P_1 at corresponding points. The intersection of these planes passes through P and P_1, and its direction-cosines $\bar{X}$, $\bar{Y}$, $\bar{Z}$, are of the form

$$\bar{X} = \cos\theta\, X_1 + \sin\theta\, X_2. \tag{11}$$

Moreover, the direction-cosines of the normal to P_1 and of the bisectors of the angles between the parametric curves on the spherical representation of P_1 are of the form

$$\left\{\begin{aligned} X' &= \cos\sigma X + \sin\sigma(\sin\theta X_1 - \cos\theta X_2), \\ X'_1 &= \sin\omega\,[\cos\sigma(\sin\theta\, X_1 - \cos\theta\, X_2) - \sin\sigma X] - \cos\omega\,\bar{X}, \\ X'_2 &= -\cos\omega\,[\cos\sigma(\sin\theta\, X_1 - \cos\theta\, X_2) - \sin\sigma X] - \sin\omega\,\bar{X}. \end{aligned}\right. \tag{12}$$

2. *Transformations Ω of surfaces V. Normal harmonic congruences.*

From the theory [4]) of transformations Ω it follows that X', Y', Z' and W_1, defined by

$$\frac{\partial}{\partial u}(W_1 w) = -w^2\frac{\partial}{\partial u}\left(\frac{W}{w}\right), \qquad \frac{\partial}{\partial v}(W_1 w) = w^2\frac{\partial}{\partial v}\left(\frac{W}{w}\right), \tag{13}$$

where w is any solution of (4), are the tangential coordinates of a surface of Voss, say V_1, such that the developables of the congruence of joins of corresponding points on V and V_1 meet the latter in the parametric geodesic conjugate system. The direction-cosines of the line L of intersection of the tangent planes to P and P_1 are necessarily $\bar{X}$, $\bar{Y}$, $\bar{Z}$, given by (11). We are interested particularly in the case where the congruence of lines L is normal. and this is the only case which will be considered hereafter. We call it the *harmonic congruence.*

The necessary and sufficient condition that the congruence of lines L be normal to a surface, say Σ, is that the function w shall be defined by

$$\frac{\partial \log w}{\partial u} = \cot\frac{\sigma}{2}\cos(\theta + \omega), \qquad \frac{\partial \log w}{\partial v} = -\tan\frac{\sigma}{2}\cos(\theta - \omega), \tag{14}$$

where θ and σ are any solutions of equations (10) [5]). In this case equations (13) are reducible to

$$\left\{\begin{aligned} \frac{\partial W_1}{\partial u} &= (W - W_1)\cot\frac{\sigma}{2}\cos(\theta + \omega) - \frac{\partial W}{\partial u}, \\ \frac{\partial W_1}{\partial v} &= (W + W_1)\tan\frac{\sigma}{2}\cos(\theta - \omega) + \frac{\partial W}{\partial v}. \end{aligned}\right. \tag{15}$$

The rectangular coordinates a_1, b_1, c_1; a_2, b_2, c_2, of the focal points F_1, F_2, of this congruence are of the respective forms

$$a_1 = x + T\frac{\cos\omega\, X_1 + \sin\omega\, X_2}{\sin(\theta - \omega)}, \quad a_2 = x + T\frac{\cos\omega\, X_1 - \sin\omega\, X_2}{\sin(\theta + \omega)}, \tag{16}$$

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4) The results of this section follow from §§ 3, 5 and 6, M., when we take $\rho = \rho_1 = 1$.

5) l. c. 2) n° 86.

where

$$(17) \qquad T = \frac{W_1 - W\cos\sigma}{\sin\sigma} - \frac{\sin(\theta-\omega)}{\sin 2\omega}\frac{\partial W}{\partial u} + \frac{\sin(\theta+\omega)}{\sin 2\omega}\frac{\partial W}{\partial v}$$ [6]).

From the theory of normal congruences it follows that the normals to the focal surfaces which are the loci of F_1 and F_2 are parallel to the tangents to the lines of curvature on Σ, which as a matter of fact correspond to the geodesic system on V and consequently are parametric. Hence if $\overline{X}_1$, $\overline{Y}_1$, $\overline{Z}_1$; $\overline{X}_2$, $\overline{Y}_2$, $\overline{Z}_2$ denote the direction-cosines of the tangents to the curves $v = \text{const.}$, $u = \text{const.}$ respectively on Σ, we find that

$$(18) \qquad \begin{cases} \overline{X}_1 = \cos\frac{\sigma}{2}(\sin\theta X_1 - \cos\theta X_2) - \sin\frac{\sigma}{2}X, \\ \overline{X}_2 = -\sin\frac{\sigma}{2}(\sin\theta X_1 - \cos\theta X_2) - \cos\frac{\sigma}{2}X. \end{cases}$$

From (11) we obtain by differentiation

$$(19) \qquad \frac{\partial \overline{X}}{\partial u} = \sqrt{E}\,\overline{X}_1, \quad \frac{\partial \overline{X}}{\partial v} = \sqrt{G}\,\overline{X}_2,$$

where

$$(20) \qquad \sqrt{E} = \frac{\sin(\theta+\omega)}{\sin\frac{\sigma}{2}}, \quad \sqrt{G} = \frac{\sin(\theta-\omega)}{\cos\frac{\sigma}{2}}, \quad F = 0.$$

Evidently E, F and G are the coefficients of the spherical representation of Σ. It is readily found that

$$(21) \qquad \frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v} = \cos(\theta+\omega), \qquad \frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u} = -\cos(\theta-\omega).$$

We express the rectangular coordinates ξ, η, ζ of Σ in the form

$$(22) \qquad \xi = \overline{W}\,\overline{X} + \overline{W}_1\overline{X}_1 + \overline{W}_2\overline{X}_2,$$

where $\overline{W}_1$ and $\overline{W}_2$ are to be determined. Since these functions are evidently tangential coordinates of the loci of F_1 and F_2 respectively, we have

$$(23) \qquad \overline{W}_1 = \sum \overline{X}_1 a_1 = \frac{W_1 - W}{2\sin\frac{\sigma}{2}}, \qquad \overline{W}_2 = \sum \overline{X}_2 a_2 = -\frac{W_1 + W}{2\cos\frac{\sigma}{2}}.$$

From these equations we find

$$(24) \qquad \frac{\partial \overline{W}_1}{\partial v} = \frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\overline{W}_2, \qquad \frac{\partial \overline{W}_2}{\partial u} = \frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\overline{W}_1,$$

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[6]) Cfr. l. c. [2]), nⁱ 78-80.

and from (18)

$$(25)\quad \begin{cases} \dfrac{\partial \overline{X}_1}{\partial u} = -\dfrac{1}{\sqrt{G}}\dfrac{\partial \sqrt{E}}{\partial v}\overline{X}_2 - \sqrt{E}\,\overline{X}, & \dfrac{\partial \overline{X}_1}{\partial v} = \dfrac{1}{\sqrt{E}}\dfrac{\partial \sqrt{G}}{\partial u}\overline{X}_2, \\ \dfrac{\partial \overline{X}_2}{\partial u} = \dfrac{1}{\sqrt{G}}\dfrac{\partial \sqrt{E}}{\partial v}\overline{X}_1, & \dfrac{\partial \overline{X}_2}{\partial v} = -\dfrac{1}{\sqrt{E}}\dfrac{\partial \sqrt{G}}{\partial u}\overline{X}_1 - \sqrt{G}\,\overline{X}. \end{cases}$$

Equations of the form (22) serve to define any surface referred to its lines of curvature. Equations (24) and (25) hold in this general case, as do also

$$(26)\qquad \frac{\partial \overline{W}}{\partial u} = \sqrt{E}\,\overline{W}_1, \quad \frac{\partial \overline{W}}{\partial v} = \sqrt{G}\,\overline{W}_2,$$

and

$$(27)\quad \begin{cases} \dfrac{\partial \xi}{\partial u} = \left(\dfrac{\partial \overline{W}_1}{\partial u} + \dfrac{1}{\sqrt{G}}\dfrac{\partial \sqrt{E}}{\partial v}\overline{W}_2 + \sqrt{E}\,\overline{W}\right)\overline{X}_1, \\ \dfrac{\partial \xi}{\partial v} = \left(\dfrac{\partial \overline{W}_2}{\partial v} + \dfrac{1}{\sqrt{E}}\dfrac{\partial \sqrt{G}}{\partial u}\overline{W}_1 + \sqrt{G}\,\overline{W}\right)\overline{X}_2. \end{cases}$$

3. ***Surfaces V_0 and surfaces Σ_0.*** — The function w defined by equations (14) is a solution of (4), and consequently there exists a surface V_0 whose tangential coordinates are X, Y, Z, w. We inquire under what conditions a surface of Voss is so defined.

If equations (14) are differentiated with respect to both u and v, we note that w satisfies the system

$$(28)\quad \begin{cases} \dfrac{\partial^2 W}{\partial u^2} - 2\cot 2\omega \dfrac{\partial \omega}{\partial u}\dfrac{\partial W}{\partial u} - 2\dfrac{B}{A}\dfrac{1}{\sin 2\omega}\dfrac{\partial \omega}{\partial u}\dfrac{\partial W}{\partial v} - \dfrac{B}{A}W = 0, \\ \dfrac{\partial^2 W}{\partial u \partial v} + \cos 2\omega\, W = 0, \\ \dfrac{\partial^2 W}{\partial v^2} - 2\dfrac{A}{B}\dfrac{1}{\sin 2\omega}\dfrac{\partial \omega}{\partial v}\dfrac{\partial W}{\partial u} - 2\cot 2\omega \dfrac{\partial \omega}{\partial v}\dfrac{\partial W}{\partial v} - \dfrac{A}{B}W = 0, \end{cases}$$

where A and B are positive constants such that

$$(29)\qquad A\cot\frac{\sigma}{2} = B\tan\frac{\sigma}{2}.$$

Comparing this result with (7), we see that the second fundamental functions of V_0 are of the form

$$(30)\quad \begin{cases} D = -\left(1 + \dfrac{B}{A}\right)\left(W + \dfrac{2}{\sin 2\omega}\dfrac{\partial \omega}{\partial u}\dfrac{\partial W}{\partial v}\right), \\ D'' = -\left(1 + \dfrac{A}{B}\right)\left(W + \dfrac{2}{\sin 2\omega}\dfrac{\partial \omega}{\partial v}\dfrac{\partial W}{\partial u}\right). \end{cases}$$

It is readily seen that these values satisfy the CODAZZI equations (8) in consequence of (28).

When equations (14) are solved for $\sin\theta$ and $\cos\theta$, we get

$$(31)\quad \begin{cases} \sin\theta = -\dfrac{1}{2\sin\omega}\left(\tan\dfrac{\sigma}{2}\dfrac{\partial \log w}{\partial u} + \cot\dfrac{\sigma}{2}\dfrac{\partial \log w}{\partial v}\right), \\ \cos\theta = \dfrac{1}{2\cos\omega}\left(\tan\dfrac{\sigma}{2}\dfrac{\partial \log w}{\partial u} - \cot\dfrac{\sigma}{2}\dfrac{\partial \log w}{\partial v}\right). \end{cases}$$

Squaring and adding, we observe that for w defined by (14) the function Φ, defined by

$$(32)\quad \Phi = W^2 - \frac{1}{\sin^2 2\omega}\left[\frac{A}{B}\left(\frac{\partial W}{\partial u}\right)^2 + 2\cos 2\omega\frac{\partial W}{\partial u}\frac{\partial W}{\partial v} + \frac{B}{A}\left(\frac{\partial W}{\partial v}\right)^2\right],$$

vanishes. As a matter of fact, it is readily shown that for any function W satisfying (28) the first derivatives of Φ are zero, so that Φ is a constant.

Suppose now that we have any solution of equations (28) for which Φ is zero. It is easily shown that σ given by (29) and θ defined by (31) satisfy equations (10). Accordingly we say that every surface of Voss whose tangential coördinate W satisfies equations (28) and makes Φ zero is a surface V_0.

Suppose we have a surface V_0 and apply to it the transformation determined by the function w of the surface itself. From (13) it follows that $W_1 = c/w$, where c is a constant. In this case equations (23) are

$$(33)\quad \overline{W}_1 = \frac{\dfrac{c}{w} - w}{2\sin\dfrac{\sigma}{2}}, \quad \overline{W}_2 = -\frac{\dfrac{c}{w} + w}{2\cos\dfrac{\sigma}{2}}$$

From these espressions we have

$$(34)\quad A\,W_1^2 - B\,W_2^2 = \text{const.}$$

Consequently we have

THEOREM I. — *When a surface V_0 is subjected to the transformation Ω determined by the function w of the surface itself, the tractrix surface of a surface Σ normal to the harmonic congruence of the transformation is a hyperbolic cylinder whose axis is the normal to Σ.*

4. ***When the tractrix surface is a paraboloid.*** — We consider now the case when the tractrix surface is a paraboloid with an equation of the form

$$(35)\quad A\overline{W}_1^2 - B\overline{W}_2^2 - 2h\overline{W} = 0.$$

If this equation be differentiated separately with respect to u and v, the resulting equations are reducible by means of (15), (23), (24) and (26) to

$$(36)\quad \begin{cases} \left[W_1\left(\cot\dfrac{\sigma}{2}A - \tan\dfrac{\sigma}{2}B\right) - W\left(\cot\dfrac{\sigma}{2}A + \tan\dfrac{\sigma}{2}B\right)\right]\dfrac{\cos(\theta+\omega)}{2} \\ \qquad + A\dfrac{\partial W}{\partial u} + h\sin(\theta+\omega) = 0, \\ \left[W_1\left(\cot\dfrac{\sigma}{2}A - \tan\dfrac{\sigma}{2}B\right) - W\left(\cot\dfrac{\sigma}{2}A + \tan\dfrac{\sigma}{2}B\right)\right]\dfrac{\cos(\theta-\omega)}{2} - \\ \qquad - B\dfrac{\partial W}{\partial v} + h\sin(\theta-\omega) = 0. \end{cases}$$

Eliminating the first parenthesis from these equations, we obtain

$$(37)\qquad A\cos(\theta-\omega)\frac{\partial W}{\partial u}+B\cos(\theta+\omega)\frac{\partial W}{\partial v}+h\sin 2\omega=0.$$

Likewise the elimination of h gives

$$(38)\qquad \begin{cases}\dfrac{\sin 2\omega}{\sin\sigma}\left[W_1\left(\cos^2\dfrac{\sigma}{2}A-\sin^2\dfrac{\sigma}{2}B\right)-W\left(\cos^2\dfrac{\sigma}{2}A+\sin^2\dfrac{\sigma}{2}B\right)\right]+\\ \qquad -A\sin(\theta-\omega)\dfrac{\partial W}{\partial u}-B\sin(\theta+\omega)\dfrac{\partial W}{\partial v}=0.\end{cases}$$

When we express the condition that this value of W_1 shall satisfy (15), we get, in consequence of (7), the expressions (30) for D and D''. When these values of D and D'' are substituted in (7), we find that W must satisfy equations (28).

Making use of this fact, we differentiate (37) separately with respect to u and v. The resulting equations are consistent only in case σ satisfies (29), and then they are equivalent to the single equation

$$(39)\qquad A\sin(\theta-\omega)\frac{\partial W}{\partial u}+B\sin(\theta+\omega)\frac{\partial W}{\partial v}+A\cot\frac{\sigma}{2}\sin 2\omega\, W=0.$$

However, when σ satisfies (29), the coefficient of W_1 in (38) vanishes, and consequently the foregoing investigation does not justify us in concluding that W is a solution of the system (28). Nevertheless we shall find that this is the case.

On the assumption that σ satisfies (29) equations (36) become

$$(40)\qquad \begin{cases}\dfrac{\partial W}{\partial u}=\quad\cot\dfrac{\sigma}{2}\cos(\theta+\omega)W-\dfrac{h}{A}\sin(\theta+\omega),\\ \dfrac{\partial W}{\partial v}=-\tan\dfrac{\sigma}{2}\cos(\theta-\omega)W+\dfrac{h}{B}\sin(\theta-\omega).\end{cases}$$

When these expressions for $\dfrac{\partial W}{\partial u}$ and $\dfrac{\partial W}{\partial v}$ are substituted in (37) and (39), these equations are satisfied identically. Moreover, if these expressions are differentiated and the results are substituted in (28), the latter are satisfied, in consequence of (10). And from (32) we have

$$(41)\qquad \Phi=-\frac{h^2}{AB}.$$

If equations (40) are solved for $\sin\theta$ and $\cos\theta$, we get

$$(42)\qquad \begin{cases}(ABW^2+h^2)\sin 2\omega\sin\theta=-A\left(B\tan\dfrac{\sigma}{2}W\cos\omega+h\sin\omega\right)\dfrac{\partial W}{\partial u}-\\ \qquad -B\left(A\cot\dfrac{\sigma}{2}W\cos\omega+h\sin\omega\right)\dfrac{\partial W}{\partial v};\\ (ABW^2+h^2)\sin 2\omega\cos\theta=A\left(B\tan\dfrac{\sigma}{2}W\sin\omega-h\cos\omega\right)\dfrac{\partial W}{\partial u}-\\ \qquad -B\left(A\cot\dfrac{\sigma}{2}W\sin\omega+h\cos\omega\right)\dfrac{\partial W}{\partial v}.\end{cases}$$

Squaring and adding, we get (41). On the assumption that W is a solution of the system (28), σ given by (29) and θ by (42) satisfy equations (10). If $\Phi = 0$, we have the case of § 3. If $\Phi \neq 0$, we determine h by (41) and the conditions of the present section are satisfied.

The *special surfaces of* Voss determined by solutions W of (28) are called V_h, where h denotes the constant obtamed from (41).

From the above results follows

THEOREM 2. — *Each solution of equations* (10) *determines a family of surfaces* V_h, *h being the parameter of the family; these surfaces are obtained by quadrature.*

Assume that we have a surface V_h for which $h \neq 0$, and that we have determined σ and θ by (29) and (42). There exists a function w given by (14) for these values of σ and θ. We use this w to determine a transformation Ω of V_h.

Substituting the values from (40) in (13), we get for the determination of W_1

$$\frac{\partial}{\partial u}(W_1 w) = \frac{h}{A} w \sin(\theta + \omega), \qquad \frac{\partial}{\partial v}(W_1 w) = \frac{h}{B} w \sin(\theta - \omega).$$

We define a function T by the quadratures

$$\frac{\partial T}{\partial u} = \cos\frac{\sigma}{2}\sqrt{E}\,w, \quad \frac{\partial T}{\partial v} = \sin\frac{\sigma}{2}\sqrt{G}\,w, \tag{43}$$

which are readily found to be consistent. Hence we have, to within an additive constant,

$$W_1 w = \tan\frac{\sigma}{2}\, T \frac{h}{A}. \tag{44}$$

From (35), (33) and (44) we get

$$\overline{W} = -\frac{TW}{w \sin\sigma}. \tag{45}$$

Reviewing the preceding results, we note that when a solution of equations (10) is known, we can find by quadratures functions W, w and T, whence we obtain a surface Σ with a hyperbolic paraboloid for tractrix. In other words we have

THEOREM 3. — *The determination of surfaces Σ with a hyperboloid paraboloid or a hypesbolic cylinder for tractrix, as given by* (35) *or* (34), *is equivalent to the determination of* BÄCKLUND *transformations of pseudospherical surfaces.*

5. ***Complete solution of equations*** (28). — The system of equations (28) is completely integrable, and consequently any solution is linearly expressible in terms of three linearly independent solutions. We investigate this problem in the light of the preceding results.

Since Φ is a constant for any solution of the system, it follows that if W and W' are two solutions, the function Ψ defined by

$$\left\{\begin{aligned} \Psi = W W' - \frac{1}{\sin^2 2\omega}\Bigg[&\frac{A}{B}\frac{\partial W}{\partial u}\frac{\partial W'}{\partial u} \\ &+ \cos 2\omega\left(\frac{\partial W}{\partial u}\frac{\partial W'}{\partial v} + \frac{\partial W}{\partial v}\frac{\partial W'}{\partial u}\right) + \frac{B}{A}\frac{\partial W}{\partial v}\frac{\partial W'}{\partial v}\Bigg]\end{aligned}\right. \tag{46}$$

is a constant.

If θ_1 and θ_2 are two solutions of equations (10) in which σ is given by (29), the two functions w_1 and w_2 defined by

$$(47)\qquad \frac{\partial \log w_i}{\partial u} = \cot\frac{\sigma}{2}\cos(\theta_i + \omega), \quad \frac{\partial \log w_i}{\partial v} = -\tan\frac{\sigma}{2}\cos(\theta_i - \omega) \quad (i = 1, 2),$$

are solutions of (28). When they are substituted in (46), we get

$$(48)\qquad w_1 w_2 [1 - \cos(\theta_1 - \theta_2)] = \text{const.}$$

If we take the logarithmic derivative of this equation, it is satisfied identically. Hence when θ_2 and w_1 are known, w_2 follows directly.

Suppose now that we have a third solution θ_3 of (10) for the same value of σ, and w_3 denotes the function given by (47) with $i = 3$. Since the left hand member of (48) cannot be equal to zero, w_3 is not a linear combination of w_1 and w_2. Thus w_1, w_2, w_3 are three linearly independent solutions of (28) in terms of which any solution is linearly expressible. Since equations (10) are reducible to the Riccati form, when a solution is known, the others can be found by quadratures. Reviewing the preceding results, we have.

Theorem 4. — *The complete solution of the system* (28) *is reducible to the integration of a* Riccati *equation and quadratures.*

If we put

$$(49)\qquad W = a_1 w_1 + a_2 w_2 + a_3 w_3,$$

where the a's are constants, equations (42) become

$$(50)\qquad \begin{cases} \left[(\sum a_i w_i)^2 + \frac{h^2}{AB}\right]\cos\theta = \sum a_i w_i \cos\theta_i . \sum a_i w_i \\ \qquad + \frac{\cot\sigma/2}{B} h \sum a_i w_i \sin\theta_i , \\ \left[(\sum a_i w_i)^2 + \frac{h^2}{AB}\right]\sin\theta = \sum a_i w_i \sin\theta_i . \sum a_i w_i - \\ \qquad - \frac{\cot\sigma/2}{B} h \sum a_i w_i \cos\theta_i . \end{cases}$$

Squaring and adding, we get

$$(51)\qquad \sum a_i a_j w_i w_j [1 - \cos(\theta_i - \theta_j)] + \frac{h^2}{AB} = 0 \qquad \begin{pmatrix} i = 1, 2, 3 \\ j = 1, 2, 3 \\ i \neq j \end{pmatrix},$$

which is consistent with (41) and (48). With the aid of (51) we show that θ defined by (50) satisfies (10).

6. ***When the tractrix is a central quadric.*** — We consider in this section the case when the tractrix surface is a central quadric with the equation

$$(52)\qquad A\overline{W}_1^2 - B\overline{W}_2^2 + C\overline{W} = K,$$

where A, B, C and K are constants.

If this equation be differentiated separately with respect to u and v, the resulting equations are reducible by means of (15), (23), (24) and (26) to

$$(53)\quad \begin{cases} \overline{W}\sin(\theta+\omega)-\frac{A}{C}\frac{\partial W}{\partial u}+\frac{\cos(\theta+\omega)}{\sin\sigma}\times \\ \times\left[W_1\left(\frac{B}{C}\sin^2\frac{\sigma}{2}-\frac{A}{C}\cos^2\frac{\sigma}{2}\right)+W\left(\frac{B}{C}\sin^2\frac{\sigma}{2}+\frac{A}{C}\cos^2\frac{\sigma}{2}\right)\right]=0, \\ \overline{W}\sin(\theta-\omega)+\frac{B}{C}\frac{\partial W}{\partial v}+\frac{\cos(\theta-\omega)}{\sin\sigma}\times \\ \times\left[W_1\left(\frac{B}{C}\sin^2\frac{\sigma}{2}-\frac{A}{C}\cos^2\frac{\sigma}{2}\right)+W\left(\frac{B}{C}\sin^2\frac{\sigma}{2}+\frac{A}{C}\cos^2\frac{\sigma}{2}\right)\right]=0. \end{cases}$$

Eliminating $\overline{W}$ from these equations, we get (38), from which, as we have seen, it follows that W must be a solution of (28), provided that the constant a defined by

$$\frac{A}{C}\cos^2\frac{\sigma}{2}-\frac{B}{C}\sin^2\frac{\sigma}{2}=a$$

is not equal to zero.

Equations (53) are equivalent to (38) and

$$(54)\qquad \overline{W}=\frac{A}{C}\frac{\cos(\theta-\omega)}{\sin 2\omega}\frac{\partial W}{\partial u}+\frac{B}{C}\frac{\cos(\theta+\omega)}{\sin 2\omega}\frac{\partial W}{\partial v}.$$

From (23) and (38) we have also

$$(55)\quad \begin{cases} a\overline{W}_1=\frac{B}{C}\sin\frac{\sigma}{2}W+\frac{\cos\frac{\sigma}{2}}{\sin 2\omega}\times \\ \times\left(\frac{A}{C}\sin(\theta-\omega)\frac{\partial W}{\partial u}+\frac{B}{C}\sin(\theta+\omega)\frac{\partial W}{\partial v}\right), \\ a\overline{W}_2=-\frac{A}{C}\cos\frac{\sigma}{2}W-\frac{\sin\frac{\sigma}{2}}{\sin 2\omega}\times \\ \left(\frac{A}{C}\sin(\theta-\omega)\frac{\partial W}{\partial u}+\frac{B}{C}\sin(\theta+\omega)\frac{\partial W}{\partial v}\right). \end{cases}$$

In order that these values satisfy equations (24) and (26), we must have $a=1$, and consequently σ must satisfy

$$(56)\qquad A\cos^2\frac{\sigma}{2}-B\sin^2\frac{\sigma}{2}=C.$$

Assuming that these conditions are satisfied, we substitute the above values of $\overline{W}$, $\overline{W}_1$, $\overline{W}_2$, in (52), and obtain

$$(57)\qquad \Phi=-\frac{CK}{AB}.$$

Hence the tractrix surface is a hyperboloid or a cone, according as W is a solution of equations (28) for which Φ is different from or equal to zero. Accordingly we have:

THEOREM 5. — *If W is a solution of equations (28), σ a constant angle not satisfying (29), and θ any solution of equations (10) in which σ has this value, equations (12) and (38) determine a transform of the surface of* Voss *corresponding to W. The harmonic normal congruence is normal to a surface Σ whose tractrix surface is an hyperboloid or a cone, according as Φ is not or is equal to zero.*

When the values (54) and (55) are substituted in (22), the result is reducible to

$$\xi = \xi_0 + \frac{W \sin\sigma}{2} \frac{A+B}{C}(X_1 \sin\theta - X_2 \cos\theta), \tag{58}$$

where

$$\xi_0 = WX + \frac{X_1}{2\sin\omega}\left(\frac{A}{C}\frac{\partial W}{\partial u} + \frac{B}{C}\frac{\partial W}{\partial v}\right) + \frac{X_2}{2\cos\omega}\left(\frac{A}{C}\frac{\partial W}{\partial u} - \frac{B}{C}\frac{\partial W}{\partial v}\right). \tag{59}$$

In consequence of (15) and (56) equation (38) may be given the form

$$W = \left(\frac{A}{C}\cos^2\frac{\sigma}{2} + \frac{B}{C}\sin^2\frac{\sigma}{2}\right) W_1$$
$$+ \frac{\sin\sigma}{\sin 2\theta}\left[\frac{A}{C}\sin(\theta-\omega)\frac{\partial W_1}{\partial u} - \frac{B}{C}\sin(\theta+\omega)\frac{\partial W_1}{\partial v}\right].$$

Hence the transform V_1 also is a surface of Voss for which its coordinate W satisfies (28) when ω is replaced by θ, a result easily verified.

From (59) and (32) it follows that

$$\xi_0^2 + \eta_0^2 + \zeta_0^2 = W^2 + \frac{AB}{C^2}(W^2 - \Phi).$$

7. *When the tractrix is a quadric of revolution. Special surfaces of* GUICHARD. — Since the parametric curves on a surface V_h for which W satisfies (28) are geodesics, the tangents to the curves of either family form a normal congruence. We consider the surfaces orthogonal to such congruences. They are special surfaces of GUICHARD.

We take first the tangents to the curves $u = \text{const.}$ From (6) it follows that the direction-cosines of these tangents are of the form

$$\overline{X} = \cos\omega\, X_1 - \sin\omega\, X_2. \tag{60}$$

By differentiation we get

$$\frac{\partial \overline{X}}{\partial u} = -2\frac{\partial\omega}{\partial u}(\sin\omega\, X_1 + \cos\omega\, X_2), \quad \frac{\partial \overline{X}}{\partial v} = \sin 2\omega\, X, \tag{61}$$

so that the coefficients of the spherical representation of the congruence are given by

$$\sqrt{\mathcal{E}} = -2\frac{\partial\omega}{\partial u}, \quad \sqrt{\mathcal{G}} = \sin 2\omega. \tag{62}$$

From these it follows that the direction-cosines of the tangents to the lines of curvature on a normal surface Σ, which in fact are the parametric lines, are of the form

$$\overline{X}_1 = \sin\omega\, X_1 + \cos\omega\, X_2, \quad \overline{X}_2 = X. \tag{63}$$

The cartesian coordinates $\bar{x}$, $\bar{y}$, $\bar{z}$, of Σ are given by equations of the form

$$\bar{x} = x + t\bar{X}, \tag{64}$$

where t is determined by the conditions

$$\sum \bar{X}\frac{\partial \bar{x}}{\partial u} = 0, \quad \sum \bar{X}\frac{\partial \bar{x}}{\partial v} = 0.$$

In consequence of (6) these equations of condition are reducible to

$$\frac{\partial t}{\partial u} = D \cot 2\omega, \quad \frac{\partial t}{\partial v} = - D'' \operatorname{cosec} 2\omega.$$

When the values of D and D'' as given by (30) are substituted in these equations, in consequence of (28) their integral, to within an additive constant, is

$$t = \left(1 + \frac{B}{A}\right)\frac{1}{\sin 2\omega}\frac{\partial W}{\partial v}, \tag{65}$$

and equation (64) reduces because of (5) to

$$\left\{\begin{aligned} \bar{x} &= WX + \frac{1}{\sin 2\omega}\frac{\partial W}{\partial u}(X_1 \cos\omega + X_2 \sin\omega) \\ &\quad + \frac{B}{A}\frac{1}{\sin 2\omega}\frac{\partial W}{\partial v}(X_1\cos\omega - X_2\sin\omega). \end{aligned}\right. \tag{66}$$

From (60), (63) and (66) we get

$$\left\{\begin{aligned} \bar{W} &= \cot 2\omega \frac{\partial W}{\partial u} + \frac{B}{A}\frac{1}{\sin 2\omega}\frac{\partial W}{\partial v}, \\ \bar{W}_1 &= \frac{\partial W}{\partial u}, \quad \bar{W}_2 = W. \end{aligned}\right. \tag{67}$$

Since W satisfies (32) and (41), we must have

$$\bar{W}^2 + \bar{W}_1^2 - \frac{B}{A}\bar{W}_2^2 = \frac{h^2}{A^2}. \tag{68}$$

We have seen that each solution of equations (28) determines a function θ which satisfies (10). For this value of θ we have (40), by means of which the above expressions are reducible to

$$\left\{\begin{aligned} \bar{W} &= -\left(\cot\frac{\sigma}{2}\sin(\theta+\omega)W + \cos(\theta+\omega)\frac{h}{A}\right), \\ \bar{W}_1 &= \cot\frac{\sigma}{2}\cos(\theta+\omega)W - \sin(\theta+\omega)\frac{h}{A}, \quad \bar{W}_2 = W, \end{aligned}\right. \tag{69}$$

and equation (66) may be written

$$\bar{x} = WX - \cot\frac{\sigma}{2}W(\sin\theta X_1 - \cos\theta X_2) - \frac{h}{A}(\cos\theta X_1 + \sin\theta X_2). \tag{70}$$

In like manner we consider the tangents to the curves $v =$ const. on V_h. The analogous functions are

$$(71) \quad \begin{cases} \overline{X} = \cos\omega X_1 + \sin\omega X_2, & \overline{X}_1 = X, \quad \overline{X}_2 = \sin\omega X_1 - \cos\omega X_2, \\ \sqrt{\overline{E}} = -\sin 2\omega, & \sqrt{\overline{G}} = -2\frac{\partial\omega}{\partial v}. \end{cases}$$

We have also

$$t = -\left(1 + \frac{A}{B}\right)\frac{1}{\sin 2\omega}\frac{\partial W}{\partial u},$$

and the coordinates of Σ are of the form

$$(72) \quad \begin{cases} \overline{x} = WX - \frac{A}{B}\frac{1}{\sin 2\omega}\frac{\partial W}{\partial u}(\cos\omega X_1 + \sin\omega X_2) - \\ \qquad - \frac{1}{\sin 2\omega}\frac{\partial W}{\partial v}(\cos\omega X_1 - \sin\omega X_2), \end{cases}$$

so that

$$(73) \quad \overline{W}_1 = W, \quad \overline{W}_2 = -\frac{\partial W}{\partial v}, \quad \overline{W} = -\frac{A}{B}\frac{1}{\sin 2\omega}\frac{\partial W}{\partial u} - \cot 2\omega\frac{\partial W}{\partial v}$$

These satisfy

$$(74) \quad \overline{W}^2 + \overline{W}_2^2 - \frac{A}{B}\overline{W}_1^2 = \frac{h^2}{B^2}.$$

In terms of θ they are

$$\overline{W} = \tan\frac{\sigma}{2}\sin(\theta - \omega)W - \cos(\theta - \omega)\frac{h}{B},$$

$$\overline{W}_1 = W, \quad \overline{W}_2 = \tan\frac{\sigma}{2}\cos(\theta - \omega)W + \sin(\theta - \omega)\frac{h}{B}.$$

Hence we have

THEOREM 6. — *The tangents to the conjugate geodesics on a surface V_h constitute congruences normal to surfaces with tractrix surfaces which are central quadrics of revolution or cones of revolution, according as h differs from or is equal to zero.*

8. *General class of surfaces Σ.* — The surfaces Σ discussed in §§ 3, 4, 6 have in common the properties that their normals are parallel to the lines of a pseudospherical congruence with the developables of the normal congruence corresponding to the asymptotic lines on the focal surfaces of the pseudospherical congruence and that the planes through the normals parallel to the tangent planes to these focal surfaces envelope surfaces of Voss. We shall show that the second property is possessed by all surfaces whose normals satisfy the first condition.

Suppose we have a normal congruence whose direction-cosines are given by (11), and with the expressions (20) for the coefficients of the spherical representation of the developables of the congruences. The fundamental functions $\overline{D}$ and $\overline{D}''$ of a surface orthogonal to this congruence satisfy the CODAZZI equations

$$(75) \quad \frac{\partial}{\partial v}\left(\frac{\overline{D}}{\sqrt{\overline{E}}}\right) = \frac{\overline{D}''}{\overline{G}}\frac{\partial\sqrt{\overline{E}}}{\partial v}, \quad \frac{\partial}{\partial u}\left(\frac{\overline{D}''}{\sqrt{\overline{G}}}\right) = \frac{\overline{D}}{\overline{E}}\frac{\partial\sqrt{\overline{G}}}{\partial u}$$

When $\overline{D}$ and $\overline{D}''$ are known, the cartesian coordinates of the surface are given by the quadratures

$$(76) \qquad \frac{\partial \overline{x}}{\partial u} = -\frac{\overline{D}}{\overline{E}} \frac{\partial \overline{X}}{\partial u}, \quad \frac{\partial \overline{x}}{\partial v} = -\frac{\overline{D}}{\overline{G}} \frac{\partial \overline{X}}{\partial v}.$$

Through the normals we draw planes parallel to the tangent planes to P and P_1. The tangential coordinates W and W_1 of the envelopes of these planes are given by

$$W = \sum \overline{x} X, \quad W_1 = \sum \overline{x} X'.$$

With the aid of (75) and (76) we find that W satisfies (4) and W_1 equations (15). Hence we have

THEOREM 7. — *If $\overline{S}$ is a surface whose normals are parallel to the lines of a pseudospherical congruence with the asymptotic lines on the focal surfaces P and P_1 of the congruence corresponding to the lines of curvature on $\overline{S}$, the planes through the normals parallel to the tangent planes to P and P_1 envelope two surfaces of* VOSS *in the relation of a transformation* Ω.

BIANCHI [7]) has shown that when any surface has a tractrix surface defined by any one of equations (34), (35), (52), in which A and B have the same signs, the normals are parallel to the lines of a pseudospherical congruence with the correspondence referred to in THEOREM 7. From the preceding results it follows that all surfaces of this type may be obtained as in §§ 3, 4, 6.

BIANCHI considers also the case where A and B have different signs, and shows that the normals to such a surface are parallel to the lines of a real pseudospherical congruence of one of three types, for all of which the focal surfaces are imaginary. Since the associated surfaces of Voss also are imaginary, there is no advantage in studying this case from our point of view.

In § 7 we found a class of surfaces whose tractrix surfaces are quadrics of revolution with equations of the forms (68) and (74). Evidently the normals to these surfaces are parallel to the tangents to the asymptotic lines on the surface P with the representation (1), and these lines correspond to the lines of curvature on Σ. BIANCHI has shown that all surfaces with this type of tractrix surface possess these properties. Accordingly we have.

THEOREM 8. *Surfaces with tractrix surfaces defined by* (34), (35), (52), (68), *or* (74), *when A and B have the same signs may be obtained by the processes of the preceding sections.*

9. *Certain properties of transformations of* RIBAUCOUR. — When in the correspondence established on the two sheets of the envelope of a two-parameter family of spheres by the points of contact on the same sphere the lines of curvature on the sheets correspond, these surfaces are said to be in the relation of a *transformation of*

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[7]) l. c. [1]).

RIBAUCOUR with one another. For brevity we say that either is obtained from the other by a *transformation* R. In this section we derive certain properties of these transformations.

The coordinates ξ, η, ζ of any surface whatever Σ referred to its lines of curvature can be put in the form (22), the functions X, X_1, X_2 satisfying (19) and (25), and the functions $\overline{W}$, $\overline{W}_1$, $\overline{W}_2$, equations (24) and (26).

Any other surface Σ' with the same spherical representation of its lines of curvature as Σ is defined by equations of the form

$$\xi' = \overline{W}'\overline{X} + \overline{W}'_1\overline{X}_1 + \overline{W}'_2\overline{X}_2, \tag{77}$$

where $\overline{W}'$, $\overline{W}'_1$, $\overline{W}'_2$ satisfy (24) and (26).

As DARBOUX has shown [8]), the function $\overline{W}$ determines a transformation R of Σ'. In fact, if E' and G' denote the first fundamental coefficients of Σ', and λ is the function defined by

$$\frac{\partial\lambda}{\partial u} = \overline{W}_1\sqrt{E'}, \quad \frac{\partial\lambda}{\partial v} = \overline{W}_2\sqrt{G'}, \tag{78}$$

the sphere of radius $\lambda/\overline{W}$ and center at the point whose coordinates are

$$x_0 = \xi' - \frac{\lambda}{\overline{W}}\overline{X}, \quad y_0 = \eta' - \frac{\lambda}{\overline{W}}\overline{Y}, \quad z_0 = \zeta' - \frac{\lambda}{\overline{W}}\overline{Z}, \tag{79}$$

touches Σ', and on the other sheet of the envelope, say Σ'_1, the lines of curvature correspond to the lines of curvature on Σ'. Moreover, the cartesian coordinates of Σ'_1 are given by equations of the form

$$\xi'_1 = \xi' - \frac{1}{m\tau}\xi, \tag{80}$$

where m is a constant and the function τ is defined by

$$\overline{W}_1^2 + \overline{W}_2^2 + \overline{W}^2 = 2m\lambda\tau. \tag{81}$$

Furthermore, the direction-cosines $(\overline{X})_1, (\overline{Y})_1, (\overline{Z})_1; (\overline{X}_1)_1, (\overline{Y}_1)_1, (\overline{Z}_1)_1; (\overline{X}_2)_1, (\overline{Y}_2)_1, (\overline{Z}_2)_1$; of the normal, and the tangents to the lines of curvature on Σ'_1 are given by

$$(\overline{X})_1 = \overline{X} - \frac{\overline{W}\xi}{m\lambda\tau}, \quad (\overline{X}_1)_1 = \overline{X}_1 - \frac{\overline{W}_1\xi}{m\lambda\tau}, \quad (\overline{X}_2)_1 = \overline{X}_2 - \frac{\overline{W}_2\xi}{m\lambda\tau} \tag{82}$$

From (80) and (82) we obtain

$$\begin{cases} (\overline{W}')_1 = \sum \xi'_1(\overline{X})_1 = \overline{W}' + \frac{\overline{W}}{m\tau}\left(1 - \frac{T}{\lambda}\right), \\ (\overline{W}'_1)_1 = \sum \xi'_1(\overline{X}_1)_1 = \overline{W}'_1 + \frac{\overline{W}_1}{m\tau}\left(1 - \frac{T}{\lambda}\right), \\ (\overline{W}'_2)_1 = \sum \xi'_1(\overline{X}_2)_1 = \overline{W}'_2 + \frac{\overline{W}_2}{m\tau}\left(1 - \frac{T}{\lambda}\right), \end{cases} \tag{83}$$

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8) G. DARBOUX, *Leçons sur la théorie générale des surfaces,* II^e partie (Paris, Gauthier-Villars, 1889), p. 383. For a full treatment of these transformations see the author's paper: *Deformable Transformations of* RIBAUCOUR [Transactions of the American Mathematical Society, vol. XVII (1916), pp. 437-458]; from the first section of which we obtain the equations used in the present paper.

where

(84) $$T = \overline{W}\,\overline{W}' + \overline{W}_1 \overline{W}_1' + \overline{W}_2 \overline{W}_2' = \xi\xi' + \eta\eta' + \zeta\zeta'.$$

We consider now the case where Σ is transformed by means of the function $\overline{W}$ appearing in the equations (22) of the surface itself. Now equations (78) are reducible to the form

$$\frac{\partial \lambda}{\partial u} = \sum \xi \overline{X}_1 \sqrt{E} = \sum \xi \frac{\partial \xi}{\partial u}, \quad \frac{\partial \lambda}{\partial v} = \sum \xi \frac{\partial \xi}{\partial v},$$

and consequently to within an addittive constant we have

$$2\lambda = \xi^2 + \eta^2 + \zeta^2.$$

From this result it follows that the spheres with centers at the points (79) pass through the origin.

Comparing this equation with (81), we see that $m\tau = 1$, and consequently equations (80) show that the origin is the transform of Σ.

Although, as we have just seen, the transformation R gives nothing of value in this case, the transformation by reciprocal radii does. In fact, the point coordinates ξ_0, η_0, ζ_0, of the transform of Σ by reciprocal radii are given by

(85) $$\xi_0 = \xi/\rho, \quad \eta_0 = \eta/\rho, \quad \zeta_0 = \zeta/\rho,$$

where

(86) $$\rho = \xi^2 + \eta^2 + \zeta^2 = 2m\lambda\tau.$$

Now

$$\frac{\partial \rho}{\partial u} = 2\sum \xi \overline{X}_1 \sqrt{E} = 2\overline{W}_1 \sqrt{E}, \quad \frac{\partial \rho}{\partial v} = 2\overline{W}_2 \sqrt{G},$$

hence

$$\frac{\partial \xi_0}{\partial u} = \frac{\sqrt{E}}{\rho}\left(\overline{X}_1 - \frac{2\overline{W}_1}{\rho}\right), \quad \frac{\partial \xi_0}{\partial v} = \frac{\sqrt{G}}{\rho}\left(\overline{X}_2 - \frac{2\overline{W}_2}{\rho}\right).$$

From these expressions and (82) it follows that the transform Σ_0 has the same spherical representation of its lines of curvature as Σ_1'. Hence we have

(87) $$(\overline{W})_0 = \sum (\overline{X})_1 \xi_0 = -\frac{\overline{W}}{\rho}, \quad (\overline{W}_1)_0 = -\frac{\overline{W}_1}{\rho}, \quad (\overline{W}_2)_0 = -\frac{\overline{W}_2}{\rho}.$$

10. ***Transformations of the surfaces*** Σ. — In § 6 we saw that if we have a special surface V_0, that is a surface of Voss determined by a solution of (28) such that $\Phi = 0$, each constant σ not satisfying (29) and a solution θ of (10) determine the normals to a surface Σ for which the tractrix surface has the equation

(88) $$A\overline{W}_1^2 - B\overline{W}_2^2 + C\overline{W}^2 = 0.$$

Again the tangents to the parametric curves on V_0 are normal to surfaces Σ whose tractrix surfaces have one of the equations

(89) $$\overline{W}^2 + \overline{W}_1^2 - \frac{B}{A}\overline{W}_2^2 = 0, \quad \overline{W}^2 + \overline{W}_2^2 - \frac{A}{B}\overline{W}_1^2 = 0.$$

Hence from (87) we have

THEOREM 9. — *Surfaces* Σ *with the tractrix surfaces* (88) *or* (89) *are transformed into surfaces of the same kind by reciprocal radii.*

If V'_0 is another special surface of Voss with the same equations (28) as V_0, the quantities σ and θ referred to above determine the normals to a surface Σ' with the same spherical representation of its lines of curvature as Σ and its tractrix surface has the same equation (88). We apply to this surface Σ' the transformation R determined by the function $\overline{W}$ of the surface Σ. From (83) we have

$$(90)\quad A(\overline{W}'_1)^2_1 - B(\overline{W}'_2)^2_1 + C(\overline{W}')^2_1 = \frac{2}{m\lambda\tau}(\lambda - T)(A\overline{W}_1\overline{W}'_1 - B\overline{W}_2\overline{W}'_2 + C\overline{W}\,\overline{W}').$$

From (54) and (55) we get

$$(91)\qquad A\overline{W}_1\overline{W}'_1 - B\overline{W}_2\overline{W}'_2 + C\overline{W}\,\overline{W}' = -\frac{AB}{C}\Psi,$$

where Ψ defined by (46) is a constant. Since W and W' belong to special surfaces V_0, they are defined by equations of the form (47), and, as shown by (48), $\Psi \neq 0$.

We inquire whether the other factor in the right-hand member of (90) can be constant, say a. This necessitates

$$a\sum\xi^2 + \sum\xi\xi' - \lambda = 0.$$

Differentiating this equation, we find that this condition is satisfied only in case

$$\frac{\overline{W}'_1}{\overline{W}_1} = \frac{\overline{W}'_2}{\overline{W}_2} = \frac{\overline{W}'}{\overline{W}} = -2a,$$

that is, when Σ and Σ' are homothetic with respect to the origin.

Suppose we have a surface V_h, as defined in § 4, with the spherical respresentation (1) of its geodesic conjugate system. The quantities θ and σ referred to above determine in the tangent planes to V_h lines forming a congruence of normals to a surface Σ', whose tractrix has the equation

$$(92)\qquad A\overline{W}'^2_1 - B\overline{W}'^2_2 + C\overline{W}'^2 = \frac{h^2}{C}.$$

As thus defined Σ' and Σ (defined above) have the same spherical representation of their lines of curvature.

We apply to Σ' the transformation R determined by $\overline{W}$ of the surface Σ. From (83) it follows that we have

$$(93)\qquad \left\{\begin{aligned} &A(\overline{W}'_1)^2_1 - B(\overline{W}'_2)^2_1 + C(\overline{W}')^2_1 \\ &= \frac{h^2}{C} + \frac{2}{m\lambda\tau}(\lambda - T)(A\overline{W}_1\overline{W}'_1 - B\overline{W}_2\overline{W}'_2 + C\overline{W}\,\overline{W}'). \end{aligned}\right.$$

Comparing the right-hand member of this equation with (91), we see that the necessary and sufficient condition that the transform Σ'_1 be a surface of the same type as Σ' is that $\Psi(W, W') = 0$.

We turn to the consideration of surfaces of GUICHARD Σ and Σ' with tractrix surfaces given by the first of (89) and (68) respectively. We apply to Σ' the transformation R determined by $\overline{W}$ of Σ. In consequence of (83) we obtain

$$(\overline{W}')_1' + (\overline{W}_1')_1^2 - \frac{B}{A}(\overline{W}_2')_1^2 = \frac{h^2}{A^2} + \frac{2}{m\lambda\tau}(\lambda - T)(\overline{W}\,\overline{W}' + \overline{W}_1\overline{W}_1' - \frac{B}{A}\overline{W}_2\overline{W}_2').$$

Substituting the values (67) and similar ones for Σ' in the left-hand member of the following equation, we get the other member:

$$\overline{W}\,\overline{W}' + \overline{W}_1\overline{W}_1' - \frac{B}{A}\overline{W}_2\overline{W}_2' = -\frac{B}{A}\Psi.$$

Hence as in the above case the determination of this type of transformations R of special surfaces of GUICHARD into surfaces of the same kind reduces to finding functions W' such that $\Psi(W, W') = 0$. We proceed to the investigation of this question.

In § 5 it was shown that W' is expressible in the form

$$W' = a_1 w_1 + a_2 w_2 + a_3 w_3,$$

where the functions w_i are defined by equations of the form (47). If a_{ij} denotes the value of Ψ for the functions w_i and w_j, then for W' we have

$$\Phi(\overline{W}') = 2(a_{12}a_1a_2 + a_{23}a_2a_3 + a_{13}a_1a_3) \neq 0.$$

In like manner

$$W = a_{01}w_1 + a_{02}w_2 + a_{03}w_3,$$

where the constants a_{01}, a_{02}, a_{03}, are such that

$$\Phi(W) = 2(a_{01}a_{02}a_{12} + a_{02}a_{03}a_{23} + a_{01}a_{03}a_{13}) = 0.$$

If the constants a_1, a_2, a_3, are chosen so that

$$a_{12}(a_1a_{02} + a_2a_{01}) + a_{23}(a_2a_{03} + a_3a_{02}) + a_{13}(a_1a_{03} + a_0a_{01}) = 0,$$

then for W and W' we have $\Psi = 0$.

Applying these results to the previous considerations, we have

THEOREM 10. — *If Σ is a surface with a tractrix surface* (88) *or* (89), *there exist ∞^2 surfaces Σ' with tractrix surfaces of the type* (92) *or* (68) *which in the transformations determined by the function W of Σ go into surfaces of the same kind as Σ'.*

Conversely it follows from the above that if W' is a function for which $\Phi \neq 0$, we can find ∞^1 functions W for which $\Psi(W, W') = 0$. Hence we have.

THEOREM 11. — *If Σ' is a surface with a tractrix of the type* (92) *or* (68), *there exist ∞^1 of the above transformations R of Σ' into surfaces of the same kind, and these can be found directly.*

11. ***Induced transformations of the surfaces V_h.*** — Now we consider the transformation of surfaces of Voss induced by the transformations R discussed in the preceding section.

In consequence of (12) the expressions (18) can be given the equivalent forms

$$\overline{X}_1 = \cos\frac{\sigma}{2}(\sin\omega X_1 - \cos\omega X_2') + \sin\frac{\sigma}{2} X',$$

$$\overline{X}_2 = \sin\frac{\sigma}{2}(\sin\omega X_1' - \cos\omega X_2') - \cos\frac{\sigma}{2} X'.$$

From these equations and (18) we find

$$X = -\left(\sin\frac{\sigma}{2}\overline{X}_1 + \cos\frac{\sigma}{2}\overline{X}_2\right),$$

$$X' = \sin\frac{\sigma}{2}\overline{X}_1 - \cos\frac{\sigma}{2}\overline{X}_2.$$

Hence for a surface Σ' with a tractrix of the type (92) the planes through the normals to Σ' which make the angles $\pi - \sigma/2$ and $\pi + \sigma/2$ with the planes whose direction-cosines are $\overline{X}_2$, $\overline{Y}_2$, $\overline{Z}_2$, envelope two surfaces of Voss in the relation of a transformation Ω.

For the surfaces Σ_1' resulting from Σ' by the transformations R discussed in the preceding section the constant σ is the same as for Σ'. Accordingly the planes trhough the normals to Σ_1' which make the angle $\pi - \sigma/2$ with the planes whose direction-cosines are $(\overline{X}_2)_1$, $(\overline{Y}_2)_1$, $(\overline{Z}_2)_1$ envelope a surface of Voss V_h. The tangential coordinates of this surface are expressible in the form

$$(X)_1 = -[\sin\sigma/2(\overline{X}_1)_1 + \cos\sigma/2(\overline{X}_2)_1],$$

$$(W')_1 = -[\sin\sigma/2(\overline{W}_1')_1 + \cos\sigma/2(\overline{W}_2')_1],$$

which in consequence of (82) and (83) are reducible to

$$(94) \qquad \begin{cases} (X)_1 = X - \dfrac{W}{m\lambda\tau}\xi, \\ (W')_1 = W' - \dfrac{W}{m\lambda\tau}(\lambda - T). \end{cases}$$

Corresponding to the case of the transformation of Σ by reciprocal radii the expressions for $(X)_1$, $(Y)_1$, $(Z)_1$ are the same as (94) and

$$(95) \qquad (W)_1 = -\frac{W}{\rho}.$$

This surface is a special surface of Voss V_0 and consequently we have thus established a new transformation of these surfaces.

We consider also the transformations of surfaces of Voss induced by the above transformations R of special surfaces of GUICHARD. If Σ_1' is the transform of Σ', it follows from the investigation that the principal planes of Σ_1' corresponding to those of Σ' envelope surfaces of Voss V_h.

Applying formulas (82) and (83) to the respective results (63), (66) and (71), (72), we get the following expressions for the tangential coordinates of the transforms of V_h:

$$(96)\qquad \begin{cases} (X)_2 = X - \dfrac{W\bar{x}_2}{m\lambda_2\tau_2}, & (W')_2 = W' + \dfrac{W}{m\lambda_2\tau_2}(\lambda_2 - T), \\ (X)_1 = X - \dfrac{W\bar{x}_1}{m\lambda_1\tau_1}, & (W')_1 = W' + \dfrac{W}{m\lambda_1\tau_1}(\lambda_1 - T), \end{cases}$$

where $\bar{x}_2$ and $\bar{x}_1$ are given by (66) and (72) respectively, and the functions λ_i are determined by quadratures of the form (78). Moreover, from (81) and (68) we get

$$2m\lambda_i\tau_i = W^2\left(1 + \frac{B}{A}\right) + \frac{h^2}{A^2},$$

When a surface Σ with either of the tractrix surfaces (89) is subjected to a transformation by reciprocal radii, the expressions for the direction-cosines are the same as above, but the W's have the values

$$(W)_2 = -\frac{W}{\rho}, \quad (W)_1 = -\frac{W}{\rho},$$

where now

$$\rho = 2m\lambda\tau = W^2\left(1 + \frac{B}{A}\right)$$

Hence in this case corresponding tangent planes to the two transforms are equidistant from the origin.

12. ***Special surfaces of*** GUICHARD, ***and congruences of*** GUICHARD. — From (66) we obtain by differentiation

$$(97)\qquad \frac{\partial \bar{x}}{\partial u} = \left(1 + \frac{B}{A}\right)W\bar{X}_1, \quad \frac{\partial \bar{x}}{\partial v} = \left(1 + \frac{B}{A}\right)\frac{\partial W}{\partial v}\bar{X}_2.$$

Hence the first fundamental coefficients $\bar{E}$ and $\bar{G}$ of Σ are of the form

$$(98)\qquad \sqrt{\bar{E}} = \left(1 + \frac{B}{A}\right)W, \quad \sqrt{\bar{G}} = \left(1 + \frac{B}{A}\right)\frac{\partial W}{\partial v}$$

The tangents to the curves $u =$ const. on Σ form a congruence of GUICHARD by definition provided that the parametric curves on the second focal surface Σ_1 of the congruence are lines of curvature. If $\bar{x}_1, \bar{y}_1, \bar{z}_1$ denote the cartesian coordinates of this focal surface, their expressions are of the form

$$(99)\qquad \bar{x}_1 = \bar{x} - \sqrt{\bar{E}}X,$$

since by differentiation we find

$$(100)\qquad \frac{\partial \bar{x}_1}{\partial u} = -\frac{\partial\sqrt{\bar{E}}}{\partial u}X, \quad \frac{\partial \bar{x}_1}{\partial v} = \sqrt{\bar{E}}(-\sin\omega X_1 + \cos\omega X_2).$$

From these expressions it follows at once that the parametric curves on this surface are orthogonal and consequently the lines form a congruence of GUICHARD.

If $(\bar{X}_1)_1, \ldots; (\bar{X}_2)_1, \ldots; (\bar{X})_1, \ldots,$ denote respectively the direction-cosines of the tangents to the curves $v =$ const., $v =$ const., and of the normals to Σ_1 we have from (100)

$$(\bar{X}_1)_1 = X, \quad (\bar{X}_2)_1 = \sin\omega X_1 - \cos\omega X_2,$$
$$(\bar{X}) = \cos\omega X_1 + \sin\omega X_2.$$

Moreover, from these expressions and (99) we obtain

$$(\bar{W})_1 = \frac{1}{\sin 2\omega}\frac{\partial W}{\partial u} + \frac{B}{A}\cot 2\omega \frac{\partial W}{\partial v},$$
$$(\bar{W}_1)_1 = -\frac{B}{A}W, \quad (\bar{W}_2)_1 = \frac{B}{A}\frac{\partial W}{\partial v}$$

In consequence of (32) and (41) we see that these functions are in the relation

$$(\bar{W})_1^2 + (\bar{W}_2)_1^2 - \frac{A}{B}(\bar{W}_1)_1^2 = \frac{h_2}{B^2}.$$

Since similar results follow when Σ is given by (72), we have

THEOREM 12. — *When for a congruence of* GUICHARD *one of the focal surfaces has a tractrix surface* (68), *the other has a tractrix surface* (74); *and vice-versa.*

From the preceding results it follows that for the case just considered the special surface of Voss associated with the new surface of GUICHARD has the tangential coordinates

$$X, \quad Y, \quad Z, \quad -\frac{B}{A}W.$$

13. *Other transformations R of surfaces Σ.*

If S_0 denotes the surfaces whose rectangular coordinates ξ_0, η_0, ζ_0, are given by (59), it follows that the distance between corresponding points on S_0 and Σ, whose coordinates are of the form (58), is $W \sin\sigma.(A+B)/2\,C$. As this value is independent of θ, if we take two solutions of (10) for the same σ, we get two surfaces Σ and Σ_1, with similar tractrix surfaces (52), in the relation of a transformation R.

Suppose now that we have a surface Σ of this type. As shown in § 11, we know how to draw planes through the normals to Σ to get a surface V_h. Making use of this result and of equations (11) and (18) we find the corresponding functions ω and θ. The constant σ is determined by (56). Since equations (10) are reducible to the RICCATI form, it follows that the further solution of these equations reduces to quadratures. Hence we have.

THEOREM 13. — *When a surface Σ with a tractrix of the form* (52) *is known, ∞^1 surfaces of the same type can be found by quadratures alone; each of these surfaces is in the relation of an transformation R with Σ.*

These transformations differ from those treated in § 10. For in the present case the planes determined by the corresponding radii of the spheres of the transformation envelope the surface of Voss referred to above. That this is not true of the transformations of § 10 is evident from (82).

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