

## ON THE ZEROS OF JACOBIAN FUNCTIONS

By H. F. BAKER.

[Received July 15th, 1911.—Read June 8th, 1911.]

M. POINCARÉ obtained, in 1883 (*Bull. de Sc. Math. de France*, t. xi, p. 132), a formula for the number and sum of the common solutions of two theta functions of two variables. Herr Wirtinger, in 1896 (*Monatsh. f. Math. u. Physik*, vii, Jahrg., pp. 1–25), obtained the corresponding results for  $n$  Jacobian functions of  $n$  variables, but under the condition that only one bilinear relation connects the periods of any two arguments. M. Humbert, in 1899 (*Liouville's Jour. de Math.*, 1899, pp. 254, 270), obtained, for two Jacobian functions of two variables based on the periods arising on a Riemann surface when subject to a further bilinear relation—so called intermediary functions—a formula for the number of solutions, deduced from Poincaré's formula for theta functions.

It remained, unless unknown to me this has already been done, to ascertain what bilinear relations among the periods, in addition to the universal bilinear relation, are necessary for the existence of Jacobian functions of  $n$  variables based on the most general possible periods; and to obtain a formula for the number, and for the sum, of the common solutions of  $n$  such functions. The results are contained in the following pages (see p. 373).

In order to make clear the fundamental assumptions it is necessary to give references, and I have ventured to refer to my own volume, *Multiply-Periodic Functions*, Cambridge, 1907, using the abbreviation *M.P.F.* This volume contains an exposition of Wirtinger's paper quoted above, which is also the basis of the process here adopted, but it will be seen that in what follows, though the case dealt with is much more general, a considerable simplification is introduced into the inductive argument. Further, as the notation of matrices, which is used throughout, offers difficulties for some readers, I have given details of the work and examples, in some cases at greater length than is strictly necessary; and for similar reasons have included two preliminary sections dealing with formulæ which essentially are well known.

The sections are as follows :—

- § 1. Known Results for Symmetric Functions, pp. 354–356.
2. A Relation connecting a System of Matrices, pp. 357–360.
3. The Conditions for the Existence of a Jacobian Function, pp. 360–372.
4. The Number and Sum of the Common Zeros of  $n$  Jacobian Functions of  $n$  Variables, pp. 372–388.
5. Some Particular Cases, pp. 388–395.

### 1. Known Results for Symmetric Functions.

Considering  $n$  fundamental quantities (roots)  $\alpha, \beta, \gamma, \dots, \lambda$ , and a set of positive integral exponents  $p, q, r, \dots$ , or  $p_1, p_2, p_3, \dots$ , we denote  $\Sigma \alpha^p$ , extended to all the roots, by  $(p)$ ; we denote  $\Sigma \alpha^p \beta^q$ , extended to every two different roots, by  $(p, q)$ ; similarly,  $\Sigma \alpha^p \beta^q \gamma^r$  by  $(p, q, r)$ ; and so on. Then, by  $(p, q)_\alpha$  is meant  $(p, q)$ , extended to all the roots except  $\alpha$ ; by  $(p, q, r)_\alpha$  is meant  $(p, q, r)$ , extended to all the roots except  $\alpha$ ; and so on.

We manifestly have

$$(p, q) = (p, q)_\alpha + \alpha^p (q)_\alpha + \alpha^q (p)_\alpha, \quad (q) = (q)_\alpha + \alpha^q, \quad (p) = (p)_\alpha + \alpha^p,$$

and hence 
$$(p, q)_\alpha = (p, q) - \alpha^p (q) - \alpha^q (p) + 2\alpha^{p+q}.$$

Again, using this last formula to express  $(q, r)_\alpha$ ,  $(r, p)_\alpha$ , and  $(p, q)_\alpha$ , and, noticing that

$$(p, q, r) = (p, q, r)_\alpha + \alpha^p (q, r)_\alpha + \alpha^q (r, p)_\alpha + \alpha^r (p, q)_\alpha,$$

we deduce that

$$(p, q, r)_\alpha = (p, q, r) - [\alpha^p (q, r) + \alpha^q (r, p) + \alpha^r (p, q)] \\ + (2!) [\alpha^{q+r} (p) + \alpha^{r+p} (q) + \alpha^{p+q} (r)] - (3!) \alpha^{p+q+r}.$$

In general, by the obvious formula,

$$(p_1, p_2, \dots, p_k)$$

$$= (p_1, p_2, \dots, p_k)_\alpha + \alpha^{p_1} (p_2, p_3, \dots, p_k)_\alpha + \dots + \alpha^{p_k} (p_1, p_2, \dots, p_{k-1})_\alpha,$$

we find

$$(p_1, p_2, \dots, p_k)_\alpha$$

$$= (p_1, p_2, \dots, p_k) - \sum_1 \alpha^{p_1} (p_2, \dots, p_k) + (2!) \sum_2 \alpha^{p_1+p_2} (p_3, \dots, p_k) \\ - (3!) \sum_3 \alpha^{p_1+p_2+p_3} (p_4, \dots, p_k) + \dots,$$

where  $\sum_i$  denotes a summation extending to every  $k-i$  of the exponents  $p_1, p_2, \dots, p_k$ ; and hence, as we obviously have

$$(p_1, p_2, \dots, p_k, p) = \alpha^p (p_1, \dots, p_k)_\alpha + \beta^p (p_1, \dots, p_k)_\beta + \dots + \lambda^p (p_1, \dots, p_k)_\lambda,$$

we find

$$\begin{aligned} (p_1, p, \dots, p_k, p) \\ = (p)(p_1, p_2, \dots, p_k) - \sum_1 (p+p_1)(p_2, \dots, p_k) \\ + (2!) \sum_2 (p+p_1+p_2)(p_3, \dots, p_k) - \dots, \quad (A) \end{aligned}$$

from which any form  $(p, q, r, \dots)$  can be calculated. For example, we find

$$\begin{aligned} (p, q) &= (p)(q) - (p+q) \\ (p, q, r) &= (p)(q)(r) - (p)(q+r) - (q)(r+p) - (r)(p+q) + 2(p+q+r) \\ (p, q, r, s) &= (p)(q)(r)(s) \\ &\quad - (p)(s)(q+r) - (q)(r)(p+s) - \dots - \dots - \dots - \dots \\ &\quad + (p+s)(q+r) + \dots + \dots \\ &\quad + 2(p)(q+r+s) + \dots + \dots \\ &\quad - 6(p+q+r+s). \end{aligned}$$

In, particular, in formula (A), if  $p_1 = p_2 = \dots = p_k = p = 1$ , and  $(p_1, \dots, p_k)$  be replaced by  $(k!) \sum a_1 a_2 \dots a_k = (k!) P_k$ , say, we obtain

$$\begin{aligned} [(k+1)!] P_{k+1} \\ = s_1 P_k (k!) - k s_2 [(k-1)!] P_{k-1} + (2!) \frac{k!}{2! (k-2)!} s_3 [(k-2)!] P_{k-2} - \dots \end{aligned}$$

$$\text{or} \quad (k+1) P_{k+1} = s_1 P_k - s_2 P_{k-1} + s_3 P_{k-2} - \dots,$$

the well known formula, usually called after Newton, where

$$s_k = \alpha^k + \beta^k + \dots,$$

from which is found

$$P_k = \frac{1}{k!} \begin{vmatrix} s_1 & s_2 & s_3 & \dots & s_k \\ k-1 & s_1 & s_2 & \dots & s_{k-1} \\ 0 & k-2 & s_1 & \dots & s_{k-2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & s_1 \end{vmatrix}.$$

Thus

$$\begin{aligned} (x-\alpha)(x-\beta)\dots(x-\lambda) \\ = x^n - x^{n-1} s_1 + x^{n-2} \frac{s_1^2 - s_2}{2!} - x^{n-3} \frac{s_1^3 - 3s_1 s_2 + 2s_3}{3!} \\ + x^{n-4} \frac{s_1^4 - 6s_1^2 s_2 + 8s_1 s_3 + 3s_2^2 - 6s_4}{4!} - \dots \end{aligned}$$

Conversely, if we replace  $\alpha$  by

$$h_1 \alpha^p + h_2 \alpha^q + h_3 \alpha^r + \dots,$$

replace  $\beta$  by

$$h_1 \beta^p + h_2 \beta^q + h_3 \beta^r + \dots,$$

and so on, the coefficient of  $h_1 h_2 \dots h_k$  in  $P_k$  is  $(p, q, r, \dots)$ , where there are  $k$  exponents. For, by this substitution,  $s_1$  becomes

$$\sigma_1 = h_1(p) + h_2(q) + \dots,$$

$s_2$  becomes

$$\sigma_2 = (2!) h_1 h_2 (p+q) + \dots,$$

where it is immaterial to write the terms involving squares of  $h_1, h_2, \dots$  and  $s_3$  becomes

$$\sigma_3 = (3!) h_1 h_2 h_3 (p+q+r) + \dots,$$

and so on; and hence the coefficient of  $h_1 h_2$  in  $\frac{\sigma_1^2 - \sigma_2}{2!}$  is  $(p)(q) - (p+q)$ , the coefficient of  $h_1 h_2 h_3$  in

$$\frac{1}{3!} (\sigma_1^3 - 3\sigma_1 \sigma_2 + 2\sigma_3)$$

is  $\frac{1}{3!} \{ (3!) (p)(q)(r) - 3 \Sigma (p) 2(q+r) + 2(3!) (p+q+r) \}$

or

$$(p)(q)(r) - \Sigma (p)(q+r) + 2(p+q+r),$$

and the coefficient of  $h_1 h_2 h_3 h_4$  in

$$\frac{1}{4!} (\sigma_1^4 - 6\sigma_1^2 \sigma_2 + 8\sigma_1 \sigma_3 + 3\sigma_2^2 - 6\sigma_4)$$

is  $\frac{1}{4!} \{ (p)(q)(r)(s) - 6 \Sigma 2(p)(q) 2(r+s) + 8 \Sigma (p)(3!)(q+r+s) + 3 \Sigma 8(p+q)(r+s) - 6(4!)(p+q+r+s) \}$

or

$$(p)(q)(r)(s) - \Sigma (p)(q)(r+s) + 2 \Sigma (p)(q+r+s) + \Sigma (p+q)(r+s) - (3!)(p+q+r+s).$$

And evidently the same is true in general.

The formula (A) is unsymmetrical with regard to the  $k$  suffixes. If we take the  $(k+1)$  possible forms of the right side and add, and afterwards change  $k+1$  into  $k$ , we have the symmetrical formula

$$\begin{aligned} & k(p_1, p_2, \dots, p_k) \\ &= \sum_i (p_1)(p_2, p_3, \dots, p_k) - (2!) \sum_i (p_1+p_2)(p_3, p_4, \dots, p_k) \\ & \quad + (3!) \sum_i (p_1+p_2+p_3)(p_4, \dots, p_k) - \dots, \quad (B) \end{aligned}$$

where, as before,  $\sum_i$  denotes a summation containing a term corresponding to every selection of  $i$  of the exponents  $p_1, p_2, \dots, p_k$ .

## 2. A Relation connecting a System of Matrices.

Let  $D$  be a matrix of  $2n$  rows and columns which is the product of two skew symmetrical matrices also of  $2n$  rows and columns, of non-vanishing determinant, say equal to  $\Delta K$ . The matrix  $\Delta - \lambda K^{-1}$  is skew symmetrical, and its determinant is a perfect square; the determinant  $|D - \lambda|$  is therefore also a perfect square, and its roots will consist, say, of  $\xi_1, \dots, \xi_n$ , each repeated. Similarly the roots of  $|D^r - \lambda|$  will consist of  $\xi_1^r, \dots, \xi_n^r$ , each repeated. Thus, if  $\sigma_r$  be the sum of  $\xi_1^r, \dots, \xi_n^r$ ,

$$\begin{aligned}\sigma_r &= \xi_1^r + \dots + \xi_n^r = \frac{1}{2} [\text{sum of } r\text{-th powers of the roots of } D] \\ &= \frac{1}{2} \sum_{l=1}^{2n} (D^r)_{l,l},\end{aligned}$$

where  $(D^r)_{l,l}$  is the  $(l, l)$ -th element of the matrix  $D^r$ , that is, the  $l$ -th diagonal element of this matrix. If, then, we take (cf. § 1 above)

$$\varpi_1 = \sigma_1, \quad \varpi_2 = \frac{1}{2!} (\sigma_1^2 - \sigma_2), \quad \varpi_3 = \frac{1}{3!} (\sigma_1^3 - 3\sigma_1\sigma_2 + 2\sigma_3),$$

and so on, the matrix  $D$  satisfies the equation

$$D^n - \varpi_1 D^{n-1} + \varpi_2 D^{n-2} - \varpi_3 D^{n-3} + \dots = 0.$$

Herein suppose that  $D$  is  $h_1 D_1 + \dots + h_n D_n$ , where  $h_1, h_2, \dots, h_n$  are undetermined parameters, and each of  $D_1, \dots, D_n$  is a matrix of  $2n$  rows and columns which is the product of two skew symmetrical matrices of the same type; in the application to be made below  $D_1, D_2, \dots$  are of the forms  $\Delta_1 K, \Delta_2 K, \dots$ , where  $K$  is the same for each. Then every term in the equation becomes a polynomial in  $h_1, h_2, \dots, h_n$  of dimension  $n$ ; we desire to pick out the coefficient of the product  $h_1 h_2 \dots h_n$  in the result.

For this let  $(D')_{12}$  denote  $D_1 D_2 + D_2 D_1$ , let  $(D')_{123}$  denote the sum of the six products of  $D_1, D_2, D_3$  in all possible orders, and so on. Then let  $\delta_1$  denote half the sum of the diagonal elements of  $D_1$ , let  $\delta_{12}$  denote half the sum, divided by  $(2!)$ , of the diagonal elements of  $(D')_{12}$ , let  $\delta_{123}$  denote half the sum, divided by  $(3!)$ , of the diagonal elements of  $(D')_{123}$ , and so on, so that

$$\delta_{12} = \frac{1}{2(2!)} \sum_{l=1}^{2n} (D_1 D_2 + D_2 D_1)_{l,l},$$

$$\begin{aligned}\delta_{123} &= \frac{1}{2(3!)} \sum_{l=1}^{2n} (D_1 D_2 D_3 + D_1 D_3 D_2 + D_2 D_3 D_1 + D_2 D_1 D_3 + D_3 D_1 D_2 \\ &\quad + D_3 D_2 D_1)_{l,l},\end{aligned}$$

and so on. Then the term  $\sigma_r$  becomes

$$\frac{1}{2} \sum_{l=1}^{2n} \{ (h_1 D_1 + \dots + h_n D_n)^r \}_{l,l}$$

or

$$(r!) \sum_r h_1 h_2 \dots h_r \delta_{12 \dots r},$$

where  $\sum_r$  denotes a summation extending to every  $r$  of the letters  $h_1, h_2, \dots, h_n$ , and terms involving powers of  $h_1, \dots, h_n$  above the first are not written down.

Also in  $D^n$  the coefficient of  $h_1 h_2 \dots h_n$  is  $(D')_{12 \dots n}$ , defined above. In  $D^{n-1}$ , the terms not containing squares or higher powers of  $h_1, \dots, h_n$  are

$$\sum_1 h_1 h_2 \dots h_{n-1} (D')_{12 \dots (n-1)};$$

multiplying this by  $\varpi_1$ , which is equal to

$$\sum_1 h_1 \delta_1,$$

the coefficient of  $h_1 h_2 \dots h_n$  in  $-\varpi_1 D^{n-1}$  is

$$-\sum_1 \delta_1 (D')_{23 \dots n},$$

where  $\sum_1$  relates to a summation of  $n$  constituents. Again, in  $D^{n-2}$  the terms not containing higher powers of  $h_1, \dots, h_n$  are

$$\sum_2 h_3 h_4 \dots h_n (D')_{34 \dots n};$$

while taking

$$\varpi_2 = \frac{1}{2!} (\sigma_1^2 - \sigma_2),$$

and retaining only terms which are of first order in each of  $h_1, \dots, h_n$ , we have to consider

$$\frac{1}{2!} \{ 2 \sum h_1 h_2 \delta_1 \delta_2 - (2!) \sum h_1 h_2 \delta_{12} \}$$

or

$$\sum h_1 h_2 (\delta_1 \delta_2 - \delta_{12}).$$

Thus the coefficient of  $h_1 h_2 \dots h_n$  in  $\varpi_2 D^{n-2}$  is

$$\sum_2 (\delta_1 \delta_2 - \delta_{12}) (D')_{34 \dots n}.$$

Similarly the coefficient of  $h_1 h_2 h_3$  in

$$\varpi_3 = \frac{1}{3!} (\sigma_1^3 - 3\sigma_1 \sigma_2 + 2\sigma_3)$$

is

$$\frac{1}{3!} \{ 3! \delta_1 \delta_2 \delta_3 - 3(2!) (\delta_1 \delta_{23} + \delta_2 \delta_{31} + \delta_3 \delta_{12}) + 2(3!) \delta_{123} \},$$

and the coefficient of  $h_1 h_2 \dots h_n$  in  $-\varpi_3 D^{n-3}$  is

$$-\sum_3 [\delta_1 \delta_2 \delta_3 - (\delta_1 \delta_{23} + \delta_2 \delta_{31} + \delta_3 \delta_{12}) + 2\delta_{123}] (D')_{45 \dots n}.$$

In general, let  $\phi_{12\dots k}$  denote the coefficient of  $h_1 h_2 \dots h_k$  in

$$\varpi_k = \frac{1}{k!} \begin{vmatrix} \sigma_1 & \sigma_2 & \dots & \sigma_k \\ k-1 & \sigma_1 & \dots & \sigma_{k-1} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 1 & \sigma_1 \end{vmatrix};$$

then, as we have  $k\varpi_k = \sigma_1\varpi_{k-1} - \sigma_2\varpi_{k-2} + \sigma_3\varpi_{k-3} - \dots$ ,

and in  $\sigma_r$  the coefficient of  $h_1 h_2 \dots h_r$  is  $(r!)\delta_{123\dots r}$ , while in  $\varpi_{k-r}$  the coefficient of  $h_{r+1} h_{r+2} \dots h_k$  is  $\phi_{r+1, \dots, k}$ , we see that the function  $\phi_{123\dots k}$  can be calculated from the formula

$$k\phi_{12\dots k} = \sum_1 \delta_1 \phi_{23\dots k} - (2!)\sum_2 \delta_{12} \phi_{34\dots k} + (3!)\sum_3 \delta_{123} \phi_{45\dots k} - \dots,$$

and has the same form in terms of  $\delta_1, \delta_2, \dots, \delta_{12}, \delta_{13}, \dots, \delta_{123}, \dots$  as has the function  $(p_1, p_2, \dots, p_k)$ , considered in § 1, in terms of

$$(p_1), (p_2), \dots, (p_1+p_2), (p_1+p_3), \dots, (p_1+p_2+p_3), \dots$$

Thus, finally, we have the identity, connecting the matrices  $D_1, \dots, D_n$ ,

$$(D')_{12\dots n} - \sum_1 \delta_1 (D')_{23\dots n} + \sum_2 \phi_{12} (D')_{34\dots n} - \sum_3 \phi_{123} (D')_{45\dots n} \\ + \dots + (-1)^n \phi_{12\dots n} = 0,$$

where  $\sum_i$  denotes a summation extending to every  $i$  of the numbers 1, 2, ...,  $n$ .

In particular, for  $n = 2$  we have

$$D_1 D_2 + D_2 D_1 - \delta_1 D_2 - \delta_2 D_1 + \delta_1 \delta_2 - \delta_{12} = 0,$$

and for  $n = 3$  we have

$$(D_1 D_2 D_3 + \dots) - \sum \delta_1 (D_2 D_3 + D_3 D_2) \\ + \sum (\delta_1 \delta_2 - \delta_{12}) D_3 - (\delta_1 \delta_2 \delta_3 - \delta_1 \delta_{23} \dots + 2\delta_{123}) = 0.$$

If we further put, symbolically,

$$(D')_{12\dots k} = D'_1 D'_2 \dots D'_k, \\ \phi_{12\dots k} = \phi'_1 \phi'_2 \dots \phi'_k,$$

the identity can be written

$$(D'_1 - \phi'_1)(D'_2 - \phi'_2) \dots (D'_n - \phi'_n) = 0.$$

*Note I.*—It is an evident consequence of what precedes that  $\phi_{12\dots n}$  is the coefficient of  $h_1 h_2 \dots h_n$  in the square root of the determinant of the matrix  $h_1 D_1 + \dots + h_n D_n$ ; and, in accordance with this, the function which  $\phi_{12\dots n}$  reduces to when the matrices  $D_1, D_2, \dots, D_n$  are all the same is equal to  $(n!)$  times the square root of the determinant of  $|D|$ .

*Note II.*—If the expression on the left side of the equation above, after multiplication by  $(-1)^n$ , be denoted by  $\psi_{12\dots n}$ , so that, for instance,

$$\psi_1 = \phi_1 - D_1,$$

$$\psi_{12} = \phi_{12} - D_1\phi_2 - D_2\phi_1 + D_1D_2 + D_2D_1,$$

we have  $\phi_{12} - D_1\psi_2 - D_2\psi_1 = \phi_{12} - D_1(\phi_2 - D_2) - D_2(\phi_1 - D_1) = \psi_{12}$ ,

$$\begin{aligned} \phi_{123} - D_1\psi_{23} - D_2\psi_{31} - D_3\psi_{12} \\ = \phi_{123} - D_1\{\phi_{23} - \phi_2D_3 - \phi_3D_2 + D_2D_3 + D_3D_2\} \\ - D_2\{\phi_{31} - \phi_3D_1 - \phi_1D_3 + D_3D_1 + D_1D_3\} \\ - D_3\{\phi_{12} - \phi_1D_2 - \phi_2D_1 + D_1D_2 + D_2D_1\} \\ = \psi_{123}, \end{aligned}$$

and so, in general,

$$\phi_{12\dots k} - D_1\psi_{23\dots k} - D_2\psi_{13\dots k} - \dots - D_k\psi_{12\dots(k-1)} = \psi_{12\dots k},$$

this depending merely on the fact that

$$D_1(D')_{23\dots k} + D_2(D')_{13\dots k} + \dots + D_k(D')_{12\dots(k-1)} = (D')_{12\dots k},$$

where, as before,  $(D')_{12\dots m}$  is the sum of the  $(m!)$  products in all possible orders of  $D_1, D_2, \dots, D_m$ .

### 3. *The Conditions for the Existence of a Jacobian Function.*

We know (M.P.F., p. 286) that for an integral function  $f$ , of  $n$  variables  $u_1, \dots, u_n$ , to satisfy  $2n$  equations of the form

$$f(u_1 + a_{1,r}, u_2 + a_{2,r}, \dots, u_n + a_{n,r}) = e^{2\pi i b^{(r)}(u + \frac{1}{2}a^{(r)}) + c_r} f(u_1, \dots, u_n),$$

where  $b^{(r)}(u + \frac{1}{2}a^{(r)}) = b_{1,r}u_1 + \dots + b_{n,r}u_n + \frac{1}{2}(b_{1,r}a_{1,r} + \dots + b_{n,r}a_{n,r})$ ,

it is necessary that the  $2n$  sets of *periods*,  $a$ , should satisfy conditions expressible in the forms

$$aK\bar{a} = 0, \quad -iaK\bar{a}_0x_0 > 0.$$

$$\bar{b}a - \bar{a}b = \Delta,$$

where  $a, b$  denote the matrices of type  $(n, 2n)$  formed with the elements  $a_{\alpha, i}, b_{\beta, i}$ ,  $\bar{a}$  and  $\bar{b}$  denote the matrices of type  $(2n, n)$  formed by transposing these,  $K$  is a skew symmetrical matrix of type  $(2n, 2n)$  whose elements are integers,  $x$  denotes any row of  $n$  quantities and  $x_0$  the row of  $n$  conjugate complex quantities, and  $\Delta$  denotes some matrix of type  $(2n, 2n)$  of which every constituent is an integer. It is a consequence of the theory that both  $K$  and  $\Delta$  have a non-vanishing determinant.



With a view to investigating these relations more in detail we first repeat the well known argument\* by which it is shown that without loss of generality we may suppose the matrix of periods  $a$  to have the form  $(e^{-1}, \sigma)$ , wherein  $e^{-1}$ , of type  $(n, n)$ , has zeros except in the diagonal, this diagonal consisting of the inverse positive integers  $e_1^{-1}, e_2^{-1}, \dots, e_n^{-1}$ , of which  $e_{r+1}/e_r$  is an integer, and wherein  $\sigma$  denotes a symmetrical matrix of type  $(n, n)$  such that, when  $\xi_1, \dots, \xi_n$  are any real quantities, the real part of the quadratic form  $i\sigma\xi^2$  is negative. To establish this result, take a unitary matrix  $g$ , of integers, of type  $(2n, 2n)$ , such that

$$gK\bar{g} = \begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix},$$

where  $e$  denotes the diagonal matrix whose elements are  $e_1, \dots, e_n$ , and  $e_{r+1}/e_r$  is an integer, as we know to be possible (M.P.F., p. 311); then put, as the definition of  $\omega$  and  $\omega'$ ,

$$a = (\omega, \omega')g;$$

the condition

$$-iaK\bar{a}_0x_0x > 0$$

gives then

$$-i(\omega, \omega') \begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix} \begin{pmatrix} \bar{\omega}_0 \\ \omega'_0 \end{pmatrix} x_0x > 0,$$

or

$$-i(\omega'e, -\omega e) \begin{pmatrix} \bar{\omega}_0 \\ \omega'_0 \end{pmatrix} x_0x > 0,$$

or

$$-i(\omega'e\bar{\omega}_0x_0x - \omega e\bar{\omega}'_0x_0x) > 0,$$

that is, if  $\xi = \bar{\omega}x$ ,  $\xi_0 = \bar{\omega}_0x_0$ ,

$$-i(\omega'e\xi_0x - \omega'_0e\xi x_0) > 0;$$

this shows that  $\omega$  is of non-vanishing determinant, since else we could choose  $x_1, \dots, x_n$  to make  $\xi$  or  $\bar{\omega}x$  consist of  $n$  zeros, in which case the left side would not be  $> 0$ .

We can hence write, in place of  $a = (\omega, \omega')g$ ,

$$a = \omega e(e^{-1}, \sigma)g,$$

where  $\sigma$  is defined by

$$\omega e\sigma = \omega';$$

then the condition

$$aK\bar{a} = 0$$

\* The quantities  $u, \sigma$  of the text are the quantities  $\frac{1}{r}\mu^{-1}u$  and  $\frac{\sigma}{r}$  of M.P.F., p. 227, and are those occurring in the theta functions of M.P.F., p. 283.

becomes  $\omega e(e^{-1}, \sigma) \begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix} \begin{pmatrix} e^{-1} \\ \bar{\sigma} \end{pmatrix} \bar{\omega} e = 0,$

or  $\omega e(\sigma e, -1) \begin{pmatrix} e^{-1} \\ \bar{\sigma} \end{pmatrix} \bar{\omega} e = 0,$

or  $\omega e(\sigma - \bar{\sigma}) \bar{\omega} e = 0,$

shewing that  $\sigma = \bar{\sigma}.$

The previous inequality then gives

$$-i(\omega e \sigma e \xi_0 x - \omega_0 e \sigma_0 e \xi_0 x_0) > 0$$

or  $-i(e \sigma e \xi_0^{\bar{e}} - e \sigma_0 e \xi_0^{\bar{e}}) > 0,$

where we have utilized the fact that, if  $\mu$  be a matrix of type  $(n, n)$  and  $p, q$  two rows each of  $n$  quantities,  $\mu p q = \bar{\mu} q p$ ; if  $\eta = e \xi$ , this is the same as

$$-i(\sigma - \sigma_0) \eta \eta_0 > 0,$$

or, if  $\sigma = \rho + i\tau$ ,  $\eta = \eta_1 + i\eta_2$ , is the same as

$$-i 2i\tau(\eta_1^2 + \eta_2^2) > 0,$$

or  $\tau(\eta_1^2 + \eta_2^2) > 0,$

which is the condition that the imaginary part of  $i\sigma\eta^2$  should be negative. The transformation is thus established.

If now,\* beside

$$a = \omega e(e^{-1}, \sigma)g,$$

in order to define  $\beta$ , of type  $(n, 2n)$ , and to define the new arguments  $v_1, \dots, v_n$ , we put  $\bar{\omega} e.b = \beta g \quad u = \omega e.v,$

\* It is shewn, M.P.F., p. 298, that, if  $R$  be the least positive integer such that  $R\Delta^{-1}$  consists of integers, and  $\gamma$  a unitary matrix such that  $\gamma R\Delta^{-1}\bar{\gamma} = \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}$ , where  $t$  is a diagonal matrix of positive integers  $t_1, \dots, t_n$  for which  $t_{n+1}/t_n$  is integral, and  $t_1 = 1$ , and  $(\omega, \omega') = \alpha\gamma^{-1}$ ,  $\tau = (\omega t)^{-1}\omega'$ , and if  $w = (\omega t)^{-1}u + \text{constant}$ , a single Jacobian function, after multiplication by the exponential of a (non-homogeneous) quadratic function of  $u$ , satisfies a defining equation

$$F(w + t^{-1}m + \tau m') = e^{-2\pi i R m' (w + i\tau m')} F(w),$$

in which  $m, m'$  are rows of  $n$  arbitrary integers, and is the sum of  $\sqrt{|\Delta|}$  theta functions of arguments  $Rw$ , with periods  $R\tau$ , that is, each of the form

$$\sum_{\mathbf{h}} \exp \left\{ 2\pi i R w \left( n + \frac{th}{R} \right) + i\pi R \tau \left( n + \frac{th}{R} \right)^2 \right\},$$

wherein  $h$  is a row of positive integers in which  $h_n < R t_n^{-1}$ .

But this reduction depends on  $\Delta$  and, therefore on the matrix  $b$  proper to the function, while the reduction of the periods given in the text depends only on the original periods  $\alpha$ , which are the same for all the Jacobian functions of the set to be considered subsequently.

and, when  $m$  is any row of  $2n$  integers, we define a row of  $2n$  integers,  $\mu$ , by means of

$$gm = \mu,$$

$$\begin{aligned} \text{we have} \quad bm(u + \tfrac{1}{2}am) &= bm \cdot wev + \tfrac{1}{2}bm \cdot we(e^{-1}, \sigma)\mu \\ &= \bar{w}e \cdot bmv + \tfrac{1}{2}\bar{w}e \cdot bm(e^{-1}, \sigma)\mu \\ &= \beta\mu v + \tfrac{1}{2}\beta\mu(e^{-1}, \sigma)\mu \\ &= \beta\mu[v + \tfrac{1}{2}(e^{-1}, \sigma)\mu], \end{aligned}$$

which is of the same form in  $v, (e^{-1}, \sigma), \beta, \mu$  as was the original in  $u, a, b, m$ .

While also

$$\begin{aligned} \bar{b}a - \bar{a}b &= \bar{b}we(e^{-1}, \sigma)g - \bar{g}\left(\begin{smallmatrix} e^{-1} \\ \sigma \end{smallmatrix}\right)\bar{w}eb \\ &= \bar{g}\left[\bar{\beta}(e^{-1}, \sigma) - \left(\begin{smallmatrix} e^{-1} \\ \sigma \end{smallmatrix}\right)\beta\right]g, \end{aligned}$$

which, since  $g$  is a unitary matrix, shews that

$$\bar{\beta}(e^{-1}, \sigma) - \left(\begin{smallmatrix} e^{-1} \\ \sigma \end{smallmatrix}\right)\beta$$

is a matrix of integers. All the conditions previously to be satisfied by  $a, b$ , and  $u$  are thus satisfied by  $(e^{-1}, \sigma), \beta$ , and  $v$ .

The condition  $aK\bar{a} = 0$  becomes, as we have seen, the identical equation

$$(e^{-1}, \sigma)\begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix}\begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix} = 0;$$

we shall put

$$E = \begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix},$$

so that the condition becomes

$$(e^{-1}, \sigma)E\begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix} = 0.$$

It may also be noticed, for the sake of a future application, that

$$(\bar{b}a - \bar{a}b)K$$

is

$$\bar{g}\left\{\left[\bar{\beta}(e^{-1}, \sigma) - \left(\begin{smallmatrix} e^{-1} \\ \sigma \end{smallmatrix}\right)\beta\right]E\right\}\bar{g}^{-1},$$

or, say,

$$\Delta K = \bar{g}\nabla E\bar{g}^{-1};$$

and hence, if

$$\Delta_1 = \bar{b}_1a - \bar{a}b_1, \quad \Delta_2 = \bar{b}_2a - a\bar{b}_2,$$

$$\Delta_1 K \cdot \Delta_2 K = \bar{g}\nabla_1 E \nabla_2 E \bar{g}^{-1}.$$

Also that

$$aK\Delta K\bar{a}$$

is equal to

$$aK(\bar{b}a - \bar{a}b)K\bar{a}$$

$$\begin{aligned} &= \omega e(e^{-1}, \sigma) gK\bar{g} \left[ \bar{\beta}(e^{-1}, \sigma) - \begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix} \beta \right] gK\bar{g} \begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix} \bar{\omega}e \\ &= \omega e \left\{ (e^{-1}, \sigma) E \nabla E \begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix} \right\} \bar{\omega}e, \end{aligned}$$

and similarly that

$$aK\Delta K\Delta K\bar{a}$$

is equal to

$$\omega e \left\{ (e^{-1}, \sigma) E \nabla E \nabla E \begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix} \right\} \bar{\omega}e.$$

In examining the conditions

$$\bar{b}a - \bar{a}b = \Delta,$$

we can thus without loss of generality suppose  $a = (e^{-1}, \sigma)$ . We shall continue to denote the arguments by  $u$ .

Then first, by multiplying  $f(u)$  by an expression of the form

$$e^{\pi i A u^2},$$

where  $A$  is a symmetrical matrix of type  $(n, n)$ , so that  $Au^2$  is a quadratic form in  $u_1, u_2, \dots, u_n$ , we can suppose that in the matrix  $b$  all the elements

$$b_{\alpha, j} \quad (\alpha = 1, 2, \dots, n; j = 1, 2, \dots, \alpha)$$

are zero, so that  $b$  has the form

$$\begin{pmatrix} 0, & b_{12}, & \dots, & b_{1n}, & b_{1, n+1}, & \dots, & b_{1, 2n} \\ 0, & 0, & \dots, & b_{2n}, & b_{2, n+1}, & \dots, & b_{2, 2n} \\ \cdot & \cdot & \dots & \cdot & \cdot & \dots & \cdot \\ 0, & 0, & \dots, & 0, & b_{n, n+1}, & \dots, & b_{n, 2n} \end{pmatrix}.$$

For, after such multiplication, when in  $f(u)$  the arguments are simultaneously increased by the rows  $am$ , where  $m$  denotes a row of  $2n$  integers, the quotient  $f(u+am)/f(u)$  is  $e^{2\pi i H}$ , where, save for parts independent of  $u$ ,

$$H = \frac{1}{2}A(u+am)^2 - \frac{1}{2}Au^2 + bm(u + \frac{1}{2}am),$$

and this is, effectively,  $(Aa+b)m(u + \frac{1}{2}am)$ ,

namely, the matrix  $b$  is replaced by  $b+Aa$ . To obtain the specified reduction in the matrix  $b$ , it is thus sufficient to take the matrix  $A$ , so that in

$$A(e^{-1}, \sigma) + b$$

the elements  $(\alpha, j)$  above described may all be zero; that is, if we write

$$b = (b_1, b_2),$$

in which each of  $b_1, b_2$  is of type  $(n, n)$ , it is sufficient to take  $A$  of such form that in

$$Ae^{-1} + b_1$$

the elements  $(\alpha, j)$  above spoken of shall all be zero; this requires only

$$A_{\alpha, j} = -b_{\alpha, j}e_{\alpha} \\ (\alpha = 1, 2, \dots, n; j = 1, 2, \dots, a),$$

whereby the matrix  $A$  is determined. It may be noticed that, whatever form is given to the symmetrical matrix  $A$ , the replacement of  $b$  in  $\bar{b}a - \bar{a}b$  by  $b + Aa$  does not alter the value of  $\bar{b}a - \bar{a}b$ .

Assume this reduction in the matrix  $b, = (b_1, b_2)$ , and now write  $\bar{b}a - \bar{a}b = \Delta$ , that is, suppose

$$\begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \end{pmatrix} (e^{-1}, \sigma) - \begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix} (b_1, b_2) = \begin{pmatrix} P & -Q \\ \bar{Q} & R \end{pmatrix}$$

or 
$$\begin{pmatrix} \bar{b}_1 e^{-1} & \bar{b}_1 \sigma \\ \bar{b}_2 e^{-1} & \bar{b}_2 \sigma \end{pmatrix} - \begin{pmatrix} e^{-1} b_1 & e^{-1} b_2 \\ \sigma b_1 & \sigma b_2 \end{pmatrix} = \begin{pmatrix} P & -Q \\ \bar{Q} & R \end{pmatrix}$$

or 
$$\begin{pmatrix} \bar{b}_1 e^{-1} - e^{-1} b_1 & \bar{b}_1 \sigma - e^{-1} b_2 \\ \bar{b}_2 e^{-1} - \sigma b_1 & \bar{b}_2 \sigma - \sigma b_2 \end{pmatrix} = \begin{pmatrix} P & -Q \\ \bar{Q} & R \end{pmatrix},$$

where  $Q$  is a matrix of type  $(n, n)$  consisting of integers, and  $P, R$  are skew symmetrical matrices of integers, also of type  $(n, n)$ ; further denote by  $P_1$  the matrix of type  $(n, n)$  whose elements to the right of the principal diagonal are the negatives of those in  $P$ , the other elements in  $P_1$  being zeros, so that

$$P = \bar{P}_1 - P_1;$$

we may express this, perhaps, by saying that  $-P_1$  is formed by the north east half of the elements of  $P$ , the remaining elements being zeros. Then the conditions to be satisfied are

$$\bar{b}_1 e^{-1} - e^{-1} b_1 = \bar{P}_1 - P_1,$$

$$e^{-1} b_2 - \bar{b}_1 \sigma = Q,$$

$$\bar{b}_2 \sigma - \sigma b_2 = R.$$

The first of these is  $e^{-1} b_1 - P_1 = \bar{b}_1 e^{-1} - \bar{P}_1$ ,

and shews that  $e^{-1} b_1 - P_1$  is a symmetrical matrix; but in  $P_1$  and in  $b_1$ , in consequence of the reduction supposed made in  $b$ , the elements below the principal diagonal are all zeros. Hence

$$b_1 = eP_1,$$

and then from the second equation above

$$b_2 = e(Q + \bar{P}_1 e\sigma),$$

so that the matrix  $b$  is entirely determined when the elements of the matrix  $\Delta$  are given, in terms indeed of the elements of the first  $n$  rows of  $\Delta$ . The remaining conditions

$$\bar{b}_2 \sigma - \sigma b_2 = R$$

require, however,

$$(\sigma e P_1 + \bar{Q}) e \sigma - \sigma e (Q + \bar{P}_1 e \sigma) = R,$$

or, as  $P = \bar{P}_1 - P_1$ , they require

$$\sigma e P e \sigma + \sigma e Q - \bar{Q} e \sigma + R = 0. \quad (C)$$

These conditions can be expressed in the form

$$(e^{-1}, \sigma) E \Delta E \begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix} = 0;$$

$$\begin{aligned} \text{for } E \Delta E &= \begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix} \begin{pmatrix} P & -Q \\ \bar{Q} & R \end{pmatrix} \begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix} = \begin{pmatrix} -e\bar{Q} & -eR \\ eP & -eQ \end{pmatrix} \begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix} \\ &= \begin{pmatrix} -eRe & e\bar{Q}e \\ -eQe & -ePe \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \text{and hence } -(e^{-1}, \sigma) E \Delta E \begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix} &= (Re + \sigma e Q e, -\bar{Q} e + \sigma e P e) \begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix} \\ &= R + \sigma e Q - \bar{Q} e \sigma + \sigma e P e \sigma. \end{aligned}$$

The matrix on the left side of (C), or the matrix  $E \Delta E$ , being skew symmetrical, the conditions (C) are equivalent to  $\frac{1}{2}n(n-1)$  conditions; in fact, a relation

$$a M \bar{a} = 0,$$

wherein  $M$  is skew symmetrical, of type  $(2n, 2n)$ , is the same as

$$\sum_{i,j}^{1 \dots 2n} M_{ij} (a_{pi} a_{qj} - a_{pj} a_{qi}) = 0,$$

and expresses the same bilinear relation connecting the periods of every pair, such as  $u_p, u_q$ , of the  $n$  arguments. But while the numerical coefficients in the bilinear relations (C) are determined when the form of the matrix  $\Delta$  is given, the converse is not true; for, evidently, in virtue of

$$(e^{-1}, \sigma) E \begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix} = 0,$$

the relation

$$(e^{-1}, \sigma) E \Delta E \begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix} = 0$$

would remain unchanged if  $\Delta$  were replaced by  $\Delta + rE^{-1}$ , which is equally a skew symmetric matrix of integers, provided  $r$  is an integer divisible by  $e_n$ . This change is equivalent to replacing  $Q$  by  $Q - re^{-1}$ , which leaves unaffected the terms

$$\sigma eQ - \bar{Q}e\sigma$$

of the equation (C). In virtue of

$$b = \{eP_1, e(Q + \bar{P}_1e\sigma)\},$$

it is also equivalent to replacing  $b$  by  $b - r(0, 1)$ , where  $0$  denotes the zero matrix of type  $(n, n)$ , and  $1$  denotes the unit matrix of type  $(n, n)$ .

Returning for a moment to the unreduced form for the periods, and the matrix  $b$ , the conditions

$$(e^{-1}, \sigma) E \begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix} = 0, \quad (e^{-1}, \sigma) E \Delta E \begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix} = 0,$$

are equivalent, as we have seen, respectively to

$$aK\bar{a} = 0, \quad aK\nabla K\bar{a} = 0,$$

where  $\nabla$  is for the present used to denote  $ba - \bar{a}b$ . It is at once evident that the latter relation, being

$$aK\bar{b}.aK\bar{a} - aK\bar{a}.bK\bar{a} = 0,$$

follows from the former  $aK\bar{a} = 0$ . It is equally evident that

$$aK\nabla K\nabla K\bar{a} = 0;$$

for this is  $aK\bar{b}.aK\nabla K\bar{a} - aK\bar{a}.bK\nabla K\bar{a} = 0$ .

Similarly it can be proved that

$$a(K\nabla)^m \nabla^{-1} \bar{a} = 0,$$

whatever integer  $m$  may be. We have previously shewn (p. 357) the existence of an equation

$$\sum_{m=1}^n p_m (K\nabla)^m = 1,$$

where  $p_m$  is a number; from this we have

$$\nabla^{-1} = \sum_{m=1}^n p_m (K\nabla)^m \nabla^{-1},$$

so that also

$$a\nabla^{-1} \bar{a} = 0.$$

And this again follows at once from

$$\nabla = \bar{b}a - \bar{a}b = \bar{A} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A,$$

where  $A = \begin{pmatrix} a \\ b \end{pmatrix}$  is of type  $(2n, 2n)$ , and 1 here denotes the unit matrix of type  $(n, n)$  (cf. M.P.F., p. 288). For this gives

$$\nabla^{-1} = A^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{A}^{-1},$$

and hence 
$$\begin{pmatrix} a \\ b \end{pmatrix} \nabla^{-1} (\bar{a}, \bar{b}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

from which we deduce

$$a \nabla^{-1} \bar{a} = 0, \quad a \nabla^{-1} \bar{b} = 1, \quad b \nabla^{-1} \bar{b} = 0.$$

*Resuming the results of this section so far obtained we thus have:* The existence of a single-valued integral function  $f(u)$ , of the  $n$  variables  $u_1, \dots, u_n$ , which with the periods  $a = (e^{-1}, \sigma)$ , subject to the relations

$$aE\bar{a} = 0,$$

satisfies, for any  $2n$  integers  $m$ , equations of the form

$$f(u+am)/f(u) = \exp 2\pi i [bm(u + \frac{1}{2}am) + H_m],$$

where  $H_m$  is independent of  $u$ , requires the existence of further  $\frac{1}{2}n(n-1)$  bilinear relations

$$aE\Delta E\bar{a} = 0,$$

where  $\Delta$ , which we write in the form

$$\Delta = \begin{pmatrix} P & -Q \\ \bar{Q} & R \end{pmatrix},$$

is a skew symmetrical matrix of integers of type  $(2n, 2n)$ , to which the matrix  $\bar{b}a - \bar{a}b$  is required to be equal. Conversely, writing  $P = \bar{P}_1 - P_1$ , where  $-P_1$  has zeros below the principal diagonal, but above this diagonal agrees with  $P$ , and assuming  $aE\bar{a} = 0$ ,  $aE\Delta E\bar{a} = 0$ , the relation

$$\bar{b}a - \bar{a}b = \Delta$$

requires, and is ensured by taking

$$b = e(P_1, Q + \bar{P}_1 e\sigma).$$

The explicit forms of the necessary bilinear relations connecting the periods, other than the universal relations expressed by  $aE\bar{a} = 0$ , do not, however, determine the matrix  $\Delta$  or the matrix  $b$ ; when these bilinear relations are given we may equally take, for corresponding forms of  $b$  and



$\Delta$ , respectively

$$b = k\beta - r(0, 1),$$

$$\Delta = k\nabla + rE^{-1},$$

where  $k$  is any integer and  $r$  is any integer divisible by  $e_n$ , and if

$$\nabla = \begin{pmatrix} p & -q \\ \bar{q} & p \end{pmatrix}, \quad p = \bar{p}_1 - p_1,$$

then

$$\beta = e(p_1, q + \bar{p}_1 e\sigma).$$

Herein  $b$  is chosen so that all the elements  $(\alpha, j)$  in which

$$\alpha = 1, 2, \dots, n \quad \text{and} \quad j = 1, 2, \dots, \alpha$$

are zero. A more general form of  $b$ , obtained by multiplying the Jacobian function by the exponential of a quadratic function of  $u_1, \dots, u_n$ , is obtained by adding  $Aa$  to the  $b$ , where  $A$  is a symmetric matrix of the type  $(n, n)$ .

In addition to the fact that the matrix  $\bar{b}a - \bar{a}b = \Delta$  must consist of integers, it is necessary (M.P.F., p. 287) that if  $z$  be any row of  $2n$  quantities satisfying  $az = 0$ , and  $z_0$  the row of conjugate complex quantities, that

$$-i\Delta zz_0 > 0.$$

When  $a = (e^{-1}, \sigma)$ , if we put  $z = (\eta, y)$ , each of  $\eta$  and  $y$  being a row of  $n$  quantities, the equation  $az = 0$  gives  $e^{-1}\eta + \sigma y = 0$ , or  $\eta = -e\sigma y$ , and therefore

$$z = \begin{pmatrix} -e\sigma \\ 1 \end{pmatrix} y,$$

where 1 denotes the unit matrix of type  $(n, n)$ , namely,

$$z = \begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix} \begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix} y = E\bar{a}y.$$

Hence the inequality becomes

$$-i\Delta E\bar{a}y \cdot E\bar{a}_0 y_0 > 0,$$

that is,

$$-i\alpha E\Delta E\bar{a}_0 y_0 y > 0.$$

Denoting the real skew symmetrical matrix  $E\Delta E$  by  $N$ , and putting  $\sigma = \sigma_1 + i\sigma_2$ , where  $\sigma_1, \sigma_2$  are real, and also

$$A = (e^{-1}, \sigma_1)N \begin{pmatrix} 0 \\ \sigma_2 \end{pmatrix}, \quad V = (0, \sigma_2)N \begin{pmatrix} 0 \\ \sigma_2 \end{pmatrix}, \quad W = (e^{-1}, \sigma_1)N \begin{pmatrix} e^{-1} \\ \sigma_1 \end{pmatrix},$$

the identity  $(e^{-1}, \sigma) N \begin{pmatrix} e^{-1} \\ \sigma \end{pmatrix} = 0$

is equivalent with  $W + iA - i\bar{A} - V = 0$ ,

so that  $W = V, \quad A = \bar{A}$ ,

or  $A$  is a symmetrical matrix, while  $V$  is obviously skew symmetrical.

Hence  $iaN\bar{a}_0 = i(e^{-1}, \sigma) N \begin{pmatrix} e^{-1} \\ \sigma_0 \end{pmatrix} = iW + A + \bar{A} + iV$ ,

and, putting  $y = x + i\xi$ ,  $y_0 = x - i\xi$ , so that each of  $x$ ,  $\xi$  is a row of  $n$  real quantities, the inequality

$$-iaE\Delta E\bar{a}_0 y_0 y > 0$$

is the same as  $(A + iV)(x - i\xi)(x + i\xi) < 0$ .

Since  $A(x\xi - \xi x) = 0, \quad Vx^2 = V\xi^2 = 0$ ;

the left side of this is  $A(x^2 + \xi^2) + V(\xi x - x\xi)$ ,

namely,  $\begin{pmatrix} A & V \\ -V & A \end{pmatrix} (x, \xi)(x, \xi)$ ,

and is a real quadratic form in  $2n$  variables; while

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left[ \begin{pmatrix} A & V \\ -V & A \end{pmatrix} - \lambda \right] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A - \lambda & V \\ -V & A - \lambda \end{pmatrix} = \begin{pmatrix} V & -A + \lambda \\ A - \lambda & V \end{pmatrix}$$

is a skew symmetrical matrix whose roots are all double. The real quadratic form in  $2n$  variables is thus capable of a form

$$\lambda_1(t_1^2 + u_1^2) + \dots + \lambda_n(t_n^2 + u_n^2),$$

wherein  $\lambda_1, \dots, \lambda_n$  are  $n$  real quantities, and  $t_1, u_1, \dots, t_n, u_n$  are  $2n$  real linear functions of  $x_1, \dots, \xi_n$ ; and the condition for the form to be constantly negative is that each of  $\lambda_1, \dots, \lambda_n$  be negative.

The reduction to this form can be made in steps as follows: let

$$A = (a_{pq}), \quad V = (v_{pq}), \quad b_{pq} = \frac{(a_{p1} + iv_{p1})(a_{1q} + iv_{1q})}{a_{11}}, \quad B = (b_{p1}),$$

the denominator  $a_{11}$  not being zero in virtue of the condition; thence

$$b_{1q} = a_{1q} + iv_{1q}, \quad b_{p1} = a_{p1} + iv_{p1},$$

and

$$B = A' + iV' = (a'_{pq}) + i(v'_{pq}),$$

the matrix  $A'$  being symmetrical, and  $V'$  skew symmetrical. We at once

find that

$$\begin{aligned} (A' + iV')(x - i\xi)(x + i\xi) \\ &= \frac{1}{a_{11}} \{ (a_{11}x_1 + \dots + a_{1n}x_n + v_{12}\xi_2 + \dots + v_{1n}\xi_n)^2 \\ &\quad + (a_{11}\xi_1 + \dots + a_{1n}\xi_n - v_{12}x_2 - \dots - v_{1n}x_n)^2 \} \\ &= \frac{1}{a_{11}} (t_1^2 + u_1^2), \end{aligned}$$

and can write

$$\begin{aligned} (A + iV)(x - i\xi)(x + i\xi) \\ &= \frac{1}{a_{11}} (t_1^2 + u_1^2) + [(A - A') + i(V - V')](x - i\xi)(x + i\xi), \end{aligned}$$

where on the right the form at the end does not contain  $x_1$  or  $\xi_1$ , but is of the same form as the original in  $x_2, \dots, x_n, \xi_2, \dots, \xi_n$ . To it the same reduction can then be applied.

If we put, as above,  $\Delta = \begin{pmatrix} P & -Q \\ \bar{Q} & R \end{pmatrix},$

the condition to be satisfied is

$$\begin{pmatrix} \sigma_2 e(Q + Pe\sigma_1) & -\sigma_2 ePe\sigma_2 \\ \sigma_2 ePe\sigma_2 & \sigma_2 e(Q + Pe\sigma_1) \end{pmatrix} (x, \xi)(x, \xi) < 0.$$

Putting  $\Delta = rE^{-1} + k\nabla, \quad P = kp, \quad Q = -re^{-1} + kq$  (see p. 369)

and taking  $p = 0$ , the condition breaks down into

$$\sigma_2(keq - r)x^2 < 0.$$

When  $n = 2$ , the conditions for this are that the coefficient of  $x_1^2$  should be negative, and that the determinant of the form in  $x_1, x_2$  should be positive. If

$$q = \begin{pmatrix} 0 & -A \\ C & -B \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} \xi & \eta \\ \eta & \xi \end{pmatrix},$$

we have

$$keq - r = \begin{pmatrix} -r & -Ae_1k \\ Ce_2k & -r - Be_2k \end{pmatrix},$$

and the former of these conditions is

$$-\xi r + Ce_2k\eta < 0,$$

while the latter is

$$|\sigma_2| |keq - r| > 0,$$

which, in virtue of  $|\sigma_2| > 0$ , reduces to

$$|keq-r| > 0,$$

that is,

$$r^2 + rkBe_2 + ACe_1e_2k^2 > 0.$$

For this case, and with  $e_1 = 1$ ,  $e_2 = 1$ , the conditions are given by Humbert, *Liouville*, 1899, pp. 259, 260.

#### 4. *The Number and Sum of the Common Zeros of $n$ Jacobian Functions of $n$ Variables.*

We consider now a set of  $n$  Jacobian functions of  $n$  variables,  $u_1, \dots, u_n$ , say

$$f_1(u-v^{(1)}), \dots, f_n(u-v^{(n)}),$$

where  $f_r(u)$  is subject to equations of the form

$$f_r(u+am) = e^{2\pi i H_r} f_r(u),$$

in which

$$H_r = b_r m(u + \frac{1}{2}am) + c_r, m,$$

$m$  denoting a row of  $2n$  integers, and  $c_r, m$  being independent of  $u_1, \dots, u_n$ ; we represent by  $\Delta_r$  the matrix of integers  $\bar{b}_r a - \bar{a} b_r$ , by  $K$  a matrix of integers of type  $(2n, 2n)$  such that the universal relation connecting the periods is

$$aK\bar{a} = 0$$

(see M.P.F., p. 286), and we put  $D_r = \Delta_r K$ . Also we put (cf. p. 357)

$$\delta_r = \frac{1}{2} \sum_{l=1}^{2n} (D_r)_{l,l}, \quad \delta_{12} = \frac{1}{2(2!)} \sum_{l=1}^{2n} (D_1 D_2 + D_2 D_1),$$

and so on, so that, for instance,  $\delta_{123}$  is one half the sum, divided by  $(3!)$  of the diagonal elements in the matrix which is the sum of all the  $(3!)$  products of  $D_1, D_2, D_3$ . Then we put

$$\phi_r = \delta_r, \quad \phi_{12} = \delta_1 \delta_2 - \delta_{12}, \quad \phi_{123} = \delta_1 \delta_2 \delta_3 - \delta_1 \delta_{23} - \delta_2 \delta_{31} - \delta_3 \delta_{12} + 2\delta_{123},$$

$$\phi_{1234} = \delta_1 \delta_2 \delta_3 \delta_4 - \sum \delta_1 \delta_2 \delta_{34} + \sum \delta_{12} \delta_{34} + 2\sum \delta_1 \delta_{234} - 6\delta_{1234},$$

and so on, the general form being obtainable by the equation (see above, p. 359)

$$k\phi_{12\dots k} = \sum \delta_1 \phi_{23\dots k} - (2!) \sum \delta_{12} \phi_{34\dots k} + (3!) \sum \delta_{123} \phi_{45\dots k} - \dots$$

Finally, we denote by  $H$  the positive integer which is the square root of the determinant of the skew symmetrical matrix  $K$ .

Then, if  $v^{(m)}$  denote a set of  $n$  constants, we prove that the  $n$  equations

$$f_1(u-v^{(1)}) = 0, \dots, f_n(u-v^{(n)}) = 0,$$

have a number of sets of solutions for  $u_1, \dots, u_n$ , no two of which are congruent in regard to the periods  $\alpha$ , given by

$$\frac{1}{H} \phi_{12 \dots n},$$

and that the sum of the values of  $u_\alpha$  at these solutions, save for quantities independent of  $v^{(1)}, \dots, v^{(n)}$ , is equal to the  $\alpha$ -th of the quantities expressed by

$$\frac{1}{H} [aK\psi_{23 \dots n} \bar{b}_1 v^{(1)} + aK\psi_{13 \dots n} \bar{b}_2 v^{(2)} + \dots + aK\psi_{12 \dots (n-1)} \bar{b}_n v^{(n)}],$$

where  $\psi_{12 \dots n}$ , as before (p. 360), denotes the matrix which can be symbolically denoted by

$$(\phi'_1 - D'_1) \dots (\phi'_k - D'_k),$$

where, after multiplication, we are to replace  $\phi'_1$  by  $\phi_1$ ,  $\phi'_1 \phi'_2$  by  $\phi_{12}$ ,  $\phi'_1 \phi'_2 \phi'_3$  by  $\phi_{123}$ , and so on, and are to replace  $D'_1$  by  $D_1$ ,  $D'_1 D'_2$  by  $(D')_{12}$ , that is,  $D_1 D_2 + D_2 D_1$ , and  $D'_1 D'_2 D'_3$  by  $(D')_{123}$ , that is, by

$$D_1 D_2 D_3 + D_1 D_3 D_2 + D_2 D_3 D_1 + D_2 D_1 D_3 + D_3 D_1 D_2 + D_3 D_2 D_1,$$

and so on. Thus the term in  $v^{(n)}$  in the sum is symbolically

$$\frac{1}{H} aK \frac{(\phi'_1 - D'_1) \dots (\phi'_n - D'_n)}{\phi'_m - D'_m} \bar{b}_m v^{(m)}.$$

As already remarked (p. 359), when the functions  $f_1(u), \dots, f_n(u)$  are the same, and therefore the matrices  $D_1, \dots, D_n$  all reduce to the same matrix  $D$ , the function  $\phi_{12 \dots n}$  reduces to  $(n!)$  times the square root of the determinant of the matrix  $D$ . In general  $\phi_{12 \dots n}$  is the coefficient of  $h_1 h_2 \dots h_n$  in the square root of the determinant of the matrix

$$h_1 D_1 + \dots + h_n D_n;$$

this is the matrix  $(\bar{B}a - \bar{a}B)K$ , where  $B$  is the matrix arising by adding periods to the arguments for the function

$$F(u) = [f_1(u)]^{h_1} [f_2(u)]^{h_2} \dots [f_n(u)]^{h_n},$$

which gives  $F(u + am)/F(u) = \exp 2\pi i \{ Bm(u + \frac{1}{2}am) + C \}$ ,

$C$  being independent of  $u$ .

If any difference  $u - v^{(m)}$  be written in the form

$$u - \frac{1}{n} [v^{(1)} + \dots + v^{(n)}] - w^{(m)},$$

we can similarly obtain a formula for the sum of the incongruent values of

$$w = u - \frac{1}{n} [v^{(1)} + \dots + v^{(n)}],$$

which satisfy the equations

$$f_1(w - v^{(1)}) = 0, \dots, f_n(w - v^{(n)}) = 0,$$

which, so far as it depends upon  $v^{(1)}, \dots, v^{(n)}$ , is linear in  $w^{(1)}, \dots, w^{(n)}$ . Each of the quantities

$$w^{(n)} = v^{(n)} - \frac{1}{n} [v^{(1)} + \dots + v^{(n)}]$$

vanishes, however, when we put

$$v^{(1)} = v^{(2)} = \dots = v^{(n)}.$$

The number of values of  $w$  being  $\phi_{12\dots n}/H$ , we thus infer that, for arbitrary  $v$ ,

$$\frac{1}{H} \phi_{12\dots n} v = \frac{1}{H} \sum_{i=1}^n aKP_i \bar{b}_i v,$$

where  $P_i$  (symbolically) =  $\frac{(\phi'_1 - D'_1) \dots (\phi'_n - D'_n)}{\phi'_i - D'_i}$ .

Now we know that any  $n$  quantities  $v$  can be written in the form  $a\xi$ , where  $\xi$  is a row of  $2n$  real quantities; we have only to equate real and imaginary parts, and utilise the known properties of the matrix  $a$  (M.P.F., p. 225); also we have  $aK\bar{a} = 0$ ,  $aKD\bar{a} = 0$ ,  $aKD_1D_2\bar{a} = 0$  (see above, p. 367), and so on, and therefore, as will abundantly appear,

$$aKP_i\bar{a} = 0;$$

hence we can write

$$aKP_i\bar{b}_i v = aKP_i\bar{b}_i a\xi = aKP_i(\Delta_i + \bar{a}b_i)\xi = aKP_i\Delta_i\xi,$$

and obtain  $a\left(\phi_{12\dots n} - \sum_{i=1}^n KP_i\Delta_i\right)\xi = 0;$

but, if  $\eta$  be a set of  $2n$  real quantities, an equation  $a\eta = 0$ , involving  $a_0\eta = 0$ , involves, in virtue of the inequality,

$$-iaK\bar{a}_0x_0x > 0$$

(M.P.F., p. 286), that  $\eta = 0$ ; thus the equation here leads to

$$\phi_{12\dots n} - \sum_{i=1}^n KP_i\Delta_i = 0; \quad (P)$$

now, for instance,  $P_1$  is  $\psi_{23\dots n}$ , namely,

$$\phi_{23\dots n} - D_2 \phi_{34\dots n} - D_3 \phi_{24\dots n} - \dots + (D_2 D_3 + D_3 D_2) \phi_{4\dots n} + \dots,$$

and the matrix obtained by transposing  $KP_1\Delta_1$  is thus

$$\begin{aligned} \Delta_1 \bar{P}_1 K &= \Delta_1 \{ \phi_{23\dots n} - K\Delta_2 \phi_{34\dots n} - K\Delta_3 \phi_{24\dots n} - \dots \\ &\quad + (K\Delta_3 K\Delta_2 + K\Delta_2 K\Delta_3) \phi_{4\dots n} + \dots \} K \\ &= D_1 \phi_{23\dots n} - D_1 D_2 \phi_{34\dots n} - D_1 D_3 \phi_{24\dots n} - \dots \\ &\quad + D_1 (D_3 D_2 + D_2 D_3) \phi_{4\dots n} + \dots \\ &= D_1 \psi_{23\dots n}, \end{aligned}$$

and so on; thus the equation (P) here leads, after transposition, to

$$\phi_{12\dots n} - D_1 \psi_{23\dots n} - D_2 \psi_{134\dots n} - \dots - D_n \psi_{12\dots(n-1)} = 0,$$

that is (see p. 360) to  $\psi_{12\dots n} = 0$ ,

the fundamental equation connecting the matrices  $D_1, \dots, D_n$ .

It thus appears that the formula for the sum of the incongruent solutions includes the formula for the number of solutions.

In order to prove these results, we utilise the theory of a Riemann surface. The Jacobian functions  $f(u)$  give rise, by taking the second partial logarithmic differential coefficients, to functions of  $u_1, \dots, u_n$ , with no essential singularities for finite values of these, which are periodic with the system of periods  $a$ . There can be constructed then a Riemann surface with  $p$  ( $\geq n$ ) everywhere finite integrals upon which  $u_1, \dots, u_n$  may be regarded as a defective system of everywhere finite integrals (M.P.F., Chap. vii). The period system of the integrals  $u$  upon this Riemann surface, a matrix  $\Pi$  of  $n$  rows and  $2p$  columns, is then capable of the form  $ak$ , where  $a$  is the period system here employed and  $k$  a matrix of integers of type  $(2n, 2p)$ . (Cf. M.P.F., p. 286.)

For this Riemann surface we require a preliminary lemma. Let  $\phi$  be a function of the  $n$  integrals of the first kind,  $u_1(x), \dots, u_n(x)$ , which is single valued on the dissected Riemann surface, and is capable of expression about any place of this as a series of integral powers of the parameter of the place, consisting either of only positive powers, or of such a series multiplied by a finite negative power; which, further, when  $(x)$  passes from a point on the right side of the  $j$ -th period loop to the corresponding point on the left side—in which case  $u$  is increased by  $ak^{(j)}$ , where  $k^{(j)}$  denotes the  $j$ -th column of the matrix  $k$ —is multiplied by  $\exp(2\pi i H_j)$ , where  $H_j = bk^{(j)}u + ck^{(j)}v + d_j$ ,

$b$  and  $c$  denoting matrices of type  $(n, 2n)$ , so that  $bk^{(j)}, ck^{(j)}$  are each

a row of  $n$  quantities,  $v$  (like  $u$ ) being a row of  $n$  quantities, and  $d_j$  a single quantity depending on  $j$ , but independent of  $(x)$  and  $v$ ; then

(1) The difference between the number of places  $(x)$  where the function  $\phi$  under consideration vanishes to the first order, and the number of places  $(x)$  where it has a pole of the first order, or the difference between the aggregate number of zeros and poles when these are multiple, is

$$\frac{1}{2} \sum_{l=1}^{2n} \{(\bar{b}a - \bar{a}b)K\}_{l,l},$$

where  $K = k\epsilon_{2p}\bar{k}$ ,  $\epsilon_{2p}$  denoting the matrix of type  $(2p, 2p)$  of which the elements  $(j, j+p)$  are each  $-1$  for  $j \leq p$ , and the elements  $(j+p, j)$  are each  $+1$ .

(2) The difference between the sum of the values of the  $\alpha$ -th of the integrals  $u_1, \dots, u_n$  at these zeros and at these poles is the  $\alpha$ -th of the quantities

$$-aK\bar{c}v,$$

save for an additive quantity not depending on  $v_1, \dots, v_n$ .

It is unnecessary to repeat the details of the demonstration (see M.P.F., p. 290). The first result is to be obtained by the integral

$$\frac{1}{2\pi i} \int d \log \phi$$

taken round the period loops of the Riemann surface, and is equal to

$$\begin{aligned} \sum_{\alpha=1}^n \frac{1}{2\pi i} \int \frac{1}{\phi} \frac{\partial \phi}{\partial u_\alpha} du_\alpha &= \frac{1}{2\pi i} \sum_{\alpha=1}^n \sum_{j=1}^p [(bk)_{\alpha j} (ak)_{\alpha, j+p} - (ak)_{\alpha j} (bk)_{\alpha, j+p}] \\ &= \frac{1}{2\pi i} \sum_{\alpha=1}^n \sum_{j=1}^p \sum_{l,m}^{1 \dots 2n} b_{\alpha l} a_{\alpha m} [k_{lj} k_{m, j+p} - k_{mj} k_{l, j+p}] \\ &= \frac{1}{2\pi i} \sum_{\alpha=1}^n \sum_{j=1}^p \sum_{l,m}^{1 \dots 2n} b_{\alpha l} a_{\alpha m} K_{m,l}, \\ &\quad (\alpha = 1, \dots, n; j = 1, \dots, p), \end{aligned}$$

which immediately gives the result above.

We may notice incidentally that another form for the result is

$$\sum_{\alpha=1}^n (aK\bar{b})_{\alpha, \alpha}.$$

To find the difference of the sum of the values of the  $\alpha$ -th of the integrals  $u(x)$  at the zeros and poles in question we are to evaluate the integral

$$-\frac{1}{2\pi i} \int du_\alpha \log \phi$$

taken round the period loops. Now for the two sides of the  $j$ -th loop



( $j \leq p$ ) the values of  $\log \phi$  differ, save possibly for a part of the form  $2\pi i N_j$ , where  $N_j$  is an integer, by

$$2\pi i [bk^{(j)}u + ck^{(j)}v + d_j],$$

and the contribution to the integral arising from the two sides of this loop is thus

$$-\int du_a [bk^{(j)}u + ck^{(j)}v + d_j]$$

taken along the positive side of this loop, that is, from the right to the left side of the loop ( $j+p$ ); a similar statement can be made for the ( $j+p$ )-th loop, but the integral obtained will be taken along the positive (or left) side of the loop ( $j+p$ ), from the left to the right side of the  $j$ -th loop. Altogether the loop pair ( $j, j+p$ ) gives rise to a contribution in which the part containing  $v$  is

$$-[(ak)_{a, j+p} ck^{(j)}v - (ak)_{a, j} ck^{(j+p)}v],$$

which is 
$$(ak)_{a, j} \sum_{\beta=1}^n (ck)_{\beta, j+p} v_{\beta} - (ak)_{a, j+p} \sum_{\beta=1}^n (ck)_{\beta, j} v_{\beta},$$

or 
$$\sum_{\beta=1}^n [(ak)_{a, j} (ck)_{\beta, j+p} - (ak)_{a, j+p} (ck)_{\beta, j}] v_{\beta},$$

or 
$$-\sum_{\beta=1}^n [(ak)_{a, j} \epsilon_{j, j+p} (ck)_{\beta, j+p} + (ak)_{a, j+p} \epsilon_{j+p, j} (ck)_{\beta, j}] v_{\beta},$$

or 
$$-\sum_{\beta=1}^n [(ak)_{a, j} (\epsilon \bar{c} k)_{j, \beta} + (ak)_{a, j+p} (\epsilon \bar{c} k)_{j+p, \beta}] v_{\beta}.$$

The sum of these for all the loop pairs is thus

$$-\sum_{\beta=1}^n (ak \epsilon \bar{c} v)_{a, \beta} v_{\beta}$$

or 
$$-(aK \bar{c} v)_a,$$

as was stated.

When the function  $\phi$  is a single Jacobian function  $f(u-v)$ , wherein  $u$  represents the integrals of the first kind on the Riemann surface, these formulæ give the number of places ( $x$ ), and the sum of the values of  $u(x)$ , at the zeros of  $f(u-v)$ , in the respective forms

$$\frac{1}{2} \sum_{l=1}^{2n} \{(\bar{b}a - \bar{a}b)K\}_{l, l}$$

and

$$-aK \bar{b}v,$$

save, in the latter case, for additive quantities independent of  $v$ .

With the help of this lemma we proceed to the proof of the general result stated at the beginning of this section. The Riemann surface being obtained, as explained, if  $u_1, \dots, u_n$  denote arbitrary values, the

number of sets of positions of  $n$  places  $(x_1), (x_2), \dots, (x_{n-1}), (x)$  which satisfy the  $n$  congruences

$$u_i(x_1) + \dots + u_i(x_{n-1}) + u_i(x) \equiv u_i \quad (i = 1, \dots, n)$$

is known (M.P.F., p. 250) to be  $H = \sqrt{|\bar{K}|}$ . Consider, then, the  $n$  equations

$$f_1[u(x_1) + \dots + u(x_{n-1}) + u(x) - v^{(1)}] = 0, \quad \dots,$$

$$f_n[u(x_1) + \dots + u(x_{n-1}) + u(x) - v^{(n)}] = 0; \quad (\text{A})$$

since to each value of  $u_i$  there correspond  $H$  sets of positions for  $(x_1), \dots, (x_{n-1}), (x)$ , and to each such set of positions there correspond, by the symmetry among  $(x_1), \dots, (x_{n-1}), (x)$ ,  $n$  positions for  $x$ , it follows from the general result enunciated at the beginning of this section that the  $n$  equations (A) are satisfied, with appropriate positions for  $(x_1), (x_2), \dots, (x_{n-1})$ , by  $n\phi_{12\dots n}$  positions for  $x$ ; and, conversely, the establishment of this result will prove the general result as to the number of solutions of the equations

$$f_1(u - v^{(1)}) = 0, \quad \dots, \quad f_n(u - v^{(n)}) = 0. \quad (\text{B})$$

Again, if  $(x_1^{(\lambda)}), \dots, (x_{n-1}^{(\lambda)}), (x^{(\lambda)})$  denote any one of the sets of solutions of equations (A), the sum of the values of  $u_a(x^{(\lambda)})$ , at the various positions of  $(x)$ , will be the same as the sum of the quantities

$$u_a(x_1^{(\lambda)}) + u_a(x_2^{(\lambda)}) + \dots + u_a(x_{n-1}^{(\lambda)}) + u_a(x^{(\lambda)}),$$

and therefore the same as the sum of the values of  $u_a$  at the solutions of the equations (B), when each such solution is counted as many times over as it arises for different sets of solutions of the equations

$$u_i(x_1) + \dots + u_i(x_{n-1}) + u_i(x) \equiv u_i \quad (i = 1, \dots, n),$$

that is,  $H$  times over. In other words, the sum of the values  $u_a(x^{(\lambda)})$  for the solutions  $(x)$  of the equations (A) is, in accordance with the general theorem enunciated above, given by the  $a$ -th element in

$$S_{n,0} = \sum_{i=1}^n \alpha K P_i \bar{b}_i v^{(i)},$$

save for quantities independent of  $v^{(1)}, \dots, v^{(n)}$ ; and, conversely, if this is proved, the general theorem enunciated is also proved as regards the sum of the values of  $u$  satisfying the equations (B).

We consider then the equations (A), but first of all under a somewhat generalised form; namely, denoting  $u(x_1), u(x_2), \dots, u(x_{n-1}), u(x)$  respec-

tively by  $u_1, u_2, \dots, u_{s-1}, u$ , we consider, as equations for  $(x)$  upon the Riemann surface, the  $s$  equations

$$f_1(u_1 + \dots + u_{s-1} - v^{(1)}) = 0, \quad \dots, \quad f_r(u_1 + \dots + u_{s-1} - v^{(r)}) = 0, \\ f_{r+1}(u_1 + \dots + u_{s-1} + u - v^{(r+1)}) = 0. \quad \dots, \quad f_s(u_1 + \dots + u_{s-1} + u - v^{(s)}) = 0,$$

in which  $r$  has one of the values  $0, 1, 2, \dots, (s-1)$ . These equations will be said to form a case  $(s, r)$ , and we denote the number of positions  $(x)$  satisfying them by  $N_{s,r}$ , and the sum of the corresponding values of  $u$  by  $S_{s,r}$ , this latter being a row of  $n$  quantities. We have seen that we desire to prove that

$$N_{n,0} = n\phi_{12\dots n}, \quad S_{n,0} = \sum_{i=1}^n aKP_i \bar{b}_i v^{(i)};$$

we obtain an explicit formula for

$$N_{s,r} - N_{s,r+1}, \quad S_{s,r} - S_{s,r+1},$$

where, if  $r+1 = s$ , both  $N_{s,r+1}$  and  $S_{s,r+1}$  are to be replaced by zero; this enables us then to find  $N_{s,0}$  and  $S_{s,0}$ .

For this consider first the equations

$$f_1(u_1 + \dots + u_{s-1} - v^{(1)}) = 0, \quad \dots, \quad f_r(u_1 + \dots + u_{s-1} - v^{(r)}) = 0, \\ f_{r+2}(u_1 + \dots + u_{s-1} + u - v^{(r+2)}) = 0, \quad \dots, \quad f_s(u_1 + \dots + u_{s-1} + u - v^{(s)}) = 0,$$

obtained from the set above by omission of the  $(r+1)$ -th of them; regard these last as equations for  $(x_1), \dots, (x_{s-1})$  in terms of  $(x)$ ; the supposed substitution of their values in the omitted equation

$$f_{r+1}(u_1 + \dots + u_{s-1} + u - v^{(r+1)}) = 0$$

leaves an equation for  $(x)$  only. The  $(s-1)$  equations for  $(x_1), \dots, (x_{s-1})$  form a case  $(s-1, 0)$ , in which, however,  $v^{(r+1)}, \dots, v^{(s-1)}$  are replaced by  $v^{(r+2)} - u, \dots, v^{(s)} - u$  respectively. The number of positions for any one of the places  $(x_1), \dots, (x_{s-1})$  which satisfy these equations being  $N_{s-1,0}$ , the number of sets of positions for  $(x_1), \dots, (x_{s-1})$  is  $\frac{1}{s-1} N_{s-1,0}$ , and the sum of the values of  $u$  at these sets of positions, or

$$\sum_{(\lambda)} (u_1^{(\lambda)} + \dots + u_{s-1}^{(\lambda)}),$$

is  $S_{s-1,0}$ , it being remembered that the  $v^{(r+1)}, \dots, v^{(s-1)}$  of the general case  $(s-1, 0)$  are here to be replaced by  $v^{(r+2)} - u, \dots, v^{(s)} - u$ . We shall assume, as the formula for  $S_{s-1,0}$  in the general case  $(s-1, 0)$ , an expression of the form

$$aKP_1 \bar{b}_1 v^{(1)} + \dots + aKP_{s-1} \bar{b}_{s-1} v^{(s-1)},$$

where  $P_1, \dots, P_{s-1}$  are matrices depending only on the functions

$f_1, \dots, f_{s-1}$  involved in that case; this is correct for  $s-1 = 1$ , and, when we have thence deduced the formula for the case  $(s, 0)$ , will be justified in general by induction. In the particular case  $(s-1, 0)$  now under consideration, instead of the functions  $f_{r+1}, \dots, f_{s-1}$  we have the functions  $f_{r+2}, \dots, f_s$ , and we shall denote the matrices consequently replacing  $P_1, \dots, P_{s-1}$  respectively by  $Q_1, \dots, Q_r, Q_{r+2}, \dots, Q_s$ ; similarly we shall denote the corresponding number  $N_{s-1, 0}$  by  $M_{s-1, 0}$ . The sum of the solutions for our particular case  $(s-1, 0)$  will then be

$$\begin{aligned} & \alpha K Q_1 \bar{b}_1 v^{(1)} + \dots + \alpha K Q_r \bar{b}_r v^{(r)} + \alpha K Q_{r+2} \bar{b}_{r+2} v^{(r+2)} + \dots + \alpha K Q_s \bar{b}_s v^{(s)} \\ & \quad - (\alpha K Q_{r+2} \bar{b}_{r+2} + \dots + \alpha K Q_s \bar{b}_s) u. \end{aligned}$$

The places  $(x_1), \dots, (x_{s-1})$  being thus supposed determined in terms of  $(x)$ , consider as a function of  $(x)$  the quotient

$$\prod_{\lambda} f_{r+1} [u_1^{(\lambda)} + \dots + u_{s-1}^{(\lambda)} + u - v^{(r+1)}] / f_{r+1} [u_1^{(\lambda)} + \dots + u_{s-1}^{(\lambda)} - v^{(r+1)}],$$

referring to the function  $f_{r+1}$  omitted in the determination of  $(x_1), \dots, (x_{s-1})$  in terms of  $(x)$ , there being as many factors in the numerator and denominator as sets  $(x_1), \dots, (x_{s-1})$  corresponding to a given position of  $(x)$ , namely,  $\frac{1}{s-1} M_{s-1, 0}$ .

When  $(x)$  changes its position from any point on the right of the  $j$ -th period loop to the corresponding place on the left side of this loop, the integral  $u$  will increase by the quantity  $ak^{(j)}$ , where  $k^{(j)}$  denotes the  $j$ -th column of the matrix  $k$ ; that is, the integral  $u_a$  will increase by the  $a$ -th of the  $n$  quantities  $ak^{(j)}$ ; the sum

$$u_1^{(\lambda)} + \dots + u_{s-1}^{(\lambda)}$$

will change in consequence into a quantity of the form

$$u_1^{(\mu)} + \dots + u_{s-1}^{(\mu)} + \alpha k t_{\mu},$$

where  $\mu$  refers to another set of positions for  $(x_1), \dots, (x_{s-1})$  belonging to the same position for  $(x)$ , and  $t_{\mu}$  denotes a set of  $2p$  integers remaining constant along the period loop, that is, the  $\alpha$ -th of the  $n$  quantities  $u_1^{(\lambda)} + \dots + u_{s-1}^{(\lambda)}$  will change into the  $\alpha$ -th of the new quantities. In consequence of this, the single factor

$$f_{r+1} [(u_1^{(\lambda)} + \dots + u_{s-1}^{(\lambda)} + u - v^{(r+1)})] / f_{r+1} [u_1^{(\lambda)} + \dots + u_{s-1}^{(\lambda)} - v^{(r+1)}]$$

will be multiplied by  $e^{2\pi i h}$ , where

$$\begin{aligned} h = & [b_{r+1} k t_{\mu} + b_{r+1} k^{(j)}] [u_1^{(\mu)} + \dots + u_{s-1}^{(\mu)} + u - v^{(r+1)}] \\ & - b_{r+1} k t_{\mu} [u_1^{(\mu)} + \dots + u_{s-1}^{(\mu)} - v^{(r+1)}] + D_j, \end{aligned}$$

$D_j$  being independent of  $(x)$  and  $v^{(1)}, \dots, v^{(s)}$ . Hence the product of such factors will be affected with a multiplier  $e^{2\pi i H}$ , where, save for quantities independent of  $(x)$  and  $v^{(1)}, \dots, v^{(s)}$ ,

$$H = \left( \sum_{(\lambda)} b_{r+1} k t_{\mu} \right) u + b_{r+1} k^{(j)} \sum_{(\lambda)} [u_1^{(\mu)} + \dots + u_{s-1}^{(\mu)}] + \frac{M_{s-1,0}}{s-1} b_{r+1} k^{(j)} [u - v^{(r+1)}].$$

Now from the formula above for the sum of the solutions, in our case  $(s-1, 0)$ , we can write down the sum of the expressions

$$u_1^{(\lambda)} + \dots + u_{s-1}^{(\lambda)},$$

and also the sum of the expressions

$$u_1^{(\mu)} + \dots + u_{s-1}^{(\mu)} + a k t_{\mu};$$

by subtraction we thence obtain

$$a k \sum_{(\lambda)} t_{\mu} = -(a K Q_{r+2} \bar{b}_{r+2} + \dots + a K Q_s \bar{b}_s) a k^{(j)};$$

we have previously remarked that  $a K \bar{a} = 0$ ; it will be seen in the course of this argument that, if we assume

$$a K P_i \bar{a} = 0 \quad (i = 1, \dots, s-1),$$

the matrices  $P_i$  being, as above explained, those that arise in the solution of the case  $(s-1, 0)$ , then it will also follow, for the case  $(s, 0)$ , that

$$a K P_i^{(s)} \bar{a} = 0 \quad (i = 1, \dots, s),$$

we shall then be justified in making the former assumption; this gives, however,

$$a K P_i \bar{b}_i a = a K P_i \bar{a} b_i + a K P_i \Delta_i = a K P_i \Delta_i,$$

and therefore, considering our particular case  $(s-1, 0)$ , the formula above, for  $a k \sum_{(\lambda)} t_{\mu}$ , gives

$$a \left[ k \sum_{(\lambda)} t_{\mu} + (K Q_{r+2} \Delta_{r+2} + \dots + K Q_s \Delta_s) k^{(j)} \right] = 0,$$

which is of the form  $a \xi = 0$ , wherein the  $2n$  quantities of the row  $\xi$  are all real, and must therefore be separately zero. Thus we can infer

$$b_{r+1} k \sum_{(\lambda)} t_{\mu} = -b_{r+1} [K Q_{r+2} \Delta_{r+2} + \dots + K Q_s \Delta_s] k^{(j)},$$

and so rewrite the formula for  $H$  in the form

$$\begin{aligned} H = & -b_{r+1} [K Q_{r+2} \Delta_{r+2} + \dots + K Q_s \Delta_s] k^{(j)} u + \frac{M_{s-1,0}}{s-1} b_{r+1} k^{(j)} [u - v^{(r+1)}] \\ & + b_{r+1} k^{(j)} [a K Q_1 \bar{b}_1 v^{(1)} + \dots + a K Q_r \bar{b}_r v^{(r)} + a K Q_{r+2} \bar{b}_{r+2} v^{(r+2)} + \dots \\ & + a K Q_s \bar{b}_s v^{(s)} - (a K Q_{r+2} \bar{b}_{r+2} + \dots + a K Q_s \bar{b}_s) u]. \end{aligned}$$

Now if  $\mu, \nu$  be matrices of types respectively  $(n, 2n), (n, n)$ , and  $x, y$  be rows respectively of  $2n$  and  $n$  quantities, we have

$$\mu x . \nu y = \bar{\mu} \nu y x = \bar{\nu} \mu x y ;$$

thus  $b_{r+1} k^{(j)}. a K Q_{r+2} \bar{b}_{r+2} u = -b_{r+2} \bar{Q}_{r+2} K \bar{a} b_{r+1} k^{(j)} u,$

and so on ; thus the coefficient of  $u$  in  $H$  is  $B k^{(j)} u$ , where

$$B = b_{r+2} \bar{Q}_{r+2} K \bar{a} b_{r+1} + \dots + b_s \bar{Q}_s K \bar{a} b_{r+1} \\ - b_{r+1} K Q_{r+2} \Delta_{r+2} - \dots - b_{r+1} K Q_s \Delta_s + \frac{M_{s-1,0}}{s-1} b_{r+1},$$

and the terms independent of  $u$  in  $H$  are

$$-b_1 \bar{Q}_1 K \bar{a} b_{r+1} k^{(j)} v^{(1)} - \dots - b_r \bar{Q}_r K \bar{a} b_{r+1} k^{(j)} v^{(r)} \\ - b_{r+2} \bar{Q}_{r+2} K \bar{a} b_{r+1} k^{(j)} v^{(r+2)} - \dots - b_s \bar{Q}_s K \bar{a} b_{r+1} k^{(j)} v^{(s)} \\ - \frac{M_{s-1,0}}{s-1} b_{r+1} k^{(j)} v^{(r+1)}.$$

Here in  $B$  a representative pair of terms is

$$b_m \bar{Q}_m K \bar{a} b_{r+1} - b_{r+1} K Q_m \Delta_m \quad (m = r+2, \dots, s);$$

if we form the matrix  $\bar{B}a - \bar{a}B$ , the corresponding terms are

$$-\bar{b}_{r+1} a K Q_m \bar{b}_m a - \Delta_m \bar{Q}_m K \bar{b}_{r+1} a \\ + \bar{a} b_{r+1} K Q_m \Delta_m - \bar{a} b_m \bar{Q}_m K \bar{a} b_{r+1};$$

replacing  $\bar{b}_m a$  by  $\bar{a} b_m + \Delta_m$ ,  $\bar{a} b_m$  by  $\bar{b}_m a - \Delta_m$ , and recalling

$$a K P_m \bar{a} = 0,$$

these terms become  $-\bar{b}_{r+1} a K Q_m \Delta_m - \Delta_m \bar{Q}_m K \bar{b}_{r+1} a \\ + \bar{a} b_{r+1} K Q_m \Delta_m + \Delta_m \bar{Q}_m K \bar{a} b_{r+1},$

or

$$-\Delta_{r+1} K Q_m \Delta_m - \Delta_m \bar{Q}_m K \Delta_{r+1}.$$

Thus by the lemma proved near the beginning of this section (p. 376), the difference between the number of zeros and poles of the function of  $(x)$  expressed by

$$\prod_{(\lambda)} f_{r+1} [u_1^{(\lambda)} + \dots + u_{s-1}^{(\lambda)} + u - v^{(r+1)}] / f_{r+1} [u_1^{(\lambda)} + \dots + u_{s-1}^{(\lambda)} - v^{(r+1)}],$$

is given by

$$\frac{M_{s-1,0}}{s-1} \frac{1}{2} \sum_{l=1}^{2n} (\Delta_{r+1} K)_l, l - \frac{1}{2} \sum_{m=r+2}^s \left\{ \sum_{l=1}^{2n} (\Delta_{r+1} K Q_m \Delta_m K + \Delta_m \bar{Q}_m K \Delta_{r+1} K)_l, l \right\}.$$

Also, by the same lemma, the difference between the sum of the values of  $u_\alpha(x)$  at the zeros and poles in question is the  $\alpha$ -th element of

$$\begin{aligned} & -aK[\bar{b}_{r+1}aKQ_1\bar{b}_1v^{(1)}+\dots+\bar{b}_{r+1}aKQ_r\bar{b}_rv^{(r)}] \\ & -aK[\bar{b}_{r+1}aKQ_{r+2}\bar{b}_{r+2}v^{(r+2)}+\dots+\bar{b}_{r+1}aKQ_s\bar{b}_sv^{(s)}]+\frac{M_{s-1,0}}{s-1}aK\bar{b}_{r+1}v^{(r+1)}, \end{aligned}$$

which itself, in virtue of  $aK\bar{a}=0$ , can be written in the form

$$\begin{aligned} & -aK[\Delta_{r+1}KQ_1\bar{b}_1v^{(1)}+\dots+\Delta_{r+1}KQ_r\bar{b}_rv^{(r)}] \\ & -aK[\Delta_{r+1}KQ_{r+2}\bar{b}_{r+2}v^{(r+2)}+\dots+\Delta_{r+1}KQ_s\bar{b}_sv^{(s)}]+\frac{M_{s-1,0}}{s-1}aK\bar{b}_{r+1}v^{(r+1)}. \end{aligned}$$

We can, however, introduce both the number of the zeros of the denominator function

$$\prod_{(\lambda)} f_{r+1}[u_1^{(\lambda)}+\dots+u_{s-1}^{(\lambda)}-v^{(r+1)}],$$

and the sum of the values of  $u(x)$  at these zeros, if we suppose that the case  $(s, r+1)$  has been considered before the case  $(s, r)$  now under consideration. For these are the zeros of the set of equations

$$f_1[u_1+\dots+u_{s-1}-v^{(1)}]=0, \quad \dots, \quad f_r[u_1+\dots+u_{s-1}-v^{(r)}]=0,$$

$$f_{r+1}[u_1+\dots+u_{s-1}-v^{(r+1)}]=0,$$

$$f_{r+2}[u_1+\dots+u_{s-1}+u-v^{(r+2)}]=0, \quad \dots, \quad f_s[u_1+\dots+u_{s-1}+u-v^{(s)}]=0,$$

of which the number is  $N_{s,r+1}$ , and the sum is  $S_{s,r+1}$ . When  $r$  is  $s-1$ , or  $r+1$  is  $s$ , this last set of equations does not contain  $(x)$  at all, and the numbers  $N_{s,s}$ ,  $S_{s,s}$  may be replaced by zeros. In general we have, adding the number and sum of the solutions of these last equations to the number and sum found above for the quotient function

$$\left. \begin{aligned} N_{s,r}-N_{s,r+1} &= \frac{M_{s-1,0}}{s-1} \frac{1}{2} \sum_{l=1}^{2n} \{ \Delta_{r+1} K \}_{l,l} \\ & - \frac{1}{2} \sum_{m=r+2}^s \left\{ \sum_{l=1}^{2n} (\Delta_{r+1} K Q_m \Delta_m K + \Delta_m \bar{Q}_m K \Delta_{r+1} K)_{l,l} \right\}, \quad (Y) \\ S_{s,r}-S_{s,r+1} &= \frac{M_{s-1,0}}{s-1} aK\bar{b}_{r+1}v^{(r+1)} - aK\Delta_{r+1}K \sum_{m=1}^s Q_m \bar{b}_m v^{(m)} \end{aligned} \right\}$$

where in  $\sum_{m=1}^s$  the value  $m=r+1$  is to be omitted, and

$$M_{s-1,0}, Q_1, \dots, Q_r, Q_{r+2}, \dots, Q_s$$

are obtained, it will be remembered, from

$$N_{s-1,0}, P_1, \dots, P_r, P_{r+1}, \dots, P_{s-1},$$

by changing the functions  $f_{r+1}, \dots, f_{s-1}$ , and the matrices deduced from them, into the functions  $f_{r+2}, \dots, f_s$ . In obtaining these formulæ, moreover, it has been assumed that

$$S_{s-1,0} = aKP_1 \bar{b}_1 v^{(1)} + \dots + aKP_{s-1} \bar{b}_{s-1} v^{(s-1)},$$

and also that

$$aKP_i \bar{a} = 0 \quad [i = 1, \dots, (s-1)].$$

By examining the formulæ for the first few cases we shall arrive at an explicit form for  $P_1, \dots, P_{s-1}$ , which can then be justified by induction. For  $s = 2, r = 1$ , the formulæ give

$$N_{2,1} = \frac{M_{1,0}}{1} \frac{1}{2} \sum_{l=1}^{2n} \{\Delta_2 K\}_{l,l},$$

$$S_{2,1} = \frac{M_{1,0}}{1} aK \bar{b}_2 v^{(2)} - aK \Delta_2 K Q_1 \bar{b}_1 v^{(1)},$$

where the  $Q_1$  refers to the corresponding case  $(s-1, 0)$  or  $(1, 0)$ . For this, however, by the lemma (p. 376),

$$N_{1,0} = \frac{1}{2} \sum_{l=1}^{2n} \{\Delta_1 K\}_{l,l}, \quad S_{1,0} = aK \bar{b}_1 v^{(1)},$$

and the substitutions to be made, of the suffixes  $r+1, \dots, s-1$  into  $r+2, \dots, s$ , are nugatory. Hence we obtain

$$N_{2,1} = \delta_1 \delta_2, \quad S_{2,1} = \delta_1 \cdot aK \bar{b}_2 v^{(2)} - aK \Delta_2 K \bar{b}_1 v^{(1)},$$

where, if  $D_i = \Delta_i K$ ,  $\delta_i = \frac{1}{2} \sum_{l=1}^{2n} (D_i)_{l,l}$ .

Next, for  $s = 2, r = 0$ , we have, by the induction formulæ,

$$N_{2,0} - N_{2,1} = \frac{M_{1,0}}{1} \delta_1 - \frac{1}{2} \sum_{l=1}^{2n} \{\Delta_1 K Q_2 \Delta_2 K + \Delta_2 \bar{Q}_2 K \Delta_1 K\}_{l,l},$$

$$S_{2,0} - S_{2,1} = \frac{M_{1,0}}{1} aK \bar{b}_1 v^{(1)} - aK \Delta_1 K Q_2 \bar{b}_2 v^{(2)},$$

where, as before,  $Q_2$  is derived from the formula relating to the case  $(1, 0)$ , and is unity, and the substitution to obtain  $M_{1,0}$  and  $Q_2$  from  $N_{1,0}$  and  $P_1$  is that of changing the suffix 1 into 2; thus  $Q_2 = P_1 = 1$ ,  $M_{1,0} = \delta_2$ , and we obtain

$$N_{2,0} - N_{2,1} = \delta_2 \delta_1 - \frac{1}{2} \sum_{l=1}^{2n} \{D_1 D_2 + D_2 D_1\}_{l,l} = \delta_1 \delta_2 - 2\delta_{12},$$

$$N_{2,0} = 2(\delta_1 \delta_2 - \delta_{12}),$$



$\delta_{12}$  being as defined before (p. 372) ; also we obtain

$$S_{2,0} - S_{2,1} = \delta_2 \cdot aK \bar{b}_1 v^{(1)} - aK \Delta_1 K \bar{b}_2 v^{(2)},$$

and hence  $S_{2,0} = (\delta_2 \cdot aK \bar{b}_1 - aK \Delta_2 K \bar{b}_1) v^{(1)} + (\delta_1 \cdot aK \bar{b}_2 - aK \Delta_1 K \bar{b}_2) v^{(2)}$

$$= aK(\delta_2 - D_2) \bar{b}_1 v^{(1)} + aK(\delta_1 - D_1) \bar{b}_2 v^{(2)},$$

which is in accordance with the assumption expressed by

$$S_{s-1,0} = aKP_1 \bar{b}_1 v^{(1)} + \dots + aKP_{s-1} \bar{b}_{s-1} v^{(s-1)},$$

with

$$P_1 = \delta_2 - D_2, \quad P_2 = \delta_1 - D_1,$$

so that  $aKP_1 \bar{a} = -aK \Delta_2 K \bar{a} = 0$ ,  $aKP_2 \bar{a} = -aK \Delta_1 K \bar{a} = 0$ .

The results are then in accord with those originally stated in this section (p. 379), namely, for  $s-1 = 1$  or  $2$ , we have shewn that

$$\frac{N_{s-1,0}}{s-1} = \phi_{1,2,\dots,(s-1)},$$

$$P_m = \psi_{1,2,\dots,m-1,m+1,\dots,(s-1)} = (\text{symbolically}) \frac{(\phi'_1 - D'_1) \dots (\phi'_{s-1} - D'_{s-1})}{\phi'_m - D'_m}$$

$$[m = 1, 2, \dots, (s-1)],$$

the notation being that explained before (pp. 373, 375). Our process of induction taking the cases in the order

$$[(1, 0)], [(2, 1), (2, 0)], [(3, 2), (3, 1), (3, 0)], \dots,$$

$$[(s-1, s-2), \dots, (s-1, 0)], [(s, s-1), \dots, (s, r+1), (s, r), \dots, (s, 0)], \dots,$$

and the substitution to be made to pass from  $N_{s-1,0}, P_1, \dots, P_{s-1}$  to  $M_{s-1,0}, Q_1, \dots, Q_s$ , being that of  $r+1, \dots, s-1$  into  $r+2, \dots, s$ , we can hence assume, in the formulæ of induction (Y),

$$\frac{M_{s-1,0}}{s-1} = \phi_{1,2,\dots,r,r+2,\dots,s},$$

$$Q_i = \frac{(\phi'_1 - D'_1) \dots (\phi'_r - D'_r)(\phi'_{r+2} - D'_{r+2}) \dots (\phi'_s - D'_s)}{\phi'_i - D'_i},$$

where  $i$  is in turn equal to  $1, 2, \dots, r, r+2, \dots, s$ , the symbol  $Q_{r+1}$  not occurring ; that is,

$$Q_i = \frac{(\phi'_1 - D'_1) \dots (\phi'_s - D'_s)}{(\phi'_i - D'_i)(\phi'_{r+1} - D'_{r+1})}.$$

Further, we have the identities

$$\Delta_m K \Delta_{r+1} K = D_m D_{r+1},$$

$$\Delta_m \bar{D}_i K \Delta_{r+1} K = \Delta_m K \Delta_i K \Delta_{r+1} K = D_m (D_i) D_{r+1},$$

$$\Delta_m \bar{D}_i \bar{D}_j K \Delta_{r+1} K = \Delta_m \bar{D}_j \bar{D}_i K \Delta_{r+1} K = D_m (D_j D_i) D_{r+1},$$

and so on; thus, as in  $Q_m$ , every term  $D_i D_j$  or  $D_i D_j D_k$  occurs associated with  $D_j D_i$  or  $D_k D_j D_i$ , and so on, we see that

$$\Delta_m \bar{Q}_m K \Delta_{r+1} K = D_m Q_m D_{r+1}.$$

Thus the induction formulæ (Y) can be written

$$\left. \begin{aligned} N_{s, r} - N_{s, r+1} &= \delta_{r+1} \phi_{1, 2, \dots, r, r+2, \dots, s} - \frac{1}{2} \sum_{m=r+2}^s \left\{ \sum_{l=1}^{2n} (D_{r+1} Q_m D_m + D_m Q_m D_{r+1})_{l, l} \right\} \\ S_{s, r} - S_{s, r+1} &= \phi_{1, 2, \dots, r, r+2, \dots, s} a K \bar{b}_{r+1} v^{(r+1)} - a K D_{r+1} \sum_{m=1}^s Q_m \bar{b}_m v^{(m)} \end{aligned} \right\}, \quad (Z)$$

where  $\sum_{m=1}^s$  omits the term for  $m = r+1$ , and symbolically

$$Q_m = \frac{(\phi'_1 - D'_1) \dots (\phi'_s - D'_s)}{(\phi'_m - D'_m)(\phi'_{r+1} - D'_{r+1})}.$$

These are the final formulæ of induction, from which the results for all the cases can be deduced. We illustrate them by obtaining explicit results for all the cases (3, 2), (3, 1), (3, 0) for  $s = 3$ ; and then we obtain the general result of this section for any case  $(s, 0)$ .

For the case (3, 2),  $s = 3$ ,  $r+1 = 3$ ,

$$N_{s, r+1} = 0, \quad S_{s, r+1} = 0, \quad \sum_{m=r+2}^s = 0,$$

and

$$N_{s, 2} = \delta_3 \cdot \phi_{1, 2},$$

$$S_{s, 2} = \phi_{1, 2} \cdot a K \bar{b}_3 v^{(3)} - a K D_3 [(\phi_2 - D_2) \bar{b}_1 v^{(1)} + (\phi_1 - D_1) \bar{b}_2 v^{(2)}].$$

For the case (3, 1),  $s = 3$ ,  $r+1 = 2$ ,

$$\begin{aligned} N_{s, 1} - N_{s, 2} &= \delta_2 \cdot \phi_{13} - \frac{1}{2} \sum_{l=1}^{2n} \{ D_2 (\phi_1 - D_1) D_3 + D_3 (\phi_1 - D_1) D_2 \}_{l, l} \\ &= \delta_2 \cdot \phi_{13} - 2\phi_1 \delta_{23} + \frac{1}{2} \sum_{l=1}^{2n} \{ D_2 D_1 D_3 + D_3 D_1 D_2 \}_{l, l}, \end{aligned}$$

so that, by the previous case,

$$N_{3,1} = \delta_2 \phi_{13} + \delta_3 \phi_{12} - 2\phi_1 \delta_{23} + \frac{1}{2} \sum_{l=1}^{2n} \{D_2 D_1 D_3 + D_3 D_1 D_2\} \iota_l,$$

while  $S_{3,1} - S_{3,2} = \phi_{13} aK \bar{b}_2 v^{(2)} - aK D_2 [(\phi_3 - D_3) \bar{b}_1 v^{(1)} + (\phi_1 - D_1) \bar{b}_3 v^{(3)}]$ ,

so that, by the previous case,

$$S_{3,1} = \phi_{12} aK \bar{b}_3 v^{(3)} + \phi_{13} aK \bar{b}_2 v^{(2)} - aK D_2 [(\phi_3 - D_3) \bar{b}_1 v^{(1)} + (\phi_1 - D_1) \bar{b}_3 v^{(3)}] \\ - aK D_3 [(\phi_2 - D_2) \bar{b}_1 v^{(1)} + (\phi_1 - D_1) \bar{b}_2 v^{(2)}].$$

Lastly, for the case (3, 0),  $s = 3$ ,  $r+1 = 1$ , we have

$$N_{3,0} - N_{3,1} = \delta_1 \phi_{23} - \frac{1}{2} \sum_{l=1}^{2n} \{D_1 Q_2 D_2 + D_2 Q_2 D_1 + D_1 Q_3 D_3 + D_3 Q_3 D_1\} \iota_l,$$

$$S_{3,0} - S_{3,1} = \phi_{23} aK \bar{b}_1 v^{(1)} - aK D_1 [Q_2 \bar{b}_2 v^{(2)} + Q_3 \bar{b}_3 v^{(3)}],$$

where

$$Q_2 = \phi_3 - D_3, \quad Q_3 = \phi_2 - D_2,$$

so that

$$N_{3,0} - N_{3,1} = \delta_1 \phi_{23} - 2\phi_3 \delta_{12} - 2\phi_2 \delta_{13} \\ + \frac{1}{2} \sum_{l=1}^{2n} \{D_1 D_3 D_2 + D_2 D_3 D_1 + D_1 D_2 D_3 + D_3 D_2 D_1\} \iota_l,$$

and hence, by the previous case,

$$N_{3,0} = \delta_2 \phi_{13} + \delta_3 \phi_{12} + \delta_1 \phi_{23} + 6\delta_{123} - 2\phi_1 \delta_{23} - 2\phi_2 \delta_{13} - 2\phi_3 \delta_{12} \\ = 3(\delta_1 \delta_2 \delta_3 - \delta_1 \delta_{23} - \delta_2 \delta_{31} - \delta_3 \delta_{12} + 2\delta_{123}) \\ = 3\phi_{123},$$

as was previously stated; while

$$S_{3,0} - S_{3,1} = \phi_{23} aK \bar{b}_1 v^{(1)} - \phi_3 aK D_1 \bar{b}_2 v^{(2)} - \phi_2 aK D_1 \bar{b}_3 v^{(3)} \\ + aK D_1 D_3 \bar{b}_2 v^{(2)} + aK D_1 D_2 \bar{b}_3 v^{(3)};$$

and therefore, by the previous case,

$$S_{3,0} = aK(\phi_{23} - \phi_2 D_3 - \phi_3 D_2 + D_2 D_3 + D_3 D_2) \bar{b}_1 v^{(1)} \\ + aK(\phi_{31} - \phi_3 D_1 - \phi_1 D_3 + D_3 D_1 + D_1 D_3) \bar{b}_2 v^{(2)} \\ + aK(\phi_{12} - \phi_1 D_2 - \phi_2 D_1 + D_1 D_2 + D_2 D_1) \bar{b}_3 v^{(3)},$$

as also was previously stated (p. 373).

Taking now from the general induction formula (Z) (p. 386)

$$S_{s,r} - S_{s,r+1} = \phi_{12\dots r, r+2, \dots, s} aK \bar{b}_{r+1} v^{(r+1)} - aK D_{r+1} \sum_{m=1}^s Q_m \bar{b}_m v^{(m)},$$

where in  $\Sigma'$  the value  $m = r+1$  is to be omitted, and adding these equations for different values of  $r$ , we obtain

$$S_{s,0} = \sum_{r=1}^s \phi_{12\dots r, r+2, \dots, s} aK \bar{b}_{r+1} v^{(r+1)} - aK \sum_{r=1}^s D_{r+1} \sum_{m=1}^s Q_m \bar{b}_m v^{(m)};$$

the right side is a sum of terms such as  $aKT_m \bar{b}_m v^{(m)}$ , where

$$T_m = \phi_{12\dots m-1, m+1, \dots, s} - D_1 Q_{m,1} - D_2 Q_{m,2} - \dots - D_{m-1} Q_{m,m-1} \\ - D_{m+1} Q_{m,m+1} - \dots - D_s Q_{m,s},$$

$Q_{m,i}$  being symbolically

$$\frac{(\phi'_1 - D'_1) \dots (\phi'_s - D'_s)}{(\phi'_i - D'_i)(\phi'_m - D'_m)}, = \psi_{1, \dots, i-1, i+1, \dots, m-1, m+1, \dots, s},$$

in accordance with the notation of p. 360. From the identity there remarked we have thence

$$T_m = \psi_{1,2,\dots,m-1,m+1,\dots,s} = \frac{(\phi'_1 - D'_1) \dots (\phi'_s - D'_s)}{\phi'_m - D'_m},$$

and so, finally,

$$S_{s,0} = \sum_{m=1}^s aK \psi_{12,\dots,m-1,m+1,\dots,s} \bar{b}_m v^{(m)},$$

which is of the same form for  $s$  as was that originally assumed for  $s-1$ ; the general result of this section, so far as regards the sum of the solutions, is thus established by induction.

We have already proved that the formula for  $N_{s,0}$  is a consequence of the formula stated for  $S_{s,0}$  (p. 375), and it is therefore unnecessary to shew that this also follows from the formula (Z).

### 5. Some Particular Cases.

(I) If, beside the universal bilinear relation connecting the periods expressed by  $aK\bar{a} = 0$ , or

$$\sum_{i,j} K_{i,j} a_{p,i} a_{q,j} = 0 \quad (p, q = 1, \dots, 2n),$$

there exist no other bilinear relations, then the various forms we have obtained

$$aK\bar{a} = 0, \quad aK\Delta K\bar{a} = 0, \quad \dots, \quad a\Delta^{-1}\bar{a} = 0,$$

must be the same; and therefore

$$K\Delta = \Delta K = r,$$

where  $r$  is an integer ( $= Rm$  in the notation of M.P.F., p. 289). This would give, from the definitions (p. 372),

$$D_i = r_i, \quad \delta_i = nr_i, \quad \delta_{ij} = nr_i r_j, \quad \delta_{ijk} = nr_i r_j r_k, \quad \dots,$$

and

$$\phi_1 = nr_1, \quad \phi_{12} = n^2 r_1 r_2 - nr_1 r_2 = n(n-1) r_1 r_2,$$

$$\phi_{123} = n^3 r_1 r_2 r_3 - 3n^2 r_1 r_2 r_3 + 2nr_1 r_2 r_3 = n(n-1)(n-2) r_1 r_2 r_3.$$

If we assume, in accordance with these, that for  $m < k$ ,

$$\phi_{12\dots m} = \frac{n!}{(n-m)!} r_1 r_2 \dots r_m,$$

the formula (p. 359)

$$k\phi_{12\dots k} = \sum_1 \delta_1 \phi_{23\dots k} - 2! \sum_2 \delta_{12} \phi_{34\dots k} + 3! \sum_3 \delta_{123} \phi_{4\dots k} - \dots$$

gives

$$\begin{aligned} k\phi_{12\dots k} &= n \left\{ k \frac{n!}{(n-k+1)!} - 2! \frac{k!}{2! (k-2)!} \frac{n!}{(n-k+2)!} \right. \\ &\quad \left. + 3! \frac{k!}{3! (k-3)!} \frac{n!}{(n-k+3)!} - \dots \right\} r_1 r_2 \dots r_k \\ &= n[(k)!] \left\{ \binom{n}{k-1} - \binom{n}{k-2} + \binom{n}{k-3} - \dots \right\} r_1 r_2 \dots r_k \\ &= k \frac{n!}{(n-k)!} r_1 \dots r_k, \end{aligned}$$

and justifies the assumption in general.

$$\text{Also } \psi_{12} = \phi_{12} - \phi_1 D_2 - \phi_2 D_1 + D_1 D_2 + D_2 D_1$$

$$= \frac{n!}{(n-2)!} r_1 r_2 - nr_1 r_2 - nr_2 r_1 + 2r_1 r_2 = (n-1)(n-2) r_1 r_2,$$

$$\psi_{123} = \phi_{123} - \phi_{23} D_1 - \dots + \phi_1 (D')_{23} + \dots - (D')_{123}$$

$$= \left\{ \frac{n!}{(n-3)!} - 3 \frac{n!}{(n-2)!} + 3 \frac{n!}{(n-1)!} 2! - 3! \right\} r_1 r_2 r_3$$

$$= (n-1)(n-2)(n-3) r_1 r_2 r_3,$$

and, in general,

$$\psi_{12\dots k} = \frac{n!}{(n-k)!} - \frac{n!}{(n-k+1)!} \frac{k!}{1!(k-1)!} + \frac{n!}{(n-k+2)!} \frac{2!}{2!(k-2)!} \\ - \frac{n!}{(n-k+3)!} \frac{3!}{3!(k-3)!} + \dots,$$

multiplied by  $r_1 r_2 \dots r_k$ , which is

$$= (k!) \left\{ \binom{n}{k} - \binom{n}{k-1} + \binom{n}{k-2} - \binom{n}{k-3} + \dots \right\} r_1 r_2 \dots r_k \\ = (n-1)(n-2) \dots (n-k) r_1 \dots r_k.$$

Thus the formula for the sum of the solutions in the case  $(n, 0)$

$$\sum_{i=1}^n aK P_i \bar{b}_i v^{(i)}$$

becomes 
$$[(n-1)!] r_1 \dots r_n \sum_{i=1}^n \frac{1}{r_i} aK \bar{b}_i v^{(i)},$$

wherein 
$$aK \bar{b}_i = a r_i \Delta_i^{-1} \bar{b}_i = r_i$$

(see p. 368, for the proof of  $a \Delta_i^{-1} \bar{b}_i = 1$ ), and so becomes

$$\Sigma v^{(i)} [(n-1)!] r_1 \dots r_n.$$

As, with the particular periods considered in § 3, we have

$$H = \sqrt{|K|} = (e_1 e_2 \dots e_n)^{-1},$$

our result is thus that for the  $n$  equations

$$f_1(u-v^{(1)}) = 0, \dots, f_n(u-v^{(n)}) = 0,$$

in the most general case, the number and sum of the solutions are respectively

$$(n!) r_1 r_2 \dots r_n / e_1 \dots e_n \quad \text{and} \quad \{[(n-1)!] r_1 \dots r_n / e_1 \dots e_n\} \Sigma v^{(i)},$$

agreeing with Wirtinger's result (M.P.F., p. 293).

(II) If, next, we take the case of other bilinear relations than  $aK\bar{a} = 0$  connecting the periods, namely (p. 369, § 3), with

$$a = (e^{-1}, \sigma), \quad K = E,$$

suppose

$$b = k\beta - r(0, 1),$$

$$\Delta = k\nabla + rE^{-1},$$

where

$$\nabla = \begin{pmatrix} p & -q \\ \bar{q} & \rho \end{pmatrix},$$

$$p = \bar{p}_1 - p_1, \quad \beta = e(p_1, q + \bar{p}_1 e\sigma),$$

then, the bilinear relations connecting the periods

$$aE\bar{a} = 0, \quad aE\nabla E\bar{a} = 0,$$

being the same for all, the different Jacobian functions will be distinguished by the values of the integers  $k, r$ ; and

$$b_i = k_i\beta - r_i(0, 1),$$

$$\Delta_i = k_i\nabla + r_iE^{-1}.$$

This gives

$$D_i = \Delta_i E = k_i\nabla E + r_i,$$

and hence

$$\delta_i = k_i\sigma_1 + r_in,$$

where

$$\sigma_1 = \frac{1}{2} \sum_{l=1}^{2n} (\nabla E)_{l,l}.$$

$$\text{Also} \quad \delta_{ij} = \frac{1}{2(2!)} \sum_{l=1}^{2n} (D_i D_j + D_j D_i)_{l,l}$$

$$= k_i k_j \sigma_2 + (k_i r_j + k_j r_i) \sigma_1 + r_i r_j n,$$

where

$$\sigma_2 = \frac{1}{2} \sum_{l=1}^{2n} \{(\nabla E)^2\}_{l,l}.$$

$$\text{Also} \quad \delta_{ijk} = k_i k_j k_k \sigma_3 + (k_i k_j r_k + \dots) \sigma_2 + (k_i r_j r_k + \dots) \sigma_1 + r_i r_j r_k n,$$

where

$$\sigma_3 = \frac{1}{2} \sum_{l=1}^{2n} \{(\nabla E)^3\}_{l,l}.$$

Hence

$$\phi_{12} = \delta_1 \delta_2 - \delta_{12} = 2k_1 k_2 P_2 + (k_1 r_2 + k_2 r_1) P_1 (n-1) + r_1 r_2 n(n-1),$$

where

$$P_1 = \sigma_1, \quad P_2 = \frac{1}{2!} (\sigma_1^2 - \sigma_2),$$

$$\text{and } \phi_{123} = \delta_1 \delta_2 \delta_3 - \delta_{12} \delta_3 - \dots + 2\delta_{123}$$

$$= 3! k_1 k_2 k_3 P_3 + (k_2 k_3 r_1 + \dots) 2! P_2 (n-2)$$

$$+ (k_1 r_2 r_3 + \dots) P_1 (n-1)(n-2) + r_1 r_2 r_3 n(n-1)(n-2),$$

where 
$$P_3 = \frac{1}{3!} (\sigma_1^3 - 3\sigma_1\sigma_2 + 2\sigma_3),$$

and so on; from these the general law for  $\phi_{12\dots k}$  is clear, and hence the formula for the number of solutions of the Jacobian equations can be put down.

Also 
$$\psi_i = \phi_i - D_i = k_i(\sigma_1 - M) + r_i(n-1),$$

where 
$$M = \nabla E,$$

and 
$$\begin{aligned} \psi_{ij} &= \phi_{ij} - \phi_i D_j - \phi_j D_i + D_i D_j + D_j D_i \\ &= 2k_i k_j P_2 + (k_i r_j + k_j r_i) P_1 (n-2) + r_i r_j (n-1)(n-2) \\ &\quad - M[2k_i k_j P_1 + (k_i r_j + k_j r_i)(n-2)] + 2M^2 k_i k_j, \end{aligned}$$

and so on.

In particular, when  $n = 2$ , we have for the number of solutions of the equations

$$f_1(u - v^{(1)}) = 0, \quad f_2(u - v^{(2)}) = 0,$$

$$\frac{\phi_{12}}{e_1 e_2} = \frac{1}{e_1 e_2} \{k_1 k_2 (\sigma_1^2 - \sigma_2) + (k_1 r_2 + k_2 r_1) \sigma_1 + 2r_1 r_2\},$$

while the sum of the solutions, save for quantities independent of  $v^{(1)}$  and  $v^{(2)}$ , is

$$\frac{1}{e_1 e_2} \{aE[k_2(\sigma_1 - M) + r_2] \bar{b}_1 v^{(1)} + aE[k_1(\sigma_1 - M) + r_1] \bar{b}_2 v^{(2)}\}.$$

We proceed to find the explicit forms for these in the notation adopted by Humbert, *Liouville*, 1899, pp. 254, 270, and *Liouville*, 1900, p. 313, footnote. Here

$$\nabla = \begin{pmatrix} 0 & D & 0 & A \\ -D & 0 & -C & B \\ 0 & C & 0 & -E \\ -A & -B & E & 0 \end{pmatrix},$$

in which each element is an integer, and the element (1, 3) is taken to be zero. If, with our notation

$$\nabla = \begin{pmatrix} p & -q \\ \bar{q} & r \end{pmatrix},$$



then 
$$p = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}, \quad p_1 = \begin{pmatrix} 0 & -D \\ 0 & 0 \end{pmatrix},$$

$$q = \begin{pmatrix} 0 & -A \\ C & -B \end{pmatrix}, \quad r = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix},$$

and hence 
$$M = \nabla E = \begin{pmatrix} p & -q \\ \bar{q} & r \end{pmatrix} \begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix} = \begin{pmatrix} -qe & -pe \\ re & -\bar{q}e \end{pmatrix},$$

which, with 
$$e = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix},$$

give 
$$pe = \begin{pmatrix} 0 & De_2 \\ -De_1 & 0 \end{pmatrix}, \quad qe = \begin{pmatrix} 0 & -Ae_2 \\ Ce_1 & -Be_2 \end{pmatrix},$$

$$\bar{q}e = \begin{pmatrix} 0 & Ce_2 \\ -Ae_1 & -Be_2 \end{pmatrix}, \quad re = \begin{pmatrix} 0 & -Ee_2 \\ Ee_1 & 0 \end{pmatrix},$$

so that 
$$M = \begin{pmatrix} 0 & Ae_2 & 0 & -De_2 \\ -Ce_1 & Be_2 & De_1 & 0 \\ 0 & -Ee_2 & 0 & -Ce_2 \\ Ee_1 & 0 & Ae_1 & Be_2 \end{pmatrix}.$$

and 
$$\sigma_1 = Be_2,$$

and 
$$\sigma_2 = \frac{1}{2} \sum_{i=1}^{2n} \{M^2\}_{i,i}$$

$$= -ACe_1e_2 - DEe_1e_2 - ACe_1e_2 + B^2e_2^2 - DEe_1e_2,$$

which give 
$$\sigma_1^2 - \sigma_2 = 2(AC + DE)e_1e_2.$$

The function 
$$\phi_{12} = k_1k_2(\sigma_1^2 - \sigma_2) + (k_1r_2 + k_2r_1)\sigma_1 + 2r_1r_2$$

thus becomes

$$\phi_{12} = 2k_1k_2e_1e_2(AC + DE) + (k_1r_2 + k_2r_1)Be_2 + 2r_1r_2,$$

and the  $\phi_{12}/e_1e_2$  agrees with the result given by Humbert for the number of solutions of

$$f_1(u - v^{(1)}) = 0, \quad f_2(u - v^{(2)}) = 0$$

when  $e_1 = 1 = e_2$ . The function  $\phi_{12}$  is the coefficient of  $\lambda_1\lambda_2$  in

$$(\lambda_1k_1 + \lambda_2k_2)^2 e_1e_2(AC + DE) + (\lambda_1k_1 + \lambda_2k_2)(\lambda_1r_1 + \lambda_2r_2)Be_2 + (\lambda_1r_1 + \lambda_2r_2)^2,$$

which, as is easily verified, is the square root of the determinant of

$$\lambda_1(k_1M + r_1) + \lambda_2(k_2M + r_2) \quad \text{or} \quad (\lambda_1k_1 + \lambda_2k_2)M + \lambda_1r_1 + \lambda_2r_2.$$

This is an example of a general remark made above (p. 959).

To find the sum of the solutions of the two equations we have to evaluate two matrices of the form

$$aE[k_2(\sigma_1 - M) + r_2]\bar{b}_1.$$

Now here

$$\begin{aligned}\beta &= e(p_1, q + \bar{p}_1 e \sigma) \\ &= e \left[ \begin{pmatrix} 0 & -D \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -A \\ C & -B \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -D & 0 \end{pmatrix} \begin{pmatrix} e_1 \sigma_{11} & e_1 \sigma_{12} \\ e_2 \sigma_{12} & e_2 \sigma_{22} \end{pmatrix} \right] \\ &= e \begin{pmatrix} 0 & -D & 0 & -A \\ 0 & 0 & C - De_1 \sigma_{11} & -B - De_1 \sigma_{12} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -De_1 & 0 & -Ae_1 \\ 0 & 0 & Ce_2 - De_1 e_2 \sigma_{11} & -Be_2 - De_1 e_2 \sigma_{12} \end{pmatrix},\end{aligned}$$

and therefore

$$\begin{aligned}b_1 &= k_1 \beta - r_1(0, 1) \\ &= \begin{pmatrix} 0 & -k_1 De_1 & -r_1 & -k_1 Ae_1 \\ 0 & 0 & Ck_1 e_2 - Dk_1 e_1 e_2 \sigma_{11} & -Bk_1 e_2 - Dk_1 e_1 e_2 \sigma_{12} - r_1 \end{pmatrix},\end{aligned}$$

Also

$$k_2(\sigma_1 - M) + r_2 = \begin{Bmatrix} k_2 Be_2 + r_2 & -Ak_2 e_2 & 0 & Dk_2 e_2 \\ Ck_2 e_1 & r_2 & -Dk_2 e_1 & 0 \\ 0 & Ek_2 e_2 & k_2 Be_2 + r_2 & Ck_2 e_2 \\ -Ek_2 e_1 & 0 & -Ak_2 e_1 & r_2 \end{Bmatrix}$$

and

$$\begin{aligned}aE &= (e^{-1}, \sigma) \begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix} = (\sigma e, -1) \\ &= \begin{pmatrix} \sigma_{11} e_1 & \sigma_{12} e_2 & -1 & 0 \\ \sigma_{21} e_1 & \sigma_{22} e_2 & 0 & -1 \end{pmatrix},\end{aligned}$$

so that, if  $aE[k_2(\sigma_1 - M) + r_2]$  be written

$$\begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} & \mu_{14} \\ \mu_{21} & \mu_{22} & \mu_{23} & \mu_{24} \end{pmatrix},$$

we have

$$\begin{aligned}\mu_{11} &= (k_2 Be_2 + r_2) \sigma_{11} e_1 + Ck_2 e_1 \sigma_{12} e_2, \\ \mu_{12} &= -Ak_2 e_2 \sigma_{11} e_1 + r_2 \sigma_{12} e_2 - Ek_2 e_2, \\ \mu_{13} &= -Dk_2 e_1 \sigma_{12} e_2 - (k_2 Be_2 + r_2) \\ \mu_{14} &= Dk_2 e_2 \sigma_{11} e_1 - Ck_2 e_2, \\ \mu_{21} &= (k_2 Be_2 + r_2) \sigma_{21} e_1 + Ck_2 e_1 \sigma_{22} e_2 + Ek_2 e_1, \\ \mu_{22} &= -Ak_2 e_2 \sigma_{21} e_1 + r_2 \sigma_{22} e_2, \\ \mu_{23} &= -Dk_2 e_1 \sigma_{22} e_2 + Ak_2 e_1, \\ \mu_{24} &= Dk_2 e_2 \sigma_{21} e_1 - r_2.\end{aligned}$$

The matrix

$$aE[k_2(\sigma_1 - M) + r_2] \bar{b}_1$$

is then

$$\begin{pmatrix} (1, 1) & (1, 2) \\ (2, 1) & (2, 2) \end{pmatrix},$$

$$\text{where } (1, 1) = -k_1 D e_1 \cdot \mu_{12} - r_1 \cdot \mu_{13} - k_1 A e_1 \cdot \mu_{14},$$

$$(1, 2) = (C k_1 e_2 - D k_1 e_1 e_2 \sigma_{11}) \mu_{13} - (B k_1 e_2 + D k_1 e_1 e_2 \sigma_{12} + r_1) \mu_{14},$$

$$(2, 1) = -k_1 D e_1 \cdot \mu_{22} - r_1 \cdot \mu_{23} - k_1 A e_1 \cdot \mu_{24},$$

$$(2, 2) = (C k_1 e_2 - D k_1 e_1 e_2 \sigma_{11}) \mu_{23} - (B k_1 e_2 + D k_1 e_1 e_2 \sigma_{12} + r_1) \mu_{24}.$$

With these forms, if we take account of the relation  $aE \nabla E \bar{a} = 0$ , which in this case can be calculated to be the single relation

$$A e_1 \sigma_{11} + B e_2 \sigma_{12} + C e_2 \sigma_{22} + D e_1 e_2 (\sigma_{12}^2 - \sigma_{11} \sigma_{22}) + E = 0,$$

in order to simplify (2, 2), we find that the formula for the sum of the solutions of the equations  $f_1(u - v^{(1)}) = 0$ ,  $f_2(u - v^{(2)}) = 0$  reduces to

$$\begin{aligned} & \frac{1}{2} \phi_{12}(v^{(1)} + v^{(2)}) \\ & + (k_2 r_1 - k_1 r_2) \begin{pmatrix} \frac{1}{2} B e_2 + D \sigma_{12} e_1 e_2 & C e_2 - D \sigma_{11} e_1 e_2 \\ -A e_1 + D \sigma_{22} e_1 e_2 & -\frac{1}{2} B e_2 - D \sigma_{12} e_1 e_2 \end{pmatrix} (v^{(1)} - v^{(2)}). \end{aligned}$$

The matrix multiplying  $v^{(1)} - v^{(2)}$  is

$$\frac{1}{2} B e_2 + (\bar{q} - \sigma e p) e.$$