THE CONDITION THAT A QUINTIC EQUATION SHOULD BE SOLUBLE BY RADICALS

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1. It is well known that the roots $\alpha, \beta, \ldots, \eta$, of an irreducible equation

$$ax^n+bx^{n-1}+\ldots+h=0,$$

can be expressed in terms of the coefficients in a finite number of steps by the four elementary operations of arithmetic, addition, subtraction, multiplication and division, combined with a finite number of operations of root extraction, when, and only when, the group of the equation is soluble. The group of an irreducible equation being always transitive, that of an irreducible quintic is a transitive substitution group of degree 5. Such groups are five in number G_{120}, G_{60}, G_{20}, G_{10}, G_5 (the order being equal to the suffix in each case), and of these the last three are soluble. The group G_{20} contains the twenty substitutions

$$U^m V^n, \quad m=1, 2, 3, 4, 5; \quad n=1, 2, 3, 4, \tag{1}$$

where $$U=(\alpha\beta\gamma\delta\epsilon), \quad V=(\alpha\beta\delta\gamma)(\epsilon),$$

so that $$U^5=V^4=1, \quad UV=VU^2.$$

G_{10}, a self-conjugate sub-group of G_{20}, containing

$$U^m W^n, \quad m=1, 2, 3, 4, 5; \quad n=1, 2, \tag{2}$$

where $$W=V^2=(\alpha\delta)(\beta\gamma)(\epsilon)$$

is defined by $$U^5=W^2=1, \quad UW=WU^4;$$

whereas G_5 consists of the single cycle

$$U, U^2, U^3, U^4, U^5\equiv 1. \tag{3}$$

In order that the quintic may be soluble by radicals in a field of rationality

R, which contains a, b, c, d, e, f, it is necessary and sufficient that some rational function of the roots $\alpha, \beta, \gamma, \delta, \epsilon$ which is unaltered in form by the substitutions of G_{20}, but by no others, should have a value rational in R.

The quintic can be reduced to the standard (Bring-Jerrard) form

$$y^5+uy+v=0,$$

with the help of soluble equations, but, in general, the coefficients u, v belong to a field R' of relative degree 6 with regard to R. It is only for a certain limited class of quintics that u, v are rational in the original field R. Adopting this standard form, Runge ("Ueber die auflösbaren Gleichungen von der Form $x^5+ux+v=0$," *Acta Mathematica*, t. VII, S. 173–186) takes the function

$$\psi=\tfrac{1}{4}(\alpha\beta+\beta\gamma+\gamma\delta+\delta\epsilon+\epsilon\alpha-\alpha\gamma-\gamma\epsilon-\epsilon\beta-\beta\delta-\delta\alpha)^2,$$

and shows that ψ is a root of the sextic

$$\Psi(\psi)\equiv(\psi-u)^4(\psi^2-6\psi u+25u^2)-5^5v^4\psi=0.$$

When this equation has a rational root in R', the quintic can be solved by radicals in R' and consequently in R. The object of this paper is to obtain a sextic equation $\Phi(\phi)=0$, satisfied by a function belonging to G_{20}, but with rational coefficients in R. Since two rational functions of the roots which belong to the same group are rational functions of each other, the equation obtained will be connected with Runge's, by a transformation of the type

$$\phi=y_0+y_1\psi+y_2\psi^2+y_3\psi^3+y_4\psi^4+y_5\psi^5,$$

or

$$\psi=z_0+z_1\phi+z_2\phi^2+z_3\phi^3+z_4\phi^4+z_5\phi^5,$$

where y_i, z_i are rational in R'. The discovery of roots of $\Phi(\phi)=0$ rational in R is, however, a simpler process than that of roots of $\Psi(\psi)=0$ rational in R'. In fact, when a, b, c, d, e, f are ordinary rational numbers and $R=1$, R' is in general a soluble sextic field, and the testing of a sextic equation whose coefficients belong to a sextic field for roots rational in that field, though possible in a finite number of steps, is a somewhat lengthy operation.

2. It is readily seen that the function

$$\phi=a^4\left[(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\delta)^2(\delta-\epsilon)^2(\epsilon-\alpha)^2\right.$$
$$\left.+(\alpha-\gamma)^2(\gamma-\epsilon)^2(\epsilon-\beta)^2(\beta-\delta)^2(\delta-\alpha)^2\right]$$

belongs to G_{20}. This number ϕ is one of a set of six conjugates which may be obtained by applying to it the six substitutions

$$1,\ (\alpha\beta\gamma),\ (\alpha\gamma\beta),\ (\beta\gamma)(\alpha),\ (\gamma\alpha)(\beta),\ (\alpha\beta)(\gamma),$$

δ, ϵ being unaltered, and, of course, the group of each of the conjugate functions is a sub-group of the symmetric group G_{120}, conjugate to G_{20}.

Now a symmetric function of the six ϕ's is an invariant of the quintic form

$$F(x, y) \equiv ax^5+bx^4y+cx^3y^2+dx^2y^3+exy^4+fy^5,$$

and the six elementary symmetric functions are rational integral invariants. We may therefore assume that ϕ is a root of a sextic equation

$$\Phi(\phi) \equiv \phi^6+A_1\phi^5+A_2\phi^4+A_3\phi^3+A_4\phi^2+A_5\phi+A_6 = 0,$$

where A_1, A_2, ..., A_6 are rational integral invariants of respective degrees 4, 8, ..., 24.

3. Using the notation of Salmon's *Higher Algebra*, the quintic form $F(x, y)$ has four irreducible invariants J, K, L, I of respective degrees 4, 8, 12, 18. When the coefficients of $F(x, y)$ are numerical, the values of the first three of these are most easily calculated in the following way.

First, taking the quadratic covariant

$$\begin{aligned} S &\equiv \tfrac{1}{100}\big[(20ae-8bd+3c^2)\,x^2+(100af-12be+2cd)\,xy \\ &\qquad\qquad +(20bf-8ce+3d^2)\,y^2\big] \\ &\equiv Ax^2+Bxy+Cy^2, \text{ say,} \end{aligned}$$

we have

$$J = B^2-4AC.$$

Next, taking the canonizant

$$\begin{aligned} T &\equiv \begin{vmatrix} 10ax+2by, & 2bx+cy, & cx+dy \\ 2bx+cy, & cx+dy, & dx+2ey \\ cx+dy, & dx+2ey, & 2ex+10fy \end{vmatrix} \div 10^3 \\ &\equiv Dx^3+Ex^2y+Fxy^2+Gy^3, \end{aligned}$$

K, the lineo-linear invariant of S and the Hessian of T, and L, the discriminant of T, are given by

$$K = 2A(3EG-F^2)-B(9DG-EF)+2C(3FD-E^2),$$

$$L = \tfrac{1}{3}\big[4(3EG-F^2)(3FD-E^2)-(9DG-EF)^2\big].$$

4. We may now assume the rational equation satisfied by ϕ to be

$$\begin{aligned}\Phi(\phi) \equiv \phi^6&+x_1 J\phi^5+(x_2 J^2+x_3 K)\,\phi^4\\ &+(x_4 J^3+x_5 JK+x_6 L)\,\phi^3+(x_7 J^4+x_8 J^2 K+x_9 K^2+x_{10} JL)\,\phi^2\\ &+(x_{11} J^5+x_{12} J^3 K+x_{13} JK^2+x_{14} J^2 L+x_{15} KL)\,\phi\\ &+x_{16} J^6+x_{17} J^4 K+x_{18} J^2 K^2+x_{19} K^3+x_{20} J^3 L+x_{21} JKL+x_{22} L^2=0,\end{aligned}$$

where $x_1, x_2, \ldots, x_{22}$ are numerical. These numerical coefficients can be found by taking a sufficient number of special quintics, whether irreducible or not, and comparing the actual coefficients of the ϕ-equation with those involving the x's.

First, taking $$x^5-n=0,$$

we have $(A, B, C)=(0, -n, 0)$, $(D, E, F, G)=(0, 0, 0, 0)$,

$$J=n^2, \quad K=L=0,$$

$$\phi_1=-625n^2, \quad \phi_2=\phi_3=\phi_4=\phi_5=\phi_6=-125n^2, \tag{4}$$

so that $(\phi+625n^2)(\phi+125n^2)^5$

$$\equiv \phi^6+x_1 n^2\phi^5+x_2 n^4\phi^4+x_4 n^6\phi^3+x_7 n^8\phi^2+x_{11} n^{10}\phi+n_{16} n^{12},$$

giving $$x_1=2.5^4, \quad x_2=7.5^7, \quad x_4=2^2.3.5^{10}, \quad x_7=5^{13}.11,$$

$$x_{11}=2.5^{15}.13, \quad x_{16}=5^{19}.$$

Again, taking $$x^5-x=0,$$

we have $$J=L=0, \quad K=2.5^{-5},$$

$$\phi_1=\phi_4=40i, \quad \phi_2=\phi_6=-40i, \quad \phi_3=\phi_5=0,$$

so that

$$\phi^6+2^7.5^2.\phi^4+2^{12}.5^4\phi^2 \equiv \phi^6+2.5^{-5}x_3\phi^4+2^2.5^{-10}x_9\phi^2+2^3.5^{-15}x_{19},$$

whence $$x_3=2^6.5^7, \quad x_9=2^{10}.5^{14}, \quad x_{19}=0.$$

Taking further

$$(\alpha, \beta, \gamma, \delta, \epsilon)=(1, -1, 3, -3, 0),\ (1, 1, -1, -1, 0),\ (1, 2, -2, 0, 0),$$
$$(1, 3, -3, 0, 0),\ (1, -1, \rho, \rho^2, 0),\ [\rho^2+\rho+1=0],$$

the remaining coefficients are found without difficulty—beyond that of the calculation. A complete verification is given by

$$(\alpha, \beta, \gamma, \delta, \epsilon) = (1, -1, 2, -2, 0).$$

The final result is

$$\begin{aligned}\Phi(\phi) \equiv \phi^6 &+ 2.5^4J\phi^5 + (5^7.7J^2 + 2^6.5^7K)\phi^4 \\ &+ (2^2.3.5^{10}J^3 + 2^6.3.5^{10}JK - 2^{11}.5^{10}L)\phi^3 \\ &+ (5^{13}.11J^4 + 2^6.3.5^{13}J^2K + 2^{10}.5^{14}K^2 - 2^{11}.5^{14}JL)\phi^2 \\ &+ (2.5^{15}.13J^5 + 2^6.5^{16}J^3K + 2^{12}.5^{15}.11JK^2 \\ &\qquad - 2^{10}.5^{15}.59J^2L - 2^{16}.5^{15}.7KL)\phi \\ &+ 5^{19}J^6 + 2^{12}.5^{19}J^2K^2 - 2^{12}.5^{19}J^3L \\ &- 2^{17}.5^{19}JKL + 2^{20}.5^{20}L^2 = 0.\end{aligned}$$

A simplification can be made by writing

$$5^3J = j, \quad 2^5.5^6K = k, \quad -2^{10}.5^9L = l;$$

the equation then becomes

$$\begin{aligned}\Phi(\phi) \equiv \phi^6 &+ 10j\phi^5 + (35j^2 + 10k)\phi^4 + (60j^3 + 30jk + 10l)\phi^3 \\ &+ (55j^4 + 30j^2k + 25k^2 + 50jl)\phi^2 \\ &+ (26j^5 + 10j^3k + 44jk^2 + 59j^2l + 14kl)\phi \\ &+ 5j^6 + 20j^2k^2 + 20j^3l + 20jkl + 25l^2 = 0.\end{aligned}$$

5. We now assume that the quintic equation

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0,$$

is irreducible in R.

When the sextic has no rational root, the group is G_{120} or G_{60} according as the discriminant

$$\Delta \equiv a^8\Pi(\alpha - \beta)^2 \equiv 5^5(J^2 - 2^7K),$$

is not or is a rational square.

The quintic is soluble by radicals in R when, and only when, $\Phi(\phi) = 0$ has a rational root, and the group is then G_{20}, G_{10}, or G_5.

Should the sextic have two different rational roots it must reduce to the form
$$\Phi(\phi) \equiv (\phi-\phi_1)(\phi-\phi_2)^5 = 0,$$
as in (4) above. For taking the group G to consist of the substitutions enumerated in (1), (2), or (3) above, according as it is of order 20, 10, 5, the function
$$\phi_1 \equiv a^4\left[(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\delta)^2(\delta-\epsilon)^2(\epsilon-\alpha)^2 + (\alpha-\gamma)^2(\gamma-\epsilon)^2(\epsilon-\beta)^2(\beta-\delta)^2(\delta-\alpha)^2\right]$$
is unaltered in form by the substitutions of G and therefore has a rational value. If also the value of $\phi_2 \equiv (\alpha\beta\gamma)(\delta)(\epsilon)\phi_1$ is rational, this rational value must be unaltered by the substitutions of G. Among these, however, there is at least one which changes ϕ_2 into each of the other ϕ's, ϕ_3, ϕ_4, ϕ_5, and ϕ_6. Hence $\phi_2 = \phi_3 = \phi_4 = \phi_5 = \phi_6$ as stated.

The sextic cannot have an irreducible quadratic or cubic or quartic factor, for then a symmetric function of two or three of the ϕ's would be rational and the group intransitive.

6. The group of the function
$$\chi \equiv a^2(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\delta)^2(\delta-\epsilon)^2(\epsilon-\alpha)^2$$
is G_{10}, whereas that of
$$\sqrt{\chi} \equiv a(\alpha-\beta)(\beta-\gamma)(\gamma-\delta)(\delta-\epsilon)(\epsilon-\alpha)$$
is G_5, the only transitive Abelian group of the fifth degree.

Now, if ϕ be a rational root of the sextic, χ is a root of the rational quadratic
$$\chi^2-\phi\chi+5^7(J^2-2^7K) = 0.$$
When this quadratic (or each of them when there are two) has no rational root, the group of the quintic is G_{20}. But when a value of χ is rational the group is G_{10}, unless one of the roots happens to be a rational square when the group reduces to G_5 and the quintic is normal and cyclical. In any case the group reduces to G_{10} when R is extended by adjunction of χ, and a further adjunction of $\sqrt{\chi}$ reduces it to G_5. When $R = 1$ and the group is G_5, the quintic is cyclotomic by a theorem due to Kronecker (Hilbert, *Zahlbericht*, § 104).

There is an extensive range of literature on soluble quintics. Many references (prior to 1886) are given in the series of papers by Young and

McClintock, *American Journal*, 6, 7, and 8. A method of obtaining radical expressions for the roots, due substantially to Lagrange, but fully developed by Kronecker, is given in Weber's *Algebra* (Bd. I, § 191 *seq.*). Complete material for writing down the actual radicals in the case of the standard quintic

$$y^5+uy+v=0$$

(when soluble) appears in Prof. Mathews' Cambridge Tract, *Algebraic Equations*. To the latter named gentleman I am indebted for some suggestions as to the method of presenting the results contained in this paper. One of the Society's referees has also kindly pointed out corrections in two of the references given in the paper, which were inaccessible to me at the time of writing.