



Cl. Circular plates of variable thickness

George D. Birkhoff Ph.D.

To cite this article: George D. Birkhoff Ph.D. (1922) Cl. Circular plates of variable thickness , Philosophical Magazine Series 6, 43:257, 953-962, DOI: [10.1080/14786442208633949](https://doi.org/10.1080/14786442208633949)

To link to this article: <http://dx.doi.org/10.1080/14786442208633949>



Published online: 08 Apr 2009.



Submit your article to this journal [↗](#)



Article views: 1



View related articles [↗](#)



Citing articles: 1 View citing articles [↗](#)

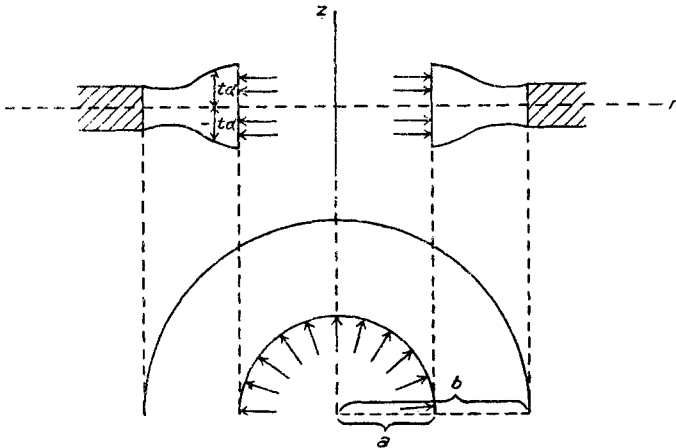
CI. *Circular Plates of Variable Thickness.* By GEORGE D. BIRKHOFF, Ph.D., Professor of Mathematics, Harvard University*.

AS far as I am aware, only plates and shells of constant thickness have been considered in the theory of elasticity. The aim of the present note is to develop a method applicable when the thickness is variable†. The method is here applied to thin circular plates, nearly plane, and clamped at the outer edge.

Case I.—The Incomplete Symmetrical Plate under Radial Pressure.

Suppose that ordinary cylindrical coordinates (fig. 1) are adopted, with the axis of z along the axis of the plate.

Fig. 1.



The plate is of inner radius a and outer radius b , and is symmetrical about the plane $z=0$. Let α be proportional to the thickness of the plate, so that $z=t\alpha$ and $z=-t\alpha$ may be taken as the equations of the upper and lower bases respectively. Here t is a small parameter, since the plate is thin.

Radial forces are applied to the inner edge, so that points

* Communicated by the Author.

† I am greatly indebted to Mr. Carl A. Garabedian, of Harvard University, for able assistance in carrying through some of the calculations and for verifying those I have made. Mr. Garabedian is undertaking the consideration of more general problems by this method.

for which $r=a$ undergo a displacement of amount ϵ . The plate is clamped at the outer edge. Thus, if U and w denote the radial and axial displacements, the boundary conditions are

$$\begin{cases} U(a, 0) = \epsilon, & U(b, 0) = 0, \\ w(b, 0) = \frac{\partial w(b, 0)}{\partial r} = 0. \end{cases}$$

Obviously such a plate is unstable, with a tendency to buckle.

It is a cardinal fact of the theory of elasticity that the actual displacement of the plate will be such as to yield the minimum potential energy consistent with the constraints imposed (Love, 'Theory of Elasticity,' third edition, p. 169).

By Love (pp. 99-100, 141) this potential energy W is given as follows:—

$$W = \pi \int_a^b \int_{-ta}^{ta} \left\{ (\lambda + 2\mu) \left[\frac{\partial U}{\partial r} + \frac{U}{r} + \frac{\partial w}{\partial z} \right]^2 + \mu \left[\left(\frac{\partial U}{\partial z} + \frac{\partial w}{\partial r} \right)^2 - 4 \frac{U}{r} \frac{\partial U}{\partial r} - 4 \frac{U}{r} \frac{\partial w}{\partial z} - 4 \frac{\partial U}{\partial r} \frac{\partial w}{\partial z} \right] \right\} r dz dr.$$

It is natural to introduce a new variable z' , such that $z = tz'$. When we replace z by z' and afterward suppress the accents there results:—

$$(A) \quad W = \pi t \int_a^b \int_{-a}^a \left\{ (\lambda + 2\mu) \left[\frac{\partial U}{\partial r} + \frac{U}{r} + \frac{1}{t} \frac{\partial w}{\partial z} \right]^2 + \mu \left[\left(\frac{1}{t} \frac{\partial U}{\partial z} + \frac{\partial w}{\partial r} \right)^2 - 4 \frac{U}{r} \frac{\partial U}{\partial r} - \frac{4}{t} \frac{U}{r} \frac{\partial w}{\partial z} - \frac{4}{t} \frac{\partial U}{\partial r} \frac{\partial w}{\partial z} \right] \right\} r dz dr.$$

Our *assumption* will be that all of the quantities involved can be expanded in ascending powers of t —in particular that

$$U = U_0 + U_1 t + U_2 t^2 + \dots, \\ w = w_0 + w_1 t + w_2 t^2 + \dots$$

It is to be noted that if in (A) we replace z by $-z$ and w by $-w$, or t by $-t$ and w by $-w$, the double integral is altered at most in sign. Since these transformations do not disturb our boundary conditions, and since W has a unique minimum, the special relations

$$(B) \quad U_m(r, z) - U_m(r, -z) = w_m(r, z) + w_m(r, -z) = 0, \\ m = 0, 1, 2, \dots,$$

$$(B') \quad U_{2m+1}(r, z) = w_{2m}(r, z) = 0, \quad m = 0, 1, 2, \dots,$$

must obtain.

The case of a plane plate shows that the energy is of

order t . Hence if W is to be a minimum, it is clear that we must make the leading term of W , namely

$$\frac{\pi\mu}{t} \int_a^b \int_{-a}^a \left[\frac{\partial U_0}{\partial z} \right]^2 r \, dz \, dr,$$

vanish if possible. But this quantity vanishes if, and only if, $U_0 = U_0(r)$, and by thus restricting U_0 the stated boundary conditions are not violated.

When U_0 is thus restricted, the double integral W will involve no negative powers of t , and will have as leading term :—

$$\pi t \int_a^b \int_{-a}^a \left\{ (\lambda + 2\mu) \left[U_0' + \frac{U_0}{r} + \frac{\partial w_1}{\partial z} \right]^2 - 4\mu \left[\frac{U_0}{r} U_0' + \frac{U_0}{r} \frac{\partial w_1}{\partial z} + U_0' \frac{\partial w_1}{\partial z} \right] \right\} r \, dz \, dr,$$

where accents denote differentiation with respect to r . The part of the integrand in braces may be written as the sum of squares :—

$$\lambda \left[U_0' + \frac{U_0}{r} + \frac{\partial w_1}{\partial z} \right]^2 + 2\mu \left[U_0'^2 + \frac{U_0^2}{r^2} + \left(\frac{\partial w_1}{\partial z} \right)^2 \right].$$

We turn next to the choice of $\frac{\partial w_1}{\partial z}$, which is an arbitrary function of r and z still at our disposal. If we call this variable x , the above expression involves x in two terms of the form

$$\lambda(m+x)^2 + 2\mu x^2.$$

Elementary calculation shows that for $x = \frac{-\lambda m}{\lambda + 2\mu}$, the foregoing expression has the minimum value $\frac{2\lambda\mu}{\lambda + 2\mu} m^2$. Hence we must take

$$\frac{\partial w_1}{\partial z} = \frac{-\lambda}{\lambda + 2\mu} \left(U_0' + \frac{U_0}{r} \right),$$

or

$$w_1 = \frac{-\lambda z}{\lambda + 2\mu} \left(U_0' + \frac{U_0}{r} \right) + s(r),$$

$s(r)$ arbitrary. But $w(r, 0) = 0$ by (B); hence $s(r) = 0$. This choice of w_1 does not interfere with the boundary conditions.

The terms written above now reduce to give

$$\pi t \int_a^b \int_{-a}^a \frac{4\mu}{\lambda + 2\mu} \left\{ (\lambda + \mu) \left(U_0'^2 + \frac{U_0^2}{r^2} \right) + \lambda U_0' \frac{U_0}{r} \right\} r \, dz \, dr,$$

and an integration with regard to z can be explicitly performed. The principal part of W takes the form $W_1 t$, where

$$W_1 = \frac{8\pi\mu}{\lambda + 2\mu} \int_a^b \left\{ (\lambda + \mu) \left(U_0'^2 + \frac{U_0^2}{r^2} \right) + \lambda U_0' \frac{U_0}{r} \right\} ar \, dr. \quad (1)$$

This integral is to be made a minimum subject to the boundary conditions

$$U_0(a) = \epsilon, \quad U_0(b) = 0.$$

Accordingly our problem is reduced to a simple problem in the Calculus of Variations. The condition $\delta W_1 = 0$ gives at once

$$\frac{d}{dr} \frac{\partial \Phi}{\partial U_0'} - \frac{\partial \Phi}{\partial U_0} = 0, \quad \dots \quad (2)$$

where Φ is the integrand in (1). Writing out this equation in full, we obtain

$$\frac{d}{dr} \left[2(\lambda + \mu) U_0' + \lambda \frac{U_0}{r} \right] ar - \left[2(\lambda + \mu) \frac{U_0}{r} + \lambda U_0' \right] a = 0.$$

Consequently U_0 must satisfy the following differential equation:—

$$U_0'' + \frac{U_0'}{r} - \frac{U_0}{r^2} = -\frac{h'}{h} \left(U_0' + \frac{\lambda}{2(\lambda + \mu)} \frac{U_0}{r} \right), \quad (3)$$

where $2h = 2t\alpha$ stands for the thickness of the plate.

In the case of a plate of constant thickness, h is constant and $h' = 0$, and (3) reduces to a well-known form (Love, p. 141).

Thus far we have determined the displacements to be

$$U = U_0(r) + U_2 t^2 + \dots, \quad (4)$$

$$w = -\frac{\lambda z}{\lambda + 2\mu} \left(U_0' + \frac{U_0}{r} \right) t + w_3 t^3 + \dots, \quad (5)$$

where it is to be remembered that the accent on z has been suppressed.

We proceed to determine U_2 from the fact that the body forces F_r , F_z must vanish. We have (Love, p. 141):

$$-\rho F_r = (\lambda + 2\mu) \left(U'' + \frac{U'}{r} - \frac{U}{r^2} \right) + \frac{\lambda + \mu}{t} \frac{\partial w}{\partial z \partial r} + \frac{\mu}{t^2} \frac{\partial^2 U}{\partial z^2} = 0, \quad (6)$$

$$\begin{aligned} -\rho F_z = & \frac{\lambda + 2\mu}{t^2} \frac{\partial^2 w}{\partial z^2} + \frac{\lambda + \mu}{t} \frac{\partial}{\partial z} \left(U' + \frac{U}{r} \right) \\ & + \mu \left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial r^2} \right) = 0. \quad (7) \end{aligned}$$

Substituting (4) and (5) in (6) and (7), the terms in t^{-2} and t^{-1} vanish. From the constant term in (6) we obtain

$$\frac{\partial^2 U_2}{\partial z^2} = -\frac{3\lambda + 4\mu}{\lambda + 2\mu} \left(U_0'' + \frac{U_0'}{r} - \frac{U_0}{r^2} \right),$$

whence, using (B),

$$U_2 = -\frac{3\lambda + 4\mu}{2(\lambda + 2\mu)} z^2 \left(U_0'' + \frac{U_0'}{r} - \frac{U_0}{r^2} \right) + g(r).$$

Also, the constant term in (7) vanishes.

Furthermore, the surface tractions must vanish on the free surfaces $z = \alpha$, $z = -\alpha$; hence (Love, p. 76) the following equations must hold on these surfaces:—

$$\left. \begin{aligned} X_\nu &= X_x \cos(x, \nu) + X_y \cos(y, \nu) + X_z \cos(z, \nu) = 0, \\ Y_\nu &= Y_x \cos(x, \nu) + Y_y \cos(y, \nu) + Y_z \cos(z, \nu) = 0, \\ Z_\nu &= Z_x \cos(x, \nu) + Z_y \cos(y, \nu) + Z_z \cos(z, \nu) = 0. \end{aligned} \right\} \quad (8)$$

If we consider a tangent plane at an arbitrary point of the surface $z = \alpha$, the direction of the normal is denoted by ν , and (8) gives the tractions across the tangent plane in terms of the stress-components across planes parallel to the rectangular coordinate planes. For cylindrical coordinates, we have (Love, pp. 100, 141):—

$$\begin{aligned} X_x &= \lambda \Delta + 2\mu e_{xx} = \lambda \Delta + 2\mu \frac{\partial U}{\partial r}, \\ Y_y &= \lambda \Delta + 2\mu e_{yy} = \lambda \Delta + 2\mu \frac{U}{r}, \\ Z_z &= \lambda \Delta + 2\mu e_{zz} = \lambda \Delta + 2\mu \frac{1}{t} \frac{\partial w}{\partial z}, \\ Y_z &= Z_y = \mu e_{yz} = 0, \\ Z_x &= X_z = \mu e_{zx} = \mu \left(\frac{1}{t} \frac{\partial U}{\partial z} + \frac{\partial w}{\partial r} \right), \\ X_y &= Y_x = \mu e_{xy} = 0, \end{aligned}$$

where

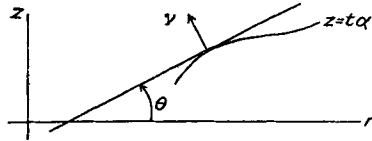
$$\Delta = U' + \frac{U}{r} + \frac{1}{t} \frac{\partial w}{\partial z}.$$

To determine the direction cosines, we return to our original variables and observe that $\tan \theta = t\alpha'$.

Hence we find along the x axis

$$\cos(x, \nu) = \frac{-t\alpha'}{\sqrt{1+t^2\alpha'^2}}, \quad \cos(y, \nu) = 0, \quad \cos(z, \nu) = \frac{1}{\sqrt{1+t^2\alpha'^2}}.$$

Fig. 2.



Hence the conditions that the tractions vanish on the free surfaces become

$$-t\alpha'X_x + X_z = 0, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$-t\alpha'Z_x + Z_z = 0. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

As we proceed to higher order terms, using (6) and (7), the relations (9) and (10) may be expected to play an important rôle in furnishing differential equations to determine the arbitrary functions that enter. In the present case there is no constant term in either left-hand member; and by virtue of (3) the terms in t also vanish.

Thus the surplus body forces per unit of volume vanish up to order t , and the surplus surface tractions per unit of area vanish up to order t^2 . Hence the total surplus applied force is of the order t^2 , whereas the total given radial force is of order t . It seems clear that when the surplus applied forces are removed, no sensible change occurs in the radial or axial displacement, and that we have a solution of our problem.

It remains to compute the principal part of the radial pressure on the inner edge of the plate. Here we make use of Saint Venant's principle and obtain the resultant traction, $X_x(r, z)$, as

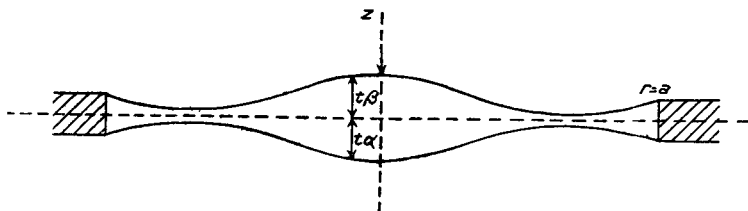
$$\int_0^{2\pi} \int_{-t\alpha(a)}^{t\alpha(a)} X_x(a, 0) \, adr \, d\theta = \frac{8\pi a\mu(\lambda + \mu)h(a)}{\lambda + 2\mu} \left[U_0'(a) + \frac{\lambda}{2(\lambda + \mu)} \frac{U_0(a)}{a} \right].$$

If we desire to take $t=1$ throughout, so that the equations of the bases are $z=\alpha$ and $z=-\alpha$, then t disappears as an explicit parameter, but the terms of the solution are still ordered according to the powers of the natural parameter, namely the ratio of the thickness of the plate to its diameter.

Case II.—The Complete Plate under Axial Pressure.

As a second example consider a thin circular plate, nearly plane, clamped at the outer edge and subject to an axial force P . Here a small force P yields a relatively large displacement, and our method is put to a more delicate test. Again adopt cylindrical coordinates (fig. 3).

Fig. 3.



The plate is not restricted to be symmetrical. Accordingly let α and β be proportional to the distances of the bases of the plate from the plane $z=0$, so that $z=t\alpha$ and $z=t\beta$ are the equations of the lower and upper bases respectively.

The boundary conditions are the following :—

$$\begin{cases} U(0, 0) = 0, & w(0, 0) = \epsilon, \\ w(a, 0) = \frac{\partial w(a, 0)}{\partial r} = 0. \end{cases}$$

Again, we take the formula for potential energy as the point of departure. The term in t^{-1} is now

$$\frac{\pi}{t} \int_0^a \int_\alpha^\beta \left\{ (\lambda + 2\mu) \left[\frac{\partial w_0}{\partial z} \right]^2 + \mu \left[\frac{\partial U_0}{\partial z} \right]^2 \right\} r \, dz \, dr.$$

This term can only be made to vanish by setting

$$w_0 = w_0(r), \quad U_0 = U_0(r),$$

and the boundary conditions are not thereby violated.

The constant term in W disappears and we can write the integrand of the term in t as the sum of squares :—

$$\begin{aligned} \left[U_0' + \frac{U_0}{r} + \frac{\partial w_1}{\partial z} \right]^2 + 2\mu \left[U_0'^2 + \frac{U_0^2}{r^2} + \left(\frac{\partial w_1}{\partial z} \right)^2 \right] \\ + \mu \left[\frac{\partial U_1}{\partial z} + w_0' \right]^2. \end{aligned}$$

In the present case we are led to attempt to make this integrand vanish, since the case of an ordinary plane plate shows

that the energy will be of order t^3 . Thus we must take

$$U_0 = 0, \quad w_1 = w_1(r), \quad U_1 = -zw_0' + p(r),$$

where $p(r)$ is arbitrary. None of these requirements violate the boundary conditions.

The physical significance of the conditions thus far obtained is immediate; the displacement is transverse, and such that filaments perpendicular to $z=0$ remain perpendicular in the displaced position and are not compressed transversely.

When these conditions are imposed, it is easy to see that the energy is of the order t^3 . More precisely, the leading term in W now becomes

$$\begin{aligned} \pi t^3 \int_0^a \int_z^\beta \left\{ (\lambda + 2\mu) \left[-zw_0'' + p' + \frac{-zw_0' + p}{r} + \frac{\partial w_2}{\partial z} \right]^2 \right. \\ \left. + \mu \left[\left(\frac{\partial U_2}{\partial z} + w_1' \right)^2 - 4 \left(\frac{-zw_0' + p}{r} \right) (-zw_0'' + p') \right. \right. \\ \left. \left. - 4 \left[\left(\frac{-zw_0' + p}{r} \right) \frac{\partial w_2}{\partial z} - 4(-zw_0'' + p') \frac{\partial w_2}{\partial z} \right] \right\} r dz dr. \end{aligned}$$

The part of the integrand in braces may be written as

$$\begin{aligned} \lambda \left[-zw_0'' + p' + \frac{-zw_0' + p}{r} + \frac{\partial w_2}{\partial z} \right]^2 \\ + 2\mu \left[(-zw_0'' + p')^2 + \frac{1}{r^2} (-zw_0' + p)^2 + \left(\frac{\partial w_2}{\partial z} \right)^2 \right] \\ + \mu \left[\frac{\partial U_2}{\partial z} + w_1' \right]^2. \end{aligned}$$

Since U_2 appears only in the last term, and since W is to be a minimum, we choose

$$U_2 = -zw_1' + q(r).$$

We proceed next to the choice of $\frac{\partial w_2}{\partial z}$, and find (by the same method as employed in the first example) :—

$$\begin{aligned} \frac{\partial w_2}{\partial z} = \frac{-\lambda}{\lambda + 2\mu} \left(-zw_0'' + p' + \frac{-zw_0' + p}{r} \right), \\ w_2 = \frac{\lambda}{\lambda + 2\mu} \left[\frac{z^2}{2} w_0'' - zp' + \frac{\frac{z^2}{2} w_0' - zp}{r} + s(r) \right], \end{aligned}$$

$s(r)$ arbitrary. It may be noted that the choice of U_2, w_2 does not interfere with the boundary conditions.

The integral written above thus becomes

$$\pi t^3 \int_0^a \int_\alpha^\beta \left\{ \frac{4\mu}{\lambda+2\mu} \left[(\lambda+2\mu) \left\{ (-zw_0'' + p')^2 + \frac{1}{r^2} (-zw_0' + p)^2 \right\} \right. \right. \\ \left. \left. + \lambda(-zw_0'' + p') \left(\frac{-zw_0' + p}{r} \right) \right] \right\} r dr dz,$$

and an integration with regard to z can be explicitly performed. To this end it is convenient to introduce a new variable z' , where.

$$z = m + z', \quad \beta = m + h, \quad \alpha = m - h;$$

at the same time we replace p by v , where

$$hv = -mw_0' + p,$$

so that hvt is the radial displacement for the mean surface $z = m$. With this notation the principal part $W_3 t^3$ of W has a coefficient.

$$W_3 = \frac{8\pi\mu}{\lambda+2\mu} \int_0^a \left\{ (\lambda+\mu) \left[\frac{w_0''^2}{3} + \frac{1}{3r^2} w_0'^2 \right. \right. \\ \left. \left. + \left(v' + \frac{h'}{h} v + \frac{m'}{h} w_0' \right)^2 + \frac{v^2}{r^2} \right] \right. \\ \left. + \lambda \left[\frac{w_0' w_0''}{3r} + \frac{v}{r} \left(v' + \frac{h'}{h} v + \frac{m'}{h} w_0' \right) \right] \right\} h^3 r dr. \quad (11)$$

It is to be observed that $\frac{h'}{h}, \frac{m'}{h}$ are independent of the thickness of the plate.

The integral is to be made a minimum subject to the boundary conditions

$$\left. \begin{aligned} w_0(0) &= \epsilon, & w_0(a) &= 0, & w_0'(a) &= 0, \\ v(0) &= v(a) &= 0. \end{aligned} \right\} \quad (12)$$

The form of W shows that $w_0'(0)$ vanishes also; otherwise the integral would be infinite.

Our problem is now one in the Calculus of Variations, and the condition $\delta W_3 = 0$ gives us

$$\frac{d}{dr} \frac{\partial \Phi}{\partial w_0'} - \frac{\partial \Phi}{\partial w_0} = 0, \quad \frac{d}{dr} \frac{\partial \Phi}{\partial v'} - \frac{\partial \Phi}{\partial v} = 0, \quad (13)$$

where Φ is the integrand in (11). We will not write out these equations. The system of two equations is of the fifth order. The five arbitrary constants in the general solution

are determined by (12). One of these is an additive constant in w_0 , since w_0 does not occur explicitly.

For the case of a *symmetrical* plate we have $m'=0$, and the variables separate. The equations assume the simpler form :—

$$\begin{aligned} \frac{d}{dr} \left[h^3 r \left\{ 2(\lambda + \mu) \frac{w_0''}{3} + \frac{\lambda w_0'}{3r} \right\} \right] - h^3 r \left\{ 2(\lambda + \mu) \frac{w_0'}{3r^2} + \frac{\lambda w_0''}{3r} \right\} &= 0, \\ \frac{d}{dr} \left[h^3 r \left\{ 2(\lambda + \mu) \left(v' + \frac{h'}{h} v \right) + \frac{\lambda v}{r} \right\} \right] - h^3 r \left\{ 2(\lambda + \mu) \frac{h'}{h} \left(v' + \frac{h'}{h} v \right) \right. \\ &\quad \left. + 2(\lambda + \mu) \frac{v}{r^2} + \frac{\lambda}{r} \left(v' + 2 \frac{h'}{h} v \right) \right\} + 0. \end{aligned}$$

If, furthermore, the plate is of constant thickness, the equations in w_0' and v become the same :—

$$r^2 v'' + r v' - v = 0, \quad r^2 w_0''' + r w_0'' - w_0' = 0.$$

This differential equation for w_0 coincides with that obtained in the usual theory (Love, p. 494).

When w_0 and v have been determined in the general case, the displacements are given by the formulas :—

$$\begin{aligned} U &= -(z-m)w_0't + hvt + \dots, \\ w &= w_0 + w_1 t + \frac{\lambda}{\lambda + 2\mu} \left[\frac{z^2}{2} \left(w_0'' + \frac{w_0'}{r} \right) - z \left\{ \frac{d}{dr} \left(m w_0' + hv \right) \right. \right. \\ &\quad \left. \left. + \frac{m w_0' + hv}{r} \right\} \right] t^2 + \dots \end{aligned}$$

It seems possible to proceed with this example, as in the earlier case, by considering the body forces and surface tractions, and determining terms in U and w of higher order.

The problem of the thin circular plate under special conditions has thus been formally treated by means of the following method, apparently applicable to thin plates, shells, and beams :—(1) introduction of a small parameter t to represent distance from a fundamental mean surface or line ; (2) expansion of the Lagrangian function $T-U$, the displacements, the body forces, and the surface tractions as power series in t ; (3) determination of the early coefficients so as to make this integral a minimum of as high order as possible in t ; and (4) determination of later coefficients so that the body forces and surface tractions have the required values.