THEOREMS CONCERNING THE SUMMABILITY OF SERIES BY BOREL'S EXPONENTIAL METHOD.

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§ 1.

Introduction.

1.1. In a paper [1]) published in the *Proceedings of the London Mathematical Society* in 1912 we proved that *a series* $\sum a_n$ *in which*

$$a_n = o\left(\frac{1}{\sqrt{n}}\right) \tag{1.1}$$

cannot be summable by Borel's *method unless it is convergent;* and we raised the question whether the theorem remains true if the o in the condition (1.1) is replaced by an O. We stated that we had no doubt as to the truth of the theorem thus suggested, but that we were unable to find a proof.

We are now able to supply the proof that was then lacking, and to do this is the principal object of the present paper. The proof is given in section 2. In sections 3 and 4 we consider some theorems of a different character but also relating to Borel's method.

§ 2.

Proof of the general Borel-Tauber [2]) Theorem.

2.1. The series $\sum a_n$ is said to be *summable* (B), *to sum* s, if

$$e^{-x} \sum s_n \frac{x^n}{n!},$$

[1]) G. H. Hardy and J. E. Littlewood, *The Relations between* Borel's *and* Cesàro's *Methods of Summation* [Proceedings of the London Mathematical Society, series II, vol. XI (1912-1913), pp. 1-16].

[2]) For an explanation of our reasons for giving this name to the theorem, see G. H. Hardy and J. E. Littlewood, *Contributions to the arithmetic Theory of Series* [Proceedings of the London Mathematical Society, series II, vol. XI (1912-1913), pp. 411-478 (p. 413)].

where

$$s_n = a_0 + a_1 + \cdots + a_n,$$

tends to the limit s when $x \to \infty$.

THEOREM 2.1. — *If* $\sum a_n$ *is summable* (B), *and*

$$a_n = O\left(\frac{1}{\sqrt{n}}\right), \tag{2.11}$$

then $\sum a_n$ *is convergent.*

The proof of this theorem, as of any TAUBERian theorem of the O type, is decidedly difficult. We shall base it on a number of preliminary lemmas.

LEMMA 2.11. — *If* $\sum a_n$ *is summable* (B), *and*

$$a_n = o(1), \tag{2.111}$$

then

$$s_n = a_0 + a_1 + \cdots + a_n = o(\sqrt{n}). \tag{2.112}$$

This is Theorem 3 if our previous paper [3]), with $k = 0$. The conclusion is true *a fortiori* if a_n satisfies (2.11).

LEMMA 2.12. — *If* $\sum a_n$ *is summable* (B) *to sum* s, *and* s_n *satisfies* (2.112), *then*

$$\frac{1}{\sqrt{2\pi\mu}} \sum_{-\infty}^{\infty} e^{-n^2/2\mu} s_{n+\mu} \to s,$$

when $\mu \to \infty$ *by integral values.*

It is to be understood that $s_{n+\mu} = 0$ when the suffix is negative.

We have

$$S = e^{-\mu} \sum_0^{\infty} s_n \frac{\mu^n}{n!} \to s. \tag{2.121}$$

We write

$$S = e^{-\mu}\left(\sum_0^{(1-H)\mu} + \sum_{(1-H)\mu}^{(1+H)\mu} + \sum_{(1+H)\mu}^{\infty}\right) = S_1 + S_2 + S_3, \tag{2.122}$$

say, H being a constant, positive, irrational, and less than unity. Then we can find a positive constant δ such that

$$S_1 = O(e^{-\delta\mu}), \qquad S_3 = O(e^{-\delta\mu}) \text{ [4])}. \tag{2.123}$$

Thus

$$S_2 = e^{-\mu} \sum_{-H\mu}^{H\mu} s_{h+\mu} \frac{\mu^{h+\mu}}{(h+\mu)!} \to s. \tag{2.124}$$

Now it is easy to deduce from STIRLING's Theorem that

$$e^{-\mu} \frac{\mu^{h+\mu}}{(h+\mu)!} = \frac{1}{\sqrt{2\pi\mu}} e^{-h^2/2\mu} \left\{1 + O\left(\frac{|h|}{\mu}\right) + O\left(\frac{|h|^3}{\mu^2}\right)\right\}, \tag{2.125}$$

uniformly for $-H\mu < h < H\mu$.

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[3]) loc. cit. [1]), p. 8.

[4]) loc. cit. [1]), p. 6.

Substituting in (2.124), we see that

$$S_2 = \frac{1}{\sqrt{2\pi\mu}} \sum_{-H\mu}^{H\mu} e^{-h^2/2\mu} s_{h+\mu} + S_2' + S_2'', \tag{2.126}$$

where

$$\left\{\begin{aligned} S_2' &= O\left(\mu^{-\frac{3}{2}} \sum_{-H\mu}^{H\mu} |h| e^{-h^2/2\mu} |s_{h+\mu}|\right) \\ &= o\left(\frac{1}{\mu} \sum_{-\infty}^{\infty} |h| e^{-h^2/2\mu}\right) \\ &= o\left(\frac{1}{\mu} \int_0^\infty x e^{-x^2/2\mu} dx\right) \\ &= o\left(\int_0^\infty y e^{-y^2} dy\right) \\ &= o(1), \end{aligned}\right. \tag{2.1261}$$

and

$$\left\{\begin{aligned} S_2'' &= O\left(\mu^{-\frac{5}{2}} \sum_{-H\mu}^{H\mu} |h|^3 e^{-h^2/2\mu} |s_{h+\mu}|\right) \\ &= o\left(\frac{1}{\mu^2} \sum_{-\infty}^{\infty} |h|^3 e^{-h^2/2\mu}\right) \\ &= o\left(\frac{1}{\mu^2} \int_0^\infty x^3 e^{-x^2/2\mu} dx\right) \\ &= o\left(\int_0^\infty y^3 e^{-y^2} dy\right) \\ &= o(1). \end{aligned}\right. \tag{2.1262}$$

From (2.124), (2.126), (2.1261), and (2.1262), it follows that

$$\bar{S}_2 = \frac{1}{\sqrt{2\pi\mu}} \sum_{-H\mu}^{H\mu} e^{-h^2/2\mu} s_{h+\mu} \rightarrow s. \tag{2.127}$$

Now

$$\left\{\begin{aligned} \bar{S} &= \frac{1}{\sqrt{2\pi\mu}} \sum_{-\infty}^{\infty} e^{-h^2/2\mu} s_{h+\mu} \\ &= \frac{1}{\sqrt{2\pi\mu}} \left(\sum_{-\infty}^{-H\mu} + \sum_{-H\mu}^{H\mu} + \sum_{H\mu}^{\infty}\right) \\ &= \bar{S}_1 + \bar{S}_2 + \bar{S}_3, \end{aligned}\right. \tag{2.128}$$

say. Also

$$\left\{\begin{aligned} \bar{S}_1 &= O\left(\frac{1}{\sqrt{\mu}} \sum_{-\infty}^{-H\mu} \sqrt{\mu}\, e^{-h^2/2\mu}\right) \\ &= O\left(\int_{H\mu}^{\infty} e^{-x^2/2\mu} dx\right) \\ &= O\left(\sqrt{\mu} \int_{H\sqrt{\mu/2}}^{\infty} e^{-y^2} dy\right) \\ &= O(e^{-\delta\mu}), \end{aligned}\right. \tag{2.1281}$$

where δ is a positive constant; and

$$(2.1282)\qquad \left\{\begin{aligned} \overline{S}_3 &= O\left(\frac{1}{\sqrt{\mu}}\sum_{H\mu}^{\infty}\sqrt{h}\,e^{-h^2/2\mu}\right)\\ &= O\left(\frac{1}{\sqrt{\mu}}\int_{H\mu}^{\infty}\sqrt{x}\,e^{-x^2/2\mu}\,dx\right)\\ &= O\left(\mu^{\frac{1}{4}}\int_{H\sqrt{\mu/2}}^{\infty}\sqrt{y}\,e^{-y^2}\,dy\right)\\ &= O(e^{-\delta\mu}). \end{aligned}\right.$$

From (2.127), (2.128), (2.1281) and (2.1282) it follows that

$$(2.129)\qquad \overline{S} \rightarrow s.$$

This completes the proof of Lemma 2.12.

LEMMA 2.13. — *Suppose that $f(x)$ is the continuous function of x defined by the* equations

$$(2.1311)\qquad f(x) = s_n + (x-n)(s_{n+1} - s_n) \qquad (n \leqq x \leqq n+1),$$

$$(2.1312)\qquad f(x) = 0 \qquad (x < 0).$$

Suppose further that $\sum a_n$ is summable (B) to sum s, and that

$$(2.111)\qquad a_n = o(1).$$

Then

$$(2.132)\qquad \frac{1}{\sqrt{2\pi x}}\int_{-\infty}^{\infty} e^{-t^2/2x} f(t+x)\,dt \rightarrow s$$

as $x \rightarrow \infty$.

We have already proved that

$$(2.133)\qquad \frac{1}{\sqrt{2\pi\mu}}\sum_{-\infty}^{\infty} e^{-h^2/2\mu} s_{h+\mu} \rightarrow s$$

when $\mu \rightarrow \infty$ by integral values. We shall prove first that (2.133) may be replaced by

$$(2.134)\qquad \frac{1}{\sqrt{2\pi\mu}}\int_{-\infty}^{\infty} e^{-t^2/2\mu} s(t+\mu)\,dt \rightarrow s,$$

where $s(x)$ is the discontinuous function which is equal to s_n when $n \leqq x < n+1$. To prove this we observe that the difference between the left hand sides of (2.133) and (2.134) may be written in the form

$$(2.1341)\qquad \frac{1}{\sqrt{2\pi\mu}}\sum_{-\infty}^{\infty}\int_{h}^{h+1}(e^{-h^2/2\mu} - e^{-t^2/2\mu}) s(t+\mu)\,dt;$$

and that $s(t+\mu)$ is of the form $o(\sqrt{\mu})$ or $o(\sqrt{t})$, according as μ or t is numerically the greater, and so in any case of the form

$$o(\sqrt{\mu}) + o(\sqrt{t}).$$

Also

$$(2.1342)\quad \left\{ \begin{aligned} &\frac{1}{\sqrt{2\pi\mu}} \sum_{-\infty}^{\infty} \int_h^{h+1} (e^{-h^2/2\mu} - e^{-t^2/2\mu})\, o(\sqrt{\mu})\, dt \\ &\quad = o\left(\frac{1}{\mu}\int_{-\infty}^{\infty} |t| e^{-t^2/2\mu}\, dt\right) = o(1), \end{aligned} \right.$$

and

$$(2.1343)\quad \left\{ \begin{aligned} &\frac{1}{\sqrt{2\pi\mu}} \sum_{-\infty}^{\infty} \int_h^{h+1} (e^{-h^2/2\mu} - e^{-t^2/2\mu})\, o(\sqrt{t})\, dt \\ &\quad = o\left(\mu^{-\frac{3}{2}} \int_{-\infty}^{\infty} |t|^{\frac{3}{2}} e^{-t^2/2\mu}\, dt\right) \\ &\quad = o(\mu^{-\frac{1}{4}}) = o(1). \end{aligned} \right.$$

Hence (2.1341) tends to zero, and (2.133) may be replaced by (2.134).

Secondly, (2.134) may be replaced by

$$(2.135)\quad \frac{1}{\sqrt{2\pi\mu}} \int_{-\infty}^{\infty} e^{-t^2/2\mu} f(t+\mu)\, dt \twoheadrightarrow s.$$

For the difference of the left hand sides of (2.134) and (2.135) is, since $a_n = o(1)$, of the form

$$(2.1351)\quad o\left(\frac{1}{\sqrt{\mu}} \int_{-\infty}^{\infty} e^{-t^2/2\mu}\, dt\right) = o(1).$$

Finally we may replace the integer μ in (2.135) by a continuous variable x. For suppose that

$$(2.136)\quad x = \mu + \theta \qquad (0 < \theta < 1).$$

Then

$$(2.1361)\quad \left\{ \begin{aligned} &\int_{-\infty}^{\infty} e^{-t^2/2\mu} f(t+\mu)\, dt - \int_{-\infty}^{\infty} e^{-t^2/2x} f(t+x)\, dt \\ &= \int_{-\infty}^{\infty} e^{-t^2/2\mu} \{f(t+\mu) - f(t+x)\}\, dt \\ &+ \int_{-\infty}^{\infty} (e^{-t^2/2\mu} - e^{-t^2/2x}) f(t+x)\, dt \\ &= o\left(\int_{-\infty}^{\infty} e^{-t^2/2\mu}\, dt\right) + o\left(\frac{1}{\sqrt{\mu}} \int_{-\infty}^{\infty} |t| e^{-t^2/2\mu}\, dt\right) + o\left(\frac{1}{\mu}\int_{-\infty}^{\infty} |t|^{\frac{3}{2}} e^{-t^2/2\mu}\, dt\right) \\ &= o(\sqrt{\mu}) + o(\sqrt{\mu}) + o(\mu^{\frac{1}{4}}) \\ &= o(\sqrt{\mu}). \end{aligned} \right.$$

Hence (2.132) follows from (2.135).

2.21. So far we have never used the full condition (2.11): the hypothesis that $a_n = o(1)$ has been sufficient for our needs. It may be inferred that we have still to face the main difficulty of the problem. This difficulty lies in the proof of the lemma which follows.

LEMMA 2.21. — *Suppose that $f(x)$ is a real continuous function of x, with a derivative $f'(x)$ continuous except at isolated points. Suppose further that*

$$f'(x) = O\left(\frac{1}{\sqrt{x}}\right). \tag{2.211}$$

Finally suppose that the relation

$$\sqrt{\frac{a}{\pi x}}\int_{-\infty}^{\infty} e^{-at^2/x} f(t+x)\,dt \rightarrow s \tag{2.212}$$

holds for some positive value of a. Then (2.212) holds for all greater values of a.

We shall prove first that

$$f(x) = O(1) \text{ [5]}. \tag{2.213}$$

We have

$$\left\{\begin{aligned} s + o(1) - f(x) &= \sqrt{\frac{a}{\pi x}}\int_{-\infty}^{\infty} e^{-at^2/x}\{f(t+x) - f(x)\}\,dt \\ &= \sqrt{\frac{a}{\pi x}}\left(\int_{-\infty}^{-Hx} + \int_{-Hx}^{Hx} + \int_{Hx}^{\infty}\right) \\ &= J_1 + J_2 + J_3, \end{aligned}\right. \tag{2.214}$$

say. It is easy to prove, as in the proof of Lemma 2.12 [6]), that

$$J_1 = O(e^{-\delta x}), \qquad J_3 = O(e^{-\delta x}). \tag{2.215}$$

But, if $-Hx < t < Hx$, we have

$$f(t+x) - f(x) = \int_x^{t+x} f'(u)\,du = O\left(\frac{|t|}{\sqrt{x}}\right), \tag{2.216}$$

and so

$$J_2 = O\left(\frac{1}{x}\int_{-\infty}^{\infty} |t| e^{-at^2/x}\,dt\right) = O(1). \tag{2.217}$$

Hence

$$s + o(1) - f(x) = O(1), \tag{2.218}$$

$$f(x) = O(1). \tag{2.219}$$

2.22. Let

$$F(x) = f(x) - s, \tag{2.221}$$

so that

$$\frac{1}{\sqrt{x}}\int_{-\infty}^{\infty} e^{-at^2/x} F(t+x)\,dt = o(1). \tag{2.222}$$

Further, let

$$I_n(x) = \frac{1}{\sqrt{x}}\int_{-\infty}^{\infty} e^{-at^2/x}\left(\frac{t^2}{x}\right)^n F(t+x)\,dt, \tag{2.223}$$

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[5]) It would be sufficient for this purpose to suppose the left hand side of (2.212) *bounded*.

[6]) See 2.1281 and 2.1282.

where n is zero or half a positive integer. Since $F(x) = O(1)$, we have

$$(2.224)\qquad I_n(x) = O\left(x^{-n-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-at^2/x} t^{2n}\, dt\right) = O(1).$$

We shall now prove that (2.224) may be replaced by

$$(2.225)\qquad I_n(x) = o(1).$$

It will be well to point out explicitly that the argument by which this transition is justified contains the kernel of the whole proof of Lemma 2.2 and of our main theorem.

2.23. Suppose first that $n > 0$. We have

$$(2.231)\qquad I_n(x) = x^{-n-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-at^2/x} t^{2n} F(t+x)\, dt$$

$$(2.232)\qquad \begin{cases} I'_n(x) = \dfrac{dI_n}{dx} = -\left(n+\dfrac{1}{2}\right) x^{-n-\frac{3}{2}} \displaystyle\int_{-\infty}^{\infty} e^{-at^2/x} t^{2n} F(t+x)\, dt \\ \quad + a x^{-n-\frac{5}{2}} \displaystyle\int_{-\infty}^{\infty} e^{-at^2/x} t^{2n+2} F(t+x)\, dt \\ \quad + x^{-n-\frac{1}{2}} \displaystyle\int_{-\infty}^{\infty} e^{-at^2/x} t^{2n} F'(t+x)\, dt \end{cases}$$ [7].

Also

$$(2.233)\qquad \begin{cases} x^{-n-\frac{1}{2}} \displaystyle\int_{-\infty}^{\infty} e^{-at^2/x} t^{2n} F'(t+x)\, dt \\ = -2n x^{-n-\frac{1}{2}} \displaystyle\int_{-\infty}^{\infty} e^{-at^2/x} t^{2n-1} F(t+x)\, dt \\ + 2a x^{-n-\frac{3}{2}} \displaystyle\int_{-\infty}^{\infty} e^{-at^2/x} t^{2n+1} F(t+x)\, dt, \end{cases}$$

by integration by parts. Substituting in (2.232), we obtain

$$(2.234)\qquad \begin{cases} I'_n = -\dfrac{2n}{\sqrt{x}} I_{n-\frac{1}{2}} - \dfrac{n+\frac{1}{2}}{x} I_n + \dfrac{2a}{\sqrt{x}} I_{n+\frac{1}{2}} + \dfrac{a}{x} I_{n+1} \\ = O\left(\dfrac{1}{\sqrt{x}}\right) + O\left(\dfrac{1}{x}\right) + O\left(\dfrac{1}{\sqrt{x}}\right) + O\left(\dfrac{1}{x}\right) \\ = O\left(\dfrac{1}{\sqrt{x}}\right); \end{cases}$$

and it is plain that a repetition of this argument will lead to

$$(2.235)\qquad I''_n = O\left(\frac{1}{x}\right)$$

[7] There is no difficulty in the differentiation under the integral sign: all the integrals concerned are absolutely and uniformly convergent.

We have therefore

$$(2.236)\qquad I_n = O(1),\quad I'_n = O\left(\frac{1}{\sqrt{x}}\right),\quad I''_n = O\left(\frac{1}{x}\right)$$

for all positive values of n.

Secondly, suppose that $n = 0$. Then it is easy to see that (2.234) must be replaced by

$$(2.237)\qquad I'_0 = -\frac{1}{2x}I_0 + \frac{2a}{\sqrt{x}}I_{\frac{1}{2}} + \frac{a}{x}I_1,$$

and so that the relations (2.236) are true even when $n = 0$.

2.24. Suppose now first that $n = 0$.

Then

$$(2.2411)\qquad I_0 = o(1),$$

by (2.222), and

$$(2.2412)\qquad x^2 I''_0 = O(x)$$

by (2.236). Hence [8])

$$(2.2413)\qquad x I'_0 = o(\sqrt{x}),\quad I'_0 = o\left(\frac{1}{\sqrt{x}}\right).$$

But we have, from (2.237),

$$(2.242)\qquad \begin{cases} \frac{2a}{\sqrt{x}}I_{\frac{1}{2}} = I'_0 + \frac{1}{2x}I_0 - \frac{a}{x}I_1 \\ = o\left(\frac{1}{\sqrt{x}}\right) + O\left(\frac{1}{x}\right) + O\left(\frac{1}{x}\right) = o\left(\frac{1}{\sqrt{x}}\right), \end{cases}$$

and so

$$(2.243)\qquad I_{\frac{1}{2}} = o(1).$$

Thus (2.225) is established when $n = \frac{1}{2}$.

But it is fairly obvious that this argument may be repeated. We have now

$$(2.2441)\qquad I_{\frac{1}{2}} = o(1),\quad x^2 I''_{\frac{1}{2}} = O(x),$$

and therefore

$$(2.2442)\qquad x I'_{\frac{1}{2}} = o(\sqrt{x}),\quad I'_{\frac{1}{2}} = o\left(\frac{1}{\sqrt{x}}\right).$$

Hence, using (2.234), we have

$$(2.245)\qquad \begin{cases} \frac{2a}{\sqrt{x}}I_1 = I'_{\frac{1}{2}} + \frac{1}{\sqrt{x}}I_0 + \frac{1}{x}I_{\frac{1}{2}} - \frac{a}{x}I_{\frac{3}{2}} \\ = o\left(\frac{1}{\sqrt{x}}\right) + o\left(\frac{1}{\sqrt{x}}\right) + o\left(\frac{1}{x}\right) + O\left(\frac{1}{x}\right) \\ = o\left(\frac{1}{\sqrt{x}}\right), \end{cases}$$

$$(2.246)\qquad I_1 = o(1).$$

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[8]) loc. cit. [2]). The Theorem used here is Theorem 6, with $f = I_0$, $\varphi = 1$, $\psi = x$, $r = 2$, $s = 1$.

And we need only repeat this argument indefinitely in order to complete the proof of (2.225).

2.25. We are now in a position to complete the proof of Lemma 2.21.

Suppose that δ is a positive number not greater than a. Then

$$(2.251)\qquad e^{-\delta t^2/x}=\sum_{\nu=1}^{n}\frac{(-1)^\nu}{\nu!}\left(\frac{\delta t^2}{x}\right)^\nu+\frac{(-1)^{n+1}}{(n+1)!}\left(\frac{\delta t^2}{x}\right)^{n+1}e^{-\delta' t^2/x},$$

where $0<\delta'<\delta$. We have therefore

$$(2.252)\qquad \frac{1}{\sqrt{x}}\int_{-\infty}^{\infty}e^{-(a+\delta)t^2/x}F(t+x)\,dt=\sum_{0}^{n}\frac{(-1)^\nu\delta^\nu}{\nu!}I_\nu(x)+\rho,$$

where

$$(2.2521)\qquad \begin{cases} |\rho|<\dfrac{K}{(n+1)!}\dfrac{\delta^{n+1}}{x^{n+\frac{3}{2}}}\displaystyle\int_{-\infty}^{\infty}e^{-at^2/x}t^{2n+2}\,dt\\ \quad=\dfrac{K}{(n+1)!}\left(\dfrac{\delta}{a}\right)^{n+1}\displaystyle\int_{-\infty}^{\infty}e^{-w^2}w^{2n+2}\,dw\\ \quad\leqq K\dfrac{\Gamma(n+\frac{3}{2})}{\Gamma(n+2)}<\dfrac{K}{\sqrt{n}},\end{cases}$$

and so ρ tends to zero as $n\to\infty$, *uniformly in* x.

Thus

$$(2.253)\qquad \frac{1}{\sqrt{x}}\int_{-\infty}^{\infty}e^{-(a+\delta)t^2/x}F(t+x)\,dt=\sum_{0}^{\infty}\frac{(-1)^\nu\delta^\nu}{\nu!}I_\nu(x).$$

The series on the right hand side is uniformly convergent in x, say for $x\geqq x_0$, and each of its terms tends to zero as $x\to\infty$. It follows that

$$(2.254)\qquad \frac{1}{\sqrt{x}}\int_{-\infty}^{\infty}e^{-(a+\delta)t^2/x}F(t+x)\,dt=o(1).$$

Thus (2.222) remains true when we substitute for a any number between a and $2a$ inclusive. As this argument may be repeated, it holds for all values of a greater than the original value. Hence (2.212) also holds, and the lemma is proved.

2.3. Theorem 2.1 is an easy deduction from Lemma 2.21. We have

$$(2.31)\qquad \begin{cases}\sqrt{\dfrac{a}{\pi x}}\displaystyle\int_{-\infty}^{\infty}e^{-at^2/x}f(t+x)\,dt-f(x)\\ =\sqrt{\dfrac{a}{\pi x}}\displaystyle\int_{-\infty}^{\infty}e^{-at^2/x}\{f(t+x)-f(x)\}\,dt.\end{cases}$$

Now

$$(2.32)\qquad |f(t+x)-f(x)|=\left|\int_{x}^{t+x}f'(u)\,du\right|<\frac{K|t|}{\sqrt{x}}.$$

Hence

$$(2.33)\qquad \begin{cases} \left|\sqrt{\frac{a}{\pi x}}\int_{-\infty}^{\infty} e^{-at^2/x} f(t+x)\,dt - f(x)\right| \\ \quad < \frac{K}{x}\sqrt{\frac{a}{\pi}}\int_{-\infty}^{\infty} e^{-at^2/x}\,|t|\,dt \\ \quad = \frac{K}{\sqrt{a}} < \varepsilon, \end{cases}$$

if a is large enough. If now we make $x \to \infty$, we obtain

$$(2.34)\qquad \overline{\lim}\, |s - f(x)| \leqq \varepsilon,$$

for all positive values of ε, so that

$$(2.35)\qquad f(x) \to s.$$

It follows that

$$s_n \to s;$$

and the proof of Theorem 2.1 is completed.

§ 3.

A new method of summation and its relations to Borel's method.

1.3. Theorem 2.1, is a generalisation of Theorem 1 of our former paper on Borel's method. It is naturally suggested that Theorems 2-5 of that paper are susceptible of similar generalisation. We shall content ourselves with stating that the generalisations thus suggested are in fact true, and with mentioning explicitly two of the most interesting of them, viz.,

Theorem 3.11. — *If* $\sum a_n$ *is summable* (B), *and* $a_n = O(1)$, *then* $\sum a_n$ *is summable* $(C, 1)$.

Theorem 3.12. — *If* $\sum a_n$ *is summable* (B), *and* $a_n = O(1)$, *then* $s_n = o(\sqrt{n})$.

We may remark that Theorem 3.12 may be deduced from Theorem 3.11 by using Theorem 11 of our paper quoted in note [2]).

3.2. We take this opportunity of correcting an error in the footnote to p. 10 of our former paper. It is stated there that the equation

$$(3.21)\qquad \frac{k!\,s_n^k}{n^k} = A + o\left(\frac{1}{\sqrt{n}}\right) \quad ^{9)}$$

gives a sufficient condition for the summability of $\sum a_n$ by Borel's method. It is easy to show that this statement is false.

Suppose that $\sum a_n n^{-s}$ is an ordinary Dirichlet's series which represents a function $f(s) = f(\sigma + it)$ regular and of finite order for all values of σ. The series is

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[9]) We write A instead of s, as we are about to use s in a different sense.

then, in virtue of a theorem of BOHR [10]), summable by CESÀRO's means all over the plane. We have therefore

$$(3.211)\qquad \frac{k!\,s_n^k}{n^k} = f(s) + o(1),$$

where k is a number which depends upon s.

Let us denote by t_n^k CESÀRO's mean of order k formed from the series

$$(3.212)\qquad \sum b_n n^{-s} = \sum \frac{a_n}{\sqrt{n}} n^{-s}.$$

It is easy to deduce from (3.211) that

$$(3.213)\qquad \frac{k!\,t_n^k}{n^k} = f\left(s + \frac{1}{2}\right) + o\left(\frac{1}{\sqrt{n}}\right) \quad ^{11)}.$$

It follows that, if the theorem suggested is true, then the series $\sum b_n n^{-s}$, and therefore the series $\sum a_n n^{-s}$, must be summable (B) all over the plane.

But there are DIRICHLET's series which represent functions regular and of finite order all over the plane and which are *not* summable (B) all over the plane. For example the series

$$1^{-s} + 0 + 0 + \cdots - 8^{-s} + 0 + \cdots + 27^{-s} + 0 + \cdots$$

represents the function

$$(1 - 2^{1-3s})\zeta(3s),$$

and is summable when, and only when, it is convergent, *i. e.* when $\sigma > 0$ [12]).

The suggested theorem is therefore certainly false.

The theorem is true when $k = 1$ [13]), and the correct generalisation is as follows.

THEOREM 3.2. — *If*

$$(3.21)\qquad \frac{(k+1)!\,s_n^{k+1}}{n^{k+1}} = A + o\left(\frac{1}{\sqrt{n}}\right),$$

then BOREL's *integral*

$$\int_0^\infty e^{-x} \sum \frac{a_n x^n}{n!}\, dx$$

is summable (C, k), *i. e.*

$$(3.22)\qquad \frac{1}{k!\,x^k}\left(\int_0^x (x-t)^k e^{-t} \sum \frac{a_n t^n}{n!}\, dt\right) \to A$$

as $x \to \infty$.

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[10]) See G. H. HARDY and M. RIESZ, *The general Theory of* DIRICHLET's *Series* [Cambridge Mathematical Tracts, no. 18, 1915], p. 56.

[11]) We cannot quote any general theorem of which this equation is a direct corollary: but the materials necessary for the proof will be found in our paper « *Contributions,* etc. », loc. cit. [2]), pp. 432 *et seq.*.

[12]) G. H. HARDY, *The Application to* DIRICHLET's *Series of* BOREL's *exponential Method of Summation* [Proceedings of the London Mathematical Society, series II, vol. VIII (1910), pp. 277-294].

[13]) G. H. HARDY, *Researches in the Theory of divergent Series and divergent Integrals* [Quarterly Journal, vol. XXXV (1904), pp. 22-66], p. 40; T. J. I'A. BROMWICH, *Infinite Series,* pp. 319-322.

We shall be content to sketch the proof of this theorem. We may suppose without loss of generality that $A = 0$. Then

$$s_0^k + s_1^k + \cdots + s_n^k = o(n^{k+\frac{1}{2}});$$

and it is easy to deduce successively

$$\frac{1}{n}\left\{s_0^k + \frac{s_1^k}{2^k} + \cdots + \frac{s_n^k}{(n+1)^k}\right\} = o\left(\frac{1}{\sqrt{n}}\right),$$

$$e^{-x}\sum \frac{s_n^k}{(n+1)^k}\frac{x^n}{n!} \rightarrow 0,$$

$$e^{-x}\sum s_n^k \frac{x^{n+k}}{(n+k)!} \rightarrow 0.$$

The last formula is equivalent to (3.22).

3 3. The work of section 2 suggests some new definitions which seem to us likely to be of considerable use in the theory of divergent series and integrals. We shall say that *the series* $\sum a_n$ *is summable* $(E,\ a)$ *to sum* s, or that

$$s_n \rightarrow s \qquad (E,\ a),$$

if

(3.31) $$\lim_{\mu \rightarrow \infty} \sqrt{\frac{a}{\pi\mu}} \sum_{-\infty}^{\infty} e^{-ah^2/\mu} s_{h+\mu} = s.$$

Similarly we shall say that

$$f(x) \rightarrow l \qquad (E,\ a)$$

if

(3.32) $$\lim_{\xi \rightarrow \infty} \sqrt{\frac{a}{\pi\xi}} \int_{-\infty}^{\infty} e^{-at^2/\xi} f(t+\xi)\,dt = l$$ [14])

The properties of these definitions are, in so far as series or integrals near to the boundary of convergence are concerned, very similar to those of BOREL's method. In particular the « TAUBERian » properties of the definition (3.31) are the same as those of BOREL's method. For example we have.

THEOREM 3.3. — *If* (3.31) *is true, i. e., if* $\sum a_n$ *is summable* $(E,\ a)$, *and*

$$a_n = O\left(\frac{1}{\sqrt{n}}\right),$$

then $\sum a_n$ *is convergent.*

3 4. BOREL's method has however a peculiarly intimate connection with the method of type $(E,\ \frac{1}{2})$. This connection is expressed by the following theorem, the proof of which is contained implicitly in the analysis of section 2.

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14) It is to be understood that $s_{h+\mu} = 0$ when $h + \mu < 0$, and $f(t+\xi) = 0$ when $t + \xi$ is less than some fixed number.

THEOREM 3.4. — *Suppose that $s_n = o(\sqrt{n})$. Then the existence of any one of the limits*

$$\lim_{x\to\infty} e^{-x}\sum_0^\infty s_n \frac{x^n}{n!}, \tag{3.411}$$

$$\lim_{\mu\to\infty} \frac{1}{\sqrt{2\pi\mu}}\sum_{-\infty}^\infty e^{-h^2/2\mu} s_{h+\mu}, \tag{3.412}$$

$$\lim_{\xi\to\infty} \frac{1}{\sqrt{2\pi\xi}}\int_{-\infty}^\infty e^{-t^2/2\xi} s(t+\xi)\,dt, \tag{3.413}$$

where $s(x)$ is the discontinuous function defined in 2.1, *involves the existence of the remainder and the equality of all. In this proposition it is indifferent whether x or ξ tends to its limit continuously or by integral values.*

The condition $s_n = o(\sqrt{n})$ is certainly satisfied if any one of the limits exist and $a_n = o(1)$. And then the existence of any one of the limits (3.411)-(3.413) *implies, and is implied by, the convergence of* BOREL'*s integral*

$$\int_0^\infty e^{-x}\left(\sum a_n \frac{x^n}{n!}\right)dx, \tag{3.414}$$

or the existence of the limit

$$\lim_{\xi\to\infty} \frac{1}{\sqrt{2\pi\xi}}\int_{-\infty}^\infty e^{-t^2/2\xi} f(t+\xi)\,dt, \tag{3.415}$$

where $f(x)$ is the continuous function defined in 2.1.

3.5. We add a few remarks as to the relations between definitions of the type (E, a) corresponding to different values of a. We proved in 2.2. that, when

$$a_n = O\left(\frac{1}{\sqrt{n}}\right), \tag{2.11}$$

summability (E, a) for a particular value of a implies summability for any larger value of a; and in 2.3 that this implies convergence. Now it is easy to prove that these definitions obey the « condition of consistency », *i. e.* that *any convergent series is summable (E, a) for any positive value of a.* We see thus that, when (2.11) is satisfied, all methods of the type (E, a) are equivalent and will sum convergent series only. But this is not a sufficient account of the matter.

Suppose that $s = 0$, so that

$$\sqrt{\frac{1}{x}}\int_{-\infty}^\infty e^{-at^2/x} s(t+x)\,dt = o(1). \tag{3.51}$$

Putting

$$t = u\sqrt{\frac{1}{2a}} = \alpha u, \tag{3.52}$$

we obtain

$$\sqrt{\frac{1}{x}}\int_{-\infty}^\infty e^{-u^2/2x} s(u+\alpha x)\,du = o(1). \tag{3.53}$$

Let us suppose, for simplicity, that α is an integer k. When $k = 1$ our method is, at any rate when $a_n = o(1)$, equivalent to BOREL's. When $k > 1$ it is equivalent to the method which consists in applying BOREL's method to the subsequence s_{kn}. Now $s_n \twoheadrightarrow 0$ implies $s_{kn} \twoheadrightarrow 0$, while of course the converse implication does not hold; and it would be natural to expect the same to be true when the limits are taken in BOREL's sense. It would therefore also be natural to expect that the truth of (3.53) for a given α should involve its truth for a larger (but not for a smaller) α either without any restriction on a_n or at any rate under some condition less narrow than the « TAUBERian » condition (2.11); in a word to expect the inference from a smaller to a larger α to be of an « ABELian » and not of a « TAUBERian » characte. And so we should expect the inference from summability (E, a) for a given a to summability for a *smaller* to be « ABELian », to be valid at any rate under a wider condition than the condition (2.11) required for the inference to a *larger* a, and to be of a less subtle character.

We shall see shortly that this conjecture is justified, and that *the inference from a larger to a smaller a holds at all events whenever* $a_n = o(1)$. We have however no direct proof of this assertion; our proof is indirect and depends on the methods of summation considered in the next section.

§ 4.

The " circle „ method.

4.1. We shall conclude this paper by giving a short account of the results of some researches which started from a suggestion made to us in 1912 by Dr. MARCEL RIESZ.

Suppose that the series

(4.11) $$f(x) = \sum a_n x^n$$

has unit radius of convergence, and that

(4.12) $$f(x) = f(\tfrac{1}{2} + y) = \sum a_n (\tfrac{1}{2} + y)^n = \sum b_n y^n,$$

so that

(4.13) $$\sum \frac{b_n}{2^n}$$

is the TAYLOR's series for $f(\frac{1}{2} + \frac{1}{2})$, *i. e.* the series obtained by putting $y = \frac{1}{2}$ in the expansion of $f(\frac{1}{2} + y)$. Then RIESZ's suggestion was that *the hypotheses (i) that* $\sum a_n$ *is summable (B), and (ii) that the series* (4.13) *is convergent, are equivalent,* at any rate under fairly general conditions as to the order of a_n.

We have succeeded in proving, by the use of some ideas already used in sections 2 and 3, that this very beautiful theorem is true whenever $s_n = o(\sqrt{n})$, and in particular whenever $a_n = o(1)$. We propose now, however, to give a proof not exactly of RIESZ's theorem, but of another theorem in which the central idea is exactly the same and which differs from RIESZ's only in certain formal respects.

4.2. Suppose that

$$F(y) = \sum a_n e^{-ny} \tag{4.21}$$

is convergent for all values of y whose real part is positive; and consider the series

$$\sum b_n = \sum \frac{(-k)^n}{n!} F^{(n)}(k) \qquad (k > 0), \tag{4.22}$$

which we may denote symbolically by

$$F(k - k). \tag{4.221}$$

Then, if (4.22) is convergent, we shall say that $\sum a_n$ is *summable with radius k*.

Suppose that this is so, and that the sum is s. Then

$$\lim_{M \to \infty} \sum_0^M \frac{k^m}{m!} \sum_0^\infty a_n n^m e^{-kn} = s, \tag{4.231}$$

or

$$\lim_{M \to \infty} \sum_0^\infty a_n e^{-kn} \sum_0^M \frac{(kn)^m}{m!} = s, \tag{4.232}$$

or

$$\lim_{M \to \infty} \sum_0^\infty s_n \Delta \left\{ e^{-kn} \sum_0^M \frac{(kn)^m}{m!} \right\} = s \quad ^{15)}, \tag{4.233}$$

or

$$\lim_{M \to \infty} \sum_0^\infty s_n \int_{kn}^{(k+1)n} e^{-t} \frac{t^M}{M!} dt = s. \tag{4.234}$$

This is the form of definition which we shall find it most convenient to adopt. It enables us to verify at once that our definition satisfies the condition of consistency, *i. e.* that *any convergent series is summable with any radius k*. More generally we have

THEOREM 4.2. — *A series which is summable with radius k is summable, to the same sum, with any smaller radius.*

This theorem is plainly equivalent to that which follows [16]).

THEOREM 4.21. — *Suppose that the series*

$$f(x) = \sum a_n x^n$$

is convergent at a point x_0 on its circle of convergence, and that $0 < \alpha < 1$. *Then the* TAYLOR's *series for*

$$f\{\alpha x_0 + (1 - \alpha) x_0\},$$

viz.,

$$\sum \frac{f^{(n)}(\alpha x_0)}{n!} \{(1 - \alpha) x_0\}^n = \sum b_n \{(1 - \alpha) x_0\}^n$$

is also convergent.

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[15]) We write $u_n - u_{n+1} = \Delta u_n$.

[16]) We do not claim this theorem as new: it is certainly contained implicitly in earlier writings; but we do not remember having seen it stated explicitly.

We may suppose without loss of generality that $x_0 = 1$.

Then

$$t_n = b_0 + (1 - \alpha) b_1 + \cdots + (1 - \alpha)^n b_n \tag{4.24}$$

is equal to the coefficient of y^n in

$$\begin{cases} \dfrac{(1-\alpha)^{n+1} - y^{n+1}}{1 - \alpha - y} \sum b_n y^n = \dfrac{(1+\alpha)^{n+1} - y^{n+1}}{1 - \alpha - y} \sum a_n (\alpha + y)^n \\ \qquad = \{(1-\alpha)^{n+1} - y^{n+1}\} \sum s_n (\alpha + y)^n, \end{cases} \tag{4.25}$$

and so

$$t_n = (1 - \alpha)^{n+1} \left\{ s_n + (n+1) \alpha s_{n+1} + \frac{(n+1)(n+2)}{1.2} \alpha^2 s_{n+2} + \cdots \right\}. \tag{4.26}$$

The theorem is a straightforward deduction from this identity.

4.3. The « circle » method of summation is related in a very simple manner to the method defined in section 3.

THEOREM 4.3. — *Suppose that $s_n = o(\sqrt{n})$. Then summability with radius k implies summability $(E, \frac{1}{2}k)$, and conversely.*

The proof of this theorem is so much like that of Lemma 2.12 that it will not be necessary to do more than summarize its general lines.

Suppose that $\sum a_n$ is summable with radius k, and that its sum is zero. Then

$$\sum s_n \int_{kn}^{k(n+1)} e^{-t} \frac{t^M}{M!} dt = o(1), \tag{4.31}$$

when $M \to \infty$ by integral values. Suppose now that

$$\frac{M}{k} = \mu = [\mu] + \nu = m + \nu,$$

so that $0 \leqq \nu < 1$, and $n = m + h$, where h, as in 2.1, ranges between $-H\mu$ and $H\mu$.

Then

$$\int_{k(m+h)}^{k(m+h+1)} e^{-t} \frac{t^{k\mu}}{(k\mu)!} dt = k \int_0^1 e^{-k(m+h+\xi)} \frac{k^{k\mu}(m+h+\xi)^{k\mu}}{(k\mu)!} d\xi. \tag{4.33}$$

But it is easy to deduce from STIRLING's Theorem that

$$e^{-k(m+h+\xi)} \frac{k^{k\mu}(m+h+\xi)^{k\mu}}{(k\mu)!} = \frac{e^{-kh^2/2\mu}}{\sqrt{2\pi k\mu}} \left\{ 1 + O\left(\frac{|h|^3}{\mu^2}\right) \right\}, \tag{4.34}$$

uniformly for $0 \leqq \nu < 1$ and $0 \leqq \xi \leqq 1$. It follows, by arguments similar to those used in the proof of Lemma 2.12, that

$$\sqrt{\frac{k}{2\pi\mu}} \sum_{-\infty}^{\infty} e^{-kh^2/2\mu} s_{h+m} = o(1), \tag{4.35}$$

and hence that $\sum a_n$ is summable $(E, \frac{1}{2}k)$ to sum zero.

A similar argument suffices to prove the converse proposition.

If in particular we suppose that $k = 1$, we obtain

THEOREM 4.31. — *If* $a_n = o(1)$, *then the existence of any one of the limits specified in Theorem* 3.4 *implies, and is implied by, the summability of the series* $\sum a_n$ *with radius* 1.

4.4. The summability of a series by BOREL's method is therefore equivalent to its summability with radius unity in all cases in which $a_n = o(1)$, and implies (though it is not necessarily implied by) its summability with any lesser radius.

We have also:

THEOREM 4.4. — *The system of* TAUBER*ian theorems which holds for the « circle » method of summation is the same as that which holds for* BOREL'*s method or for a method of type* (E, a). *In particular, if a series* $\sum a_n$ *is summable by the circle method, then*

(i) *the condition*

$$a_n = O\left(\frac{1}{\sqrt{n}}\right)$$

implies convergence, and

(ii) *the condition*

$$a_n = O(1)$$

implies summability $(C, 1)$, *and*

(iii) *the condition*

$$a_n = O(1)$$

implies

$$s_n = o(\sqrt{n}).$$

The proof of these theorems involves no difficulty beyond that implied in an adaptation and rearrangement of arguments used already; and the same applies to

THEOREM 4.41. (RIESZ'*s Theorem*). — *If* $a_n = o(1)$ *then the summability* (B) *of* $\sum a_n$ *implies, and is implied by, the convergence of the series for* $f\left(\frac{1}{2} + y\right)$ *when* $y = \frac{1}{2}$.

4.5. We conclude with a few miscellaneous remarks.

(i) We asserted at the end of section 3 that summability (E, a) for a particular value of a involved summability for any smaller a whenever $a_n = o(1)$. The truth of this assertion follows now from Theorems 4.2 and 4.3.

(ii) The analysis employed in the proof of Theorem 4.2 suggests yet another definition of the sum of a divergent series viz., as the limit of

$$(1 - \alpha)^{n+1}\left\{s_n + (n + 1)\alpha s_{n+1} + \frac{(n + 1)(n + 2)}{1.2}\alpha^2 s_{n+2} + \cdots\right\},$$

where α is any number between 0 and 1. The properties of this definition would naturally resemble those of the other definitions which we have been discussing.

(iii). The « circle » method may be generalised by supposing that

$$F(y) = \sum a_n e^{-\lambda_n y}$$

where (λ_n) is any ascending sequence which has the limit ∞ and is such that the series is convergent for all values of y whose real part is positive. The « sum » of

$\sum a_n$ is then again defined as being the sum of (4.22), or as

$$\lim_{M\to\infty} \sum s_n \int_{\lambda_n}^{\lambda_{n+1}} e^{-t} \frac{t^M}{M!} dt.$$

The definition reduces to that already considered when λ_n is a constant multiple of n. These methods are connected with those used by RIESZ [17]) for effecting the analytic continuation of a function represented partially by a DIRICHLET's series.

The « TAUBERian » condition which corresponds to $a_n = o\left(\frac{1}{\sqrt{n}}\right)$ is now

$$a_n = o\left(\frac{\lambda_n - \lambda_{n-1}}{\sqrt{\lambda_n}}\right).$$

(iv) It is of some interest to find a theorem which shall enable us to infer the summability (B) of $\sum a_n$ from the properties of the analytic function $f(x)$. The following theorem, which we state without proof [18]), was found independently by RIESZ and by ourselves.

THEOREM 4.5. — *If $\sum a_n x^n$ is a power series whose radius of convergence is unity, and the function $f(x)$ which it represents satisfies the condition*

$$|f(x) - s| < A|1 - x|^\alpha,$$

where $\alpha > 0$, at all points inside a circle which lies inside the circle of convergence and touches it at the point $x = 1$, then $\sum a_n$ is summable (B) to sum s.

Cambridge, September 1915.

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[17]) M. RIESZ, *Sur la représentation analytique des fonctions définies par des séries de* DIRICHLET [Acta Mathematica, t. XXXV (1912), pp. 253-270].

[18]) The proof depends on considerations of function theory and differs entirely in character from those given in this paper.