Some problems of 'Partitio Numerorum': IV. The singular series in Waring's Problem and the value of the number $G(k)$.

By

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1. Introduction.

1. 1. In this memoir we continue the investigations initiated in two earlier memoirs bearing a similar title, and complete the proof of all the assertions which they contain[1]). We shall assume throughout that the reader is acquainted with the notation and terminology of these memoirs.

The fundamental theorem of Hilbert[2]) asserts the existence of the numbers $g(k)$ and $G(k)$. In our first memoir we proved that, if

$$a=\frac{1}{k}, \quad K=2^{k-1}, \quad \varkappa=1-\frac{1}{K}, \quad s>2K+1,$$

then

$$r_{k,s}(n)=Cn^{sa-1}S+O(n^{sa\varkappa+\varepsilon}), \tag{1.11}$$

where S is the 'singular series'

$$S=\sum\left(\frac{S_{p,q}}{q}\right)^s e_q(-np). \tag{1.12}$$

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[1]) G. H. Hardy and J. E. Littlewood, Some problems of 'Partitio Numerorum': I. A new solution of Waring's Problem, Göttinger Nachrichten 1920, S. 33—54; II. Proof that every large number is the sum of at most 21 biquadrates, Mathematische Zeitschrift 9 (1921), S. 14—27.

The third memoir of the series (Some problems of 'Partitio Numerorum': III. On the expression of a number as a sum of primes) will appear shortly in the Acta Mathematica. The problems considered in this memoir are of a somewhat different character. We refer to these memoirs as P. N. 1, P. N. 2, P. N. 3.

[2]) D. Hilbert, Beweis für die Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl n-ter Potenzen, Göttinger Nachrichten 1909, S. 17—36: reprinted with certain changes in Mathematische Annalen, 67 (1909), S. 281—300.

The sum of the series is positive, and indeed greater than $\frac{1}{2}$, if s is sufficiently large; and $sa\varkappa < sa - 1$ if $s > kK$. Thus

$$r_{k,s}(n) \sim C n^{sa-1} S, \tag{1. 13}$$

as $n \to \infty$, for all large enough values of s, say for $s \geqq G_1(k)$. It is plain that Hilbert's theorem follows as a corollary.

Important simplifications of our method have been effected by Landau[3]) and Weyl[4]). These improvements relate to our treatment of the 'major arcs'. In particular Weyl has shown that, if we are concerned with an existence theorem only, so that it is not important to obtain the best possible upper bound for $G(k)$, the rather difficult analysis which we used may be replaced by an argument of a much more elementary character.

We proved nothing in this memoir about the values of $G_1(k)$ or $G(k)$, though our analysis suggested very forcibly that

$$G(k) \leqq G_1(k) \leqq kK + 1 = s_0. \tag{1. 14}$$

In order to prove this it is necessary to examine the singular series more closely, and to prove that

$$S > \sigma = \sigma(k, s) > 0 \tag{1. 15}$$

for $s \geqq s_0$. This would be sufficient; but in fact, as Herr Ostrowski has shown[5]), the truth of (1. 15) for $s = s_0$ will involve

$$S > \sigma = \sigma(k) > 0 \tag{1. 151}$$

(the value of σ being independent of s) for $s \geqq s_0$. In our second memoir, however, we effected an improvement in (1. 11), showing that

$$r_{k,s}(n) = C n^{sa-1} S + O(n^{(s-4)a\varkappa+2a+\varepsilon}) \tag{1. 16}$$

(a better result if only $k > 2$). If now we can prove that (1. 15), and therefore (1. 151), is true for $s > (k-2)K + 4$, we shall have proved that

$$G(k) \leqq G_1(k) \leqq (k-2)K + 5. \tag{1. 17}$$

This we proved before when $k = 4$, in some ways the most interesting case. It is the general proof of (1. 17) that is our primary object now.

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3) E. Landau, Zur Hardy-Littlewoodschen Lösung des Waringschen Problems, Göttinger Nachrichten 1921, S. 88—92.

4) H. Weyl, Bemerkung zur Hardy-Littlewoodschen Lösung des Waringschen Problems, Göttinger Nachrichten 1922.

5) A. Ostrowski, Bemerkung zur Hardy-Littlewoodschen Lösung des Waringschen Problems, Mathematische Zeitschrift, 9 (1921), S. 28—34. We return to this point in § 6. 3.

Our principal theorem then, will be:

Theorem 1. *There is a positive number* $\sigma = \sigma(k, s)$ *such that* $S > \sigma$ *for* $s \geqq (k-2)K + 5$, *so that*

$$r_{k,s}(n) \sim C n^{sa-1} S$$

for all such values of s. *In particular,* $r_{k,s}(n)$ *is positive for all such values of* s *and all sufficiently large values of* n, *so that*

$$G(k) \leqq (k-2)K + 5.$$

1. 2. We have in any event to undertake a detailed examination of the singular series; and we shall push our analysis a good deal further than is necessary for our immediate purpose. We do so primarily because the analysis is interesting in itself. But it must be remembered also that the inequality (1. 17) is, in all probability, far short of the actual truth. It is not unlikely that the order of the error term in (1. 16), which is the obstacle to further progress at present, may before long be materially reduced. The discussion of the singular series, for values of s smaller than those contemplated in Theorem 1, will then become of immediate importance, as every improvement in (1. 16) will give a corresponding improvement in the value of $G(k)$.

It may be useful if we summarise at this stage the existing state of knowledge as regards the values of $g(k)$ and $G(k)$. This is exhibited in the following table.

$k =$	2	3	4	5	6	7	8
$g(k) \leqq$	4	9	37	58	478	3806	31353
$g(k) \geqq \left[\left(\frac{3}{2}\right)^k\right] + 2^k - 2 =$	4	9	19	37	73	143	279
$G(k) \leqq$	4	8	37	58	478	3806	31353
$G(k) \leqq (k-2)2^{k-1} + 5 =$	(5)	(9)	21	53	133	325	773
$G(k) \geqq$	4	4	16	6	9	8	32

The numbers in the first line are the upper bounds for $g(k)$ which have been obtained by elementary arguments, and are due in order to Lagrange, Wieferich, Wieferich, Baer, Baer, Wieferich, and Kempner respectively[6]). Those in the third line are the corresponding

[6]) The names are those of the authors who found the actual numbers quoted. The proofs of 'Waring's Theorem' for the cases in question are due to Lagrange, Maillet, Liouville, Maillet, Fleck, Wieferich, and Hurwitz (and Maillet) respectively. For detailed references see A. J. Kempner, Über das Waringsche Problem und einige Verallgemeinerungen. Inaugural-Dissertation, Göttingen 1912, and W. S. Baer, Beiträge zum Waringschen Problem, Inaugural-Dissertation,

upper bounds for $G(k)$, and are identical with the numbers in the first except when $k=3$. The inequality $G(3)\leqq 8$ is due to Landau[7]).

The fourth line contains the upper bounds given by Theorem 1. It will be observed that the numbers for $k=2$ and $k=3$ are inferior to those already known, but that there is a very substantial improvement for all larger values of k.

The second and fifth lines contain the best known lower bounds for $g(k)$ and $G(k)$ respectively. It was observed by Euler, and later by Bretschneider[8]), that the number $2^k l-1$, where l is determined by

$$3^k=2^k l+m, \qquad 0<m<2^k,$$

requires $l+2^k-2$ powers; and this observation gives the numbers tabulated. The numbers 4, 9, 19 are mentioned by Waring, but there is nothing to show that he had recognised the general law[9]).

The numbers in the fifth line are more interesting and require further explanation. It was proved by Hurwitz[10]) and Maillet[11]) that

$$G(k)\geqq k+1$$

for every k; and in some cases, *e. g.* for $k=3$, 5 and 7, no more than this is known.

In other cases it is possible to prove a good deal more by the consideration of simple congruence relations. The simplest case is $k=4$. Every biquadrate is congruent to 0 or to 1 to modulus 16, so that a number $16m+15$ requires at least 15 biquadrates. Thus (as was observed by Landau) $G(k)\geqq 15$, and Kempner, considering numbers $16^{\beta}.31$,

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Göttingen 1913. The numbers for $k=7$ and $k=8$ could no doubt be substantially reduced.

Proofs of the existence of $g(k)$, from which an upper bound for $g(k)$ could be calculated, have also been given for $k=10$ (I. Schur), $k=12$ (Kempner) and $k=14$ (Kempner).

7) E. Landau, Über eine Anwendung der Primzahltheorie auf das Waringsche Problem in der elementaren Zahlentheorie, Mathematische Annalen, **66** (1909), S. 102—105.

8) See Kempner, loc. cit., S. 44—45.

9) Waring asserts quite explicitly, not merely that $g(k)$ exists, but that $g(2)=4$, $g(3)=9$, $g(4)=19$, 'et sic deinceps'. Nothing is known, so far as we are aware, inconsistent with the view that the numbers in question are the actual values of $g(k)$ for every k.

10) A. Hurwitz, Über die Darstellung der ganzen Zahlen als Summen von n-ter Potenzen ganzer Zahlen, Mathematische Annalen, **65** (1908), S. 424—427.

11) E. Maillet, Sur la décomposition d'un entier en une somme de puissances huitièmes d'entiers, Bulletin de la société mathématique de France, **36** (1908), p. 69—77

where β is large, improved this inequality to $G(4) \geqq 16$. He also proved that $G(k) \geqq 4k$ whenever k is a power of 2, and that $G(6) \geqq 9$. This is the origin of the remaining numbers in our table. Again, every ninth power is congruent to 0, 1, or -1, to modulus 27, so that a number $27m \pm 13$ requires at least 13 ninth powers: thus $G(9) \geqq 13$.

Considerations of this character concerning cubes lead only to the Hurwitz-Maillet inequality; and when $k = 5$ or $k = 7$ the resulting inequalities are entirely trivial, for any residue to modulus 25 can be generated by 3 fifth powers, and any residue to modulus 49 by 4 seventh powers. It will be found that these simple facts have a very interesting bearing on the structure of our singular series.

2. The formal theory of the singular series.

2. 1. The singular series is absolutely convergent for sufficiently large values of s,[12]) and is then expressible as an infinite product

(2. 11) $$S = 1 + A_2 + A_3 + \ldots = \sum A_q = \chi_2 \chi_3 \chi_5 \ldots = \Pi \chi_\pi,$$

where π is a prime and

(2. 12) $$\chi_\pi = 1 + A_\pi + A_{\pi^2} + \ldots = \sum A_{\pi^\lambda}.$$ [13])

The sum χ_π is a finite sum, for A_{π^λ} is always zero from a certain value of λ onwards[14]).

The question of the absolute convergence of the series and product will be discussed more precisely later. Our immediate object is to determine the form of the factors χ_π.

2. 2. We suppose that $q = \pi^\lambda \, (\lambda \geqq 1)$, and we denote by $\nu(\xi, q, n)$ the number of solutions of the congrence

(2. 21) $$\sum_{r=1}^{s} x_r^k \equiv n \pmod{q}$$

for which

(2. 211) $$0 \leqq x_r < \xi \qquad (r = 1, 2, \ldots, s).$$

We write

(2. 22) $$\nu(q, q, n) = M(q, n) = M(q).$$ [15])

Finally, we denote by

(2.23) $$N(q, n) = N(q)$$

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[12]) P. N. 1, S. 40.

[13]) P. N. 2, S. 18.

[14]) P. N. 2, S. 22 (f. n. 7). This will also appear incidentally later (S. 374).

[15]) When it is unnecessary to show explicitly the dependence of M on n.

the number of solutions of (2.21) for which $0 \leqq x_r < q$ $(r \leqq s)$, and for which not every x is divisible by π. Such a solution we may call a *primitive* solution.

Following Landau, we write '$y \mid z$' and '$y \nmid z$' for 'z is divisible by y' and 'z is not divisible by y'. We shall find it convenient, moreover, to have a special notation to express '$y^r \mid z$, $y^{r+1} \nmid z$', *i. e.* 'y^r is the highest power of y that divides z'. In these circumstances we shall write '$y^r \| z$'.

This being so, the value of χ_π is given, in terms of the numbers N, by the following theorem.

Theorem 2. *Suppose that*

$$k > 2, \qquad \pi^\theta \| k \quad (\theta \geqq 0), \qquad (\pi^k)^\beta \| n \quad (\beta \geqq 0),\text{[16]} \tag{2.24}$$

and let

$$\varphi = \theta + 1 \quad (\pi > 2), \qquad \varphi = \theta + 2 \quad (\pi = 2). \tag{2.25}$$

Then

$$\chi_\pi = B\pi^{\varphi(1-s)} N(\pi^\varphi, 0) + \pi^{\beta(k-s)+\varphi(1-s)} N\left(\pi^\varphi, \frac{n}{\pi^{\beta k}}\right), \tag{2.26}$$

where

$$B = 0 \qquad (\beta = 0), \tag{2.2611}$$

$$B = 1 + \pi^{k-s} + \pi^{2(k-s)} + \pi^{(\beta-1)(k-s)} \qquad (\beta > 0). \tag{2.2612}$$

The proof of this theorem rests on a series of lemmas.

2.3. Lemma 1. *If*

$$\pi^\theta \| k \quad (\theta \geqq 0), \qquad q = \pi^\lambda, \qquad \lambda > \theta + 1 \quad (\pi > 2), \qquad \lambda > \theta + 2 \quad (\pi = 2),$$

$$x = \xi + \alpha\pi^{\lambda-\theta-1}, \tag{2.31}$$

then

$$x^k \equiv \xi^k + \frac{k}{\pi^\theta}\alpha\xi^{k-1}\pi^{\lambda-1} \pmod q. \tag{2.32}$$

We have

$$x^k = \sum_{r=0}^{k} \binom{k}{r} \alpha^r \xi^{k-r} \pi^{r(\lambda-\theta-1)}.$$

The terms $r = 0, 1$ are those which occur in (2.32).

Suppose then $r \geqq 2$. The index of the highest power of π that divides $r!$ is

$$\left[\frac{r}{\pi}\right] + \left[\frac{r}{\pi^2}\right] + \cdots < \frac{r}{\pi - 1}.$$

[16]) $(\pi^k)^\beta \| n$ means, of course, $\pi^{\beta k} \mid n$, $\pi^{(\beta+1)k} \nmid n$. Its meaning is different from that of $\pi^{\beta k} \| n$, which would mean $\pi^{\beta k} \mid n$, $\pi^{\beta k+1} \nmid n$.

Hence the r-th term is divisible by π^c, where

$$c > \theta - \frac{r}{\pi-1} + r(\lambda - \theta - 1) = \lambda + \frac{\pi-2}{\pi-1} r + (r-1)(\lambda - \theta - 2) - 2.$$

If $\pi > 2$, $c - \lambda > \frac{1}{2} r - 2 \geqq -1$. If $\pi = 2$, $\lambda \geqq \theta + 3$, and so $c - \lambda > r - 1 - 2 \geqq -1$. In either case $c - \lambda > -1$, or $c - \lambda \geqq 0$.

2.4. Lemma 2: $\sum_{\lambda=0}^{\mu} A_{\pi^\lambda}(n) = \pi^{\mu(1-s)} M(\pi^\mu, n)$.

Writing, as usual, $q = \pi^\lambda$, we have

$$A_q = A_q(n) = \sum_p{}' \left(\frac{S_{p,q}}{q}\right) e_q(-np)$$

$$= q^{-s} \sum_p{}' \sum_{x_1, x_2, \ldots, x_s = 0}^{q-1} e_q(p(x_1^k + x_2^k + \ldots + x_s^k - n)) = q^{-s} \sum_{x_1, x_2, \ldots, x_s}{}' c_q(X),$$

where $X = x_1^k + x_2^k + \ldots + x_s^k - n$ and $c_q(X)$ is Ramanujan's sum[17])

$$c_q(X) = \sum_p{}' e_q(pX).$$

If $\lambda = 1$,

$$q = \pi, \; c_\pi(X) = -1 \quad (\pi \nmid X), \; c_\pi(X) = \pi - 1 \quad (\pi \mid X),$$

and

$$(2.41) \quad A_\pi = \pi^{-s}\Big(\sum_x (-1) + \sum_{\pi \mid X} \pi\Big) = \pi^{-s}(-\pi^s + \pi M(\pi)) = \pi^{1-s} M(\pi) - 1.$$

If $\lambda > 1$,

$$c_{\pi^\lambda}(X) = 0 \quad (\pi^{\lambda-1} \nmid X), \quad c_{\pi^\lambda}(X) = -\pi^{\lambda-1} \quad (\pi^{\lambda-1} \| X),$$

$$c_{\pi^\lambda}(X) = \pi^{\lambda-1}(\pi - 1) \quad (\pi^\lambda \mid X),$$

$$A_{\pi^\lambda} = \pi^{-\lambda s}\Big(\sum_{\pi^{\lambda-1} \mid X} (-\pi^{\lambda-1}) + \sum_{\pi^\lambda \mid X} \pi^\lambda\Big)$$

$$= \pi^{\lambda - \lambda s} M(\pi^\lambda) - \pi^{\lambda - \lambda s - 1} \nu(\pi^\lambda, \pi^{\lambda-1}, n).$$

If now

$$x_r = \xi_r + \alpha_r \pi^{\lambda-1} \quad (0 \leqq \xi_r < \pi^{\lambda-1}, \; 0 \leqq \alpha_r < \pi),$$

we have

$$\sum x_r^k \equiv \sum \xi_r^k \pmod{\pi^{\lambda-1}},$$

and so

$$\nu(\pi^\lambda, \pi^{\lambda-1}, n) = \sum_{\alpha_1, \alpha_2, \ldots, \alpha_s} \nu(\pi^{\lambda-1}, \pi^{\lambda-1}, n) = \pi^s M(\pi^{\lambda-1}),$$

$$(2.42) \quad A_{\pi^\lambda} = \pi^{\lambda(1-s)} M(\pi^\lambda) - \pi^{(\lambda-1)(1-s)} M(\pi^{\lambda-1}).$$

[17]) See S. Ramanujan, On certain trigonometrical sums and their applications in the theory of numbers, Transactions of the Cambridge Philosophical Society, 22 (1918), pp. 259–276; G. H. Hardy, Note on Ramanujan's function $c_q(n)$, Proceedings of the Cambridge Philosophical Society, 20 (1921), pp. 263–271; and P. N. 3.

The lemma follows from (2.41) and (2.42). As corollaries we have

Lemma 3: $\chi_\pi \geqq 0$.

Lemma 4: *If n is representable in any manner as a sum of s (positive, negative, or zero) k-th powers, then $\chi_\pi > 0$.*

Lemma 3 is an immediate consequence of Lemma 2. To prove Lemma 4 we have only to observe that $A_{\pi\lambda} = 0$ from a certain value of λ onwards[18]), and that, under the hypothesis of the lemma, $M(\pi^\lambda) > 0$ for every λ.

2.5. Lemma 5. *If $\pi^\theta | k$ $(\theta \geqq 0)$, then*

(2.51) $$N(\pi^\mu, m) = \pi^{(\mu-\varphi)(s-1)} N(\pi^\varphi, m),$$

where φ is defined as in Theorem 2, $\mu \geqq \varphi$, and m is arbitrary.

We may suppose $\mu > \varphi$, and write

(2.52) $$x_r = \xi_r + \alpha_r \pi^{\mu-\theta-1} \quad (0 \leqq \xi_r < \pi^{\mu-\theta-1},\ 0 \leqq \alpha_r < \pi^{\theta+1}).$$

Let $k = \pi^\theta k_0$. Then $(k_0, \pi) = 1$. Also

(2.53) $$x_r^k \equiv \xi_r^k + k_0 \alpha_r \xi_r^{k-1} \pi^{\mu-1} \pmod{\pi^\mu},$$

by Lemma 1. If now

(2.54) $$m = m_1 + m_2 \pi^{\mu-1} \quad (0 \leqq m_1 < \pi^{\mu-1}),$$

the congruence

(2.55) $$\sum x_r^k \equiv m \quad (\mathrm{mod}\ \pi^\mu,\ 0 \leqq x_r < \pi^\mu),$$

s equivalent to the pair of congruences

(2.551) $$\sum \xi_r^k = m_1 \equiv m \quad (\mathrm{mod}\ \pi^{\mu-1},\ 0 \leqq \xi_r < \pi^{\mu-\theta-1}),$$

and

(2.552) $$\sum k_0 \alpha_r \xi_r^{k-1} \equiv m_2 - \frac{\Sigma \xi_r^k - m_1}{\pi^{\mu-1}} \quad (\mathrm{mod}\ \pi,\ 0 \leqq \alpha_r < \pi^{\theta+1}).$$

In what follows we take into consideration only primitive solutions of (2.55) and (2.551). In such a solution of (2.551) some ξ, say ξ_1, is not divisible by π. This being so, the values of $\alpha_2, \alpha_3, \ldots$ in (2.552) may be assigned arbitrarily, and then, since $(k_0 \xi_1^{k-1}, \pi) = 1$, the value of α_1 will be determined uniquely to modulus π. There will therefore be π^θ possible values of α_1 less than $\pi^{\theta+1}$, and $\pi^\theta (\pi^{\theta+1})^{s-1} = \pi^{(\theta+1)s-1}$ sets of α's associated with every solution of (2.551). That is to say we have

(2.56) $$N(\pi^\mu, m) = \pi^{(\theta+1)s-1} N_1,$$

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[18]) See S. 369, footnote [14]).

where $N(\pi^\mu, m)$ is the number of primitive solutions of (2.55) and N_1 the number of primitive solutions of (2.551).

Again, $N(\pi^{\mu-1}, m)$ is the number of primitive solutions of

(2.57) $$\sum x_r^k \equiv m \quad (\operatorname{mod} \pi^{\mu-1},\ 0 \leqq x_r < \pi^{\mu-1}).$$

If here we write

$$x_r = \xi_r + \alpha_r \pi^{\mu-\theta-1} \quad (0 \leqq \xi_r < \pi^{\mu-\theta-1},\ 0 \leqq \alpha_r < \pi^\theta),$$

and use Lemma 1 and the hypothesis $\mu > \varphi$, we obtain

$$x_r^k \equiv \xi_r^k \qquad (\operatorname{mod} \pi^{\mu-1}).$$

Hence

(2.58) $$N(\pi^{\mu-1}, m) = \sum_{\alpha_1, \alpha_2, \ldots, \alpha_s} N_1 = \pi^{\theta s} N_1.$$

From (2.56) and (2.58) we deduce

(2.59) $$N(\pi^\mu, m) = \pi^{s-1} N(\pi^{\mu-1}, m) \qquad (\mu > \varphi),$$

and the lemma follows immediately.

Proof of Theorem 2.

2.61. Let ν be the integer such that $\pi^{\beta k+\nu} | n$, so that $0 \leqq \nu < k$; let

(2.611) $$\lambda_0 = \operatorname{Max}(\beta k + \nu + 1,\ \beta k + \varphi);$$

and suppose that $\lambda \geqq \lambda_0$.

We divide the solutions (primitive or imprimitive) of

(2.612) $$\sum x_r^k \equiv n \qquad (\operatorname{mod} \pi^\lambda,\ 0 \leqq x_r < \pi^\lambda),$$

into classes as follows. In the first class we put the primitive solutions, $N(\pi^\lambda, n)$ in number; in the second class the solutions in which every x is divisible by π but not every x by π^2; in the third those in which every x is divisible by π^2 but not every x by π^3; and so on.

The second class of solutions is correlated with the class of primitive solutions of

(2.613) $$\sum y_r^k \equiv \frac{n}{\pi^k} \quad (\operatorname{mod} \pi^{\lambda-k},\ 0 \leqq y_r < \pi^{\lambda-1}).$$

If we write

$$y_r = \xi_r + \alpha_r \pi^{\lambda-k} \quad (0 \leqq \xi_r < \pi^{\lambda-k},\ 0 \leqq \alpha_r < \pi^{k-1}),$$

then

$$\sum y_r^k \equiv \sum \xi_r^k \qquad (\operatorname{mod} \pi^{\lambda-k}),$$

and the number of primitive solutions of (2.613) is plainly $\pi^{(k-1)s}$ times the number of similar solutions of

$$\sum \xi_r^k \equiv \frac{n}{\pi^k} \qquad (\operatorname{mod} \pi^{\lambda-k},\ 0 \leqq \xi_r < \pi^{\lambda-k}),$$

or is

$$\pi^{(k-1)s} N\left(\pi^{\lambda-k}, \frac{n}{\pi^k}\right).$$

Similarly the number of solutions of the $(\alpha+1)$-th class, where $\alpha \leqq \beta$, is

$$\pi^{\alpha(k-1)s} N\left(\pi^{\lambda-\alpha k}, \frac{n}{\pi^{\alpha k}}\right). \tag{2.615}$$

There are no solutions of any higher class, since $(\beta+1)k \geqq \beta k + \nu + 1$ and $\pi^{\beta k+\nu+1} \nmid n$. Hence, if $\lambda \geqq \beta k + \nu + 1$, and so certainly if $\lambda \geqq \lambda_0$, we have

$$M(\pi^\lambda, n) = \sum_{\alpha=0}^{\beta} \pi^{\alpha(k-1)s} N\left(\pi^{\lambda-\alpha k}, \frac{n}{\pi^{\alpha k}}\right). \tag{2.616}$$

2.62. Again, if $\lambda - \alpha k \geqq \varphi$, and so certainly if $\lambda \geqq \lambda_0$ and $\alpha \leqq \beta$, we have, by Lemma 5,

$$N\left(\pi^{\lambda-\alpha k}, \frac{n}{\pi^{\alpha k}}\right) = \pi^{(\lambda-\alpha k-\varphi)(s-1)} N\left(\pi^{\varphi}, \frac{n}{\pi^{\alpha k}}\right). \tag{2.621}$$

Making this substitution in (2.616), and multiplying by $\pi^{\lambda(1-s)}$, we obtain

$$\begin{aligned} \pi^{\lambda(1-s)} M(\pi^\lambda, n) &= \sum_{\alpha=0}^{\beta} \pi^{\lambda(1-s)} \cdot \pi^{\alpha(k-1)s} \cdot \pi^{(\lambda-\alpha k-\varphi)(s-1)} N\left(\pi^{\varphi}, \frac{n}{\pi^{\alpha k}}\right) \\ &= \sum_{\alpha=0}^{\beta} \pi^{\alpha(k-s)+\varphi(1-s)} N\left(\pi^{\varphi}, \frac{n}{\pi^{\alpha k}}\right). \end{aligned} \tag{2.622}$$

If $\alpha < \beta$ and $\varphi \leqq k$, $\frac{n}{\pi^{\alpha k}}$ is divisible by π^φ, and we may replace it in N by 0. If $\pi > 2$, $\varphi = \theta + 1 \leqq 2^\theta \leqq \pi^\theta \leqq k$. If $\pi = 2$, $\varphi = \theta + 2 \leqq 2^\theta \leqq k$ unless $\theta = 0$ or $\theta = 1$, in which cases $\varphi \leqq 3 \leqq k$. Hence we may replace every N in (2.622), except that for which $\alpha = \beta$, by 0.

It follows that the right hand side of (2.622), is equal, when $\lambda \geqq \lambda_0$, to the value for χ_π given in Theorem 2. It is also independent of λ, and therefore, by Lemma 2, equal to

$$\lim_{\lambda\to\infty} \pi^{\lambda(1-s)} M(\pi^\lambda, n) = \chi_\pi. \tag{2.623}$$

This completes the proof of the theorem. We may observe that we have shown incidentally that

$$A_{\pi^\lambda} = 0 \qquad (\lambda \geqq \lambda_0). \tag{2.624}$$

3. Some properties of the sums $S_{p,q}$.

3.1. In this section we establish certain properties of the Gaussian sums

$$(3.11)\qquad S_{p,q} = S_{p,q,k} = \sum_{j=0}^{q-1} e_q(j^k p)$$

which will be useful for the further study of the singular series[19]). We have not attempted to make the theory complete, though we have developed it a little further than is absolutely necessary.

We denote by

$$\chi = \chi_\varkappa = \chi_\varkappa(m) \qquad (1 \leqq \varkappa \leqq h = \varphi(q))$$

the h Dirichlet's ,characters' to modulus q.[20]) χ_1 is the principal character, and $\bar{\chi}_\varkappa$ is the character conjugate to $\chi_\varkappa$. We shall be concerned only with the case $q = \pi^\lambda$, where $\pi > 2$ and $\lambda \geqq 1$.

It will be convenient to write

$$(3.12)\qquad S'_{p,q} = \sum_{j=0}^{q-1} \chi_1(j)\, e_q(j^k p) = \sum_{(j,q)=1} e_q(j^k p).$$

It is plain that, if $\lambda \leqq k$,

$$(3.13)\qquad S_{p,q} = S'_{p,q} + \sum_{\pi | j} 1 = S'_{p,q} + \pi^{\lambda-1}.$$

3.2. Lemma 6. *If $(l, q) = 1$ then*

$$\sum_\varkappa \bar{\chi}_\varkappa(l)\, \chi_\varkappa(m) = 0$$

unless $m \equiv l \pmod q$, in which case the sum is h.

The result is obvious if $(m, q) > 1$. If $(m, q) = 1$, we determine m from the congruence $mm' \equiv 1 \pmod q$. We have then

$$\bar{\chi}_\varkappa(l)\, \chi_\varkappa(m) = \bar{\chi}_\varkappa(l)\, \bar{\chi}_\varkappa(m') = \bar{\chi}_\varkappa(lm'),$$

and

$$\sum_\varkappa \bar{\chi}_\varkappa(lm') = 0$$

unless $lm' \equiv 1$ or $m \equiv l$, in which case the sum is h.

---

[19]) What we do is, in effect, to develop from our own point of view certain portions of the theory of the division of the circle (*Kreisteilung*). It is not unlikely that the substance of our analysis is to be found elsewhere; but it is not altogether easy to extract, from the classical accounts of the theory, the particular parts which we require.

[20]) A systematic account of the theory will be found in Landau's *Handbuch*, 1 (Zweites Buch)

We write

(2.21) $$\delta = (h, k) = (\varphi(q), k) = (\pi^{\lambda-1}(\pi-1), k).$$

3.3. Lemma 7. *There are just δ characters $\chi_\varkappa$ which possess the property*

(3.31) $$\chi_\varkappa^k = \chi_1.$$

These characters are given by

(3.32) $$\chi_\varkappa(l) = e\left(\frac{\varrho z}{\delta}\right),$$

where $\varrho = 0, 1, 2, \ldots, \delta - 1$ and z is the index of l.

We have generally

(3.33) $$\chi_\varkappa(l) = e\left(\frac{yz}{h}\right),$$

where y is the index which specifies $\varkappa$.[21] The necessary and sufficient condition for (3.31) is that $kyz \equiv 0 \pmod h$ for every z, or that

(3.34) $$ky \equiv 0 \pmod h.$$

From (3.34) we deduce

(3.35) $$\frac{ky}{\delta} \equiv 0 \left(\operatorname{mod} \frac{h}{\delta}\right),$$

which has the single solution $y \equiv 0$ to modulus $\frac{h}{\delta}$. Thus (3.35) has the δ solutions

$$y \equiv \frac{\varrho h}{\delta} \qquad (\varrho = 0, 1, \ldots, \delta - 1)$$

to modulus h. These are all solutions of (3.34), and are plainly the only solutions.

We shall call the characters $\chi' = \chi_{\varkappa'}$ which satisfy (3.31) the *special* characters. It is clear that $\bar{\chi}_{\varkappa'}$ is a special character.

Lemma 8. *We have*

(3.36) $$\sum_{\varkappa'} \bar{\chi}_{\varkappa'}(l) = 0 \;\; (\delta \nmid z), \qquad \sum_{\varkappa'} \bar{\chi}_{\varkappa'}(l) = \delta \;\; (\delta \mid z).$$

For

$$\sum_{\varkappa'} \bar{\chi}_{\varkappa'}(l) = \sum_{\varrho=0}^{\delta-1} e\left(-\frac{\varrho z}{\delta}\right).$$

Lemma 9. *Suppose that $q = \pi^\lambda$ $(\lambda > 1)$, and that $k \mid \pi - 1$, so that $\delta = k$. Then*

(3.37) $$\sum_l e_q(lp) = 0,$$

[21]) Landau, S. 401–402.

if $(p, q) = 1$ *and the summation is extended over those residues* l *of* q *for which* $\delta \mid z$.

We denote by

$$G = g + m\pi$$

the primitive root $(\bmod q)$ to which the indices refer, g being a primitive root $(\bmod \pi)$.[22])

Suppose first that $\delta = k = \pi - 1$. Then the indices of the l's in question are

$$0, \quad \pi - 1, \quad 2(\pi - 1), \quad \ldots, \quad (\pi^{\lambda-1} - 1)(\pi - 1).$$

Suppose that z_1 and z_2 are any two of these $\pi^{\lambda-1}$ indices, $z_2 > z_1$, and l_1 and l_2 the corresponding values of l. Then

$$l_2 - l_1 \equiv G^{z_1}(G^{z_2 - z_1} - 1) = G^{z_1}(G^{\mu\delta} - 1) \qquad (\bmod \pi),$$

where μ is an integer, and

$$G^{\mu\delta} - 1 \equiv g^{\mu\delta} - 1 \equiv 0 \qquad (\bmod \pi).$$

Hence $l_2 - l_1 \equiv 0 \ (\bmod \pi)$. On the other hand, l_1 and l_2 are incongruent to modulus q, since $\mu\delta = z_2 - z_1 < \pi^{\lambda-1}(\pi - 1)$ and G is a primitive root for q. It follows that the l's in question are the numbers of the arithmetical progression

$$1, \quad \pi + 1, \quad 2\pi + 1, \quad \ldots, \quad (\pi^{\lambda-1} - 1)\pi + 1,$$

so that

$$\sum_l e_q(-lp) = e_q(-p) \sum_{r=0}^{\pi^{\lambda-1}-1} e\left(-\frac{rp}{\pi^{\lambda-1}}\right) = 0.$$

The lemma is therefore proved when $\delta = \pi - 1$. The extension to the general case is immediate. The indices of the l's in question are now

$$0, \ \delta, \ 2\delta, \ldots, \ \pi - 1, \ldots, \ \pi^{\lambda-1}(\pi - 1) - \delta$$

and form $\frac{\pi - 1}{\delta}$ arithmetical progressions of the type

$$A, \ A + \pi - 1, \ldots, \ A + (\pi^{\lambda-1} - 1)(\pi - 1),$$

where A is one of $0, \delta, 2\delta \ldots, \pi - 1 - \delta$. The l's corresponding to the indices contained in any one of these progressions form an arithmetical pregression of difference π, and the sum of the lemma splits up into $\frac{\pi - 1}{\delta}$ sums which vanish individually.

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[22]) Landau, *Handbuch*, S. 394.

3.4. Lemma 10. *We have*

$$(3.41)\qquad S'_{p,q}=\sum_{l=0}^{q-1} e_q(lp)\sum_{\varkappa'}\bar{\chi}_{\varkappa'}(l),$$

the summation with respect to $\varkappa'$ extending over all special characters.

We may plainly restrict l to values prime to q. If $(l,q)=1$, $(m,q)=1$, we have, by Lemma 6,

$$\sum_l e_q(lp)\sum_\varkappa \bar{\chi}_\varkappa(l)\chi_\varkappa(m)=\sum_{l\equiv m}\sum_\varkappa+\sum_{l\not\equiv m}\sum_\varkappa = h\,e_q(mp).$$

Hence, if j runs through values less than and prime to q,

$$S'_{p,q}=\sum_j e_q(j^k p)=\frac{1}{h}\sum_j\sum_l\sum_\varkappa e_q(lp)\,\bar{\chi}_\varkappa(l)\,\chi_\varkappa(j^k)$$
$$=\frac{1}{h}\sum_l e_q(lp)\sum_\varkappa \bar{\chi}_\varkappa(l)\sum_j(\chi_\varkappa(j))^k.$$

The sum with respect to j is zero unless $\chi_\varkappa$ is special, when it is h: whence the lemma.

Lemma 11. *If $q=\pi^\lambda$ $(1\leqq\lambda\leqq k)$ and $\delta=(h,k)$, then*

$$(3.42)\qquad S'_{p,q,k}=S'_{p,q,\delta}.$$

This is an immediate consequence of Lemma 10. For the right hand side of (3.41) involves k only in so far as the special characters are fixed by k, and is therefore unaltered when k is replaced by δ.

Lemma 12. *If $q=\pi^\lambda$ $(1<\lambda\leqq k)$ and $\pi \nmid k$, then*

$$(3.43)\qquad S_{p,q,k}=\pi^{\lambda-1}.$$ [23]

It is plain from (3.13) that what we have to prove is

$$(3.44)\qquad S'_{p,q,k}=0,$$

or, by Lemma 11,

$$(3.45)\qquad S'_{p,q,\delta}=0.$$

By Lemmas 10 and 8, we have

$$S'_{p,q,\delta}=\sum_l e_q(lp)\sum_{\varkappa'}\bar{\chi}_{\varkappa'}(l)=\delta\sum_l e_q(lp),$$

where the last summation is restricted to values of l whose indices are multiples of δ; and this sum is zero, by Lemma 9 [24]).

[23]) This has been proved already, in a different manner, in P. N. 2, S. 19–21; but it is interesting to see how the result arises from our present point of view.

[24]) Since $\delta\mid\pi-1$ when $\pi\nmid k$.

3.5. Lemma 13. *If $\lambda = 1$, $q = \pi$, and $\delta = 1$, then*

(3.51) $$S_{p,q,k} = 0.$$

But if $\delta > 1$ then

(3.52) $$S_{p,q,k} = \sum_{\varkappa'} \tau_{\bar{\varkappa}'} \chi_{\varkappa'}(p),$$

where

(3.53) $$\tau_{\bar{\varkappa}} = \sum_{l} e_q(l) \bar{\chi}_{\varkappa}(l),$$

and the summation with respect to $\varkappa'$ extends over the special characters χ', exclusive of the principal character χ_1. Also

(3.54) $$|S_{p,q,k}| \leqq (\delta - 1)\sqrt{q}.$$

We may regard (3.52) as including (3.51), since its right hand side disappears when $\delta = 1$.

We have, by (3.13) and (3.41),

$$S_{p,q,k} = 1 + S'_{p,q,k} = 1 + \sum_{l} e_q(lp)\bar{\chi}_1(l) + \sum_{l} e_q(lp) \sum_{\varkappa'} \bar{\chi}_{\varkappa'}(l),$$

where the principal character is now excluded from the summation with respect to $\varkappa'$, and l runs from 0 to $q-1$. The sum of the first two terms is

$$1 + c_q(p) = 1 + \mu(q) = 0.$$

The third term is

$$\sum_{\varkappa'} \chi_{\varkappa'}(p) \sum_{l} e_q(lp) \bar{\chi}_{\varkappa'}(lp).$$

Since lp runs through the residues of q when l does so, the inner sum is $\tau_{\bar{\varkappa}'}$, whence the result of the lemma.

Finally, to prove (3.54), we have only to observe that, q being prime, $\bar{\chi}_{\varkappa}$ is primitive (*eigentlich*)[25]), and

$$|\tau_k| = \sqrt{q}.$$

4. The behaviour of χ_π for large values of π.

4.1. In this section we are concerned with large values of π, and may suppose $\pi > k$, so that $\theta = 0$, $\varphi = 1$. The O's which occur refer to the passage of π to infinity; the constants which they imply depend upon k and s, but not upon n.

We suppose that $k \geqq 3$.

Lemma 14. *We have*

(4.11) $$A_\pi = \pi^{-s} \sum_{p} e_\pi(-np) \Big(\sum_{\varkappa'} \tau_{\bar{\varkappa}'} \chi_{\varkappa'}(p) \Big)^s,$$

[25]) Landau, *Handbuch*. S. 479.

where the summation with respect to $\varkappa'$ extends over all special characters other than the principal character.

This follows at once from (3.52).

Lemma 15. *If* $s \geqq 1$, $\beta = 0$, *then*

$$\chi_\pi = 1 + O(\pi^{\frac{1}{2} - \frac{1}{2}s}). \tag{4.12}$$

We suppose first that $\pi \nmid n$, so that $\nu = 0$. Then

$$\chi_\pi = 1 + A_\pi. \tag{4.13}$$

Here we replace A_π by the right hand side of (4.11). Any product of χ's is a χ and so, when we expand by the multinomial theorem and invert the order of summation, we obtain

$$A_\pi = \pi^{-s} \sum\nolimits_1 T \sum_p \chi(p) e_\pi(-np),$$

where T is a product of s τ's, χ a product of s χ's, and the number of terms in $\sum_1$ is $O(1)$. The inner sum is $O(\sqrt{\pi})$ for every χ and all values of n in question[26]), and so

$$A_\pi = O(\pi^{-s} \cdot (\sqrt{\pi})^s \cdot \sqrt{\pi}) = O(\pi^{\frac{1}{2} - \frac{1}{2}s}),$$

which proves the lemma when $\pi \nmid n$.

Next suppose $\pi \mid n$, so that $0 < \nu < k$. In this case $\lambda_0 = \nu + 1$ and

$$\chi_\pi = 1 + A_\pi + \sum_2^{\nu+1} A_{\pi^\lambda}. \tag{4.14}$$

Now $S_{p,\pi^\lambda} = \pi^{\lambda - 1}$ for $2 \leqq \lambda \leqq \nu + 1 \leqq k$, by Lemma 12; and so

$$A_{\pi^\lambda} = \pi^{-s} \sum_p e_{\pi^\lambda}(-np) = \pi^{-s} c_{\pi^\lambda}(n),$$

$$A_{\pi^\lambda} = \pi^{\lambda - s - 1}(\pi - 1) \quad (2 \leqq \lambda \leqq \nu), \qquad A_{\pi^\lambda} = -\pi^{\lambda - s - 1} \quad (\lambda = \nu + 1),$$

$$\sum_2^{\nu+1} A_{\pi^\lambda} = -\pi^{1-s}.$$

Thus

$$\chi_\pi = 1 + O(\pi^{\frac{1}{2} - \frac{1}{2}s}) - \pi^{1-s} = 1 + O(\pi^{\frac{1}{2} - \frac{1}{2}s}).$$

This completes the proof of Lemma 15.

If n is fixed, $\pi \nmid n$ from a certain value of π onwards. Hence we obtain

Theorem 3. *The singular series* $S = \Sigma A_q$, *and the product* $P = \Pi \chi_\pi$, *are absolutely convergent for* $s \geqq 4$, *and* $S = P$.

---

[26]) It is -1 if χ is the principal character, and the product of a χ and a τ if χ is non-principal (and so primitive: Landau, *Handbuch*, S. 480).

4.2. Lemma 16. *If* $s \geqq 1$ *then*

(4.21) $$1 + O(\pi^{\frac{1}{2}-\frac{1}{2}s}) < \chi_\pi < (1 + \pi^{k-s} + \ldots + \pi^{\beta(k-s)})(1 + O(\pi^{\frac{1}{2}-\frac{1}{2}s})).$$

This is proved already if $\beta = 0$, and we may suppose $\beta > 0$. From Theorem 2 we have, on the one hand

(4.22) $$\chi_\pi(n) \geqq \pi^{1-s} N(\pi, 0),$$

and on the other

(4.23) $$\chi_\pi(n) \leqq (1 + \pi^{k-s} + \ldots + \pi^{(\beta-1)(k-s)})\pi^{1-s} N(\pi, 0) + \pi^{\beta(k-s)+1-s} N(\pi, n'),$$

where $n' = \frac{n}{\pi^{\beta k}}$. Since neither π nor n' is divisible by π^k, we have

$$\pi^{1-s} N(\pi, 0) = \pi^{1-s} N(\pi, \pi) = \chi_\pi(\pi), \qquad \pi^{1-s} N(\pi, n') = \chi_\pi(n'),$$

and each of these is, by Lemma 15, of the form $1 + O(\pi^{\frac{1}{2}-\frac{1}{2}s})$. Thus (4.21) follows from (4.22) and (4.23).

As a corollary we have

Lemma 17. *If* $s \geqq k + 2$ *then* $\chi_\pi = 1 + O(\pi^{-2})$.

5. The numbers γ_π, $\Gamma(k)$.

5.1. Given k and π, and any positive integer m, there are two possibilities. Either (i) there is a number

(5.11) $$h_\pi = h(k, s, \pi) > 0$$

such that

(5.12) $$\chi_\pi \geqq h_\pi$$

for $s \geqq m$ and all values of n, or (ii) there is no such number. We define

$$\gamma_\pi = \gamma(k, \pi)$$

as the least value of m for which (i) is true, and $\Gamma(k)$ by

(5.13) $$\Gamma(k) = \operatorname*{Max}_\pi \gamma_\pi.$$

Further, we define

$$\gamma'_\pi = \gamma'(k, \pi).$$

as the least value of m such that

(5.14) $$\chi_\pi > 0$$

for $s \geqq m$ and all values of n.

It is evident that $\gamma'_\pi \leqq \gamma_\pi$.

Lemma 18. *If* $\chi_\pi > 0$ *for all sufficiently large values of* n, *then* $\chi_\pi > 0$ *for all values of* n.

In proving this Lemma we leave out of account for the moment the special case $k=4$, $\pi=2$. That the result is still true in this case will appear incidentally later.

It is easy to see that, apart from the exceptional case, $\varphi < k$. Thus if $\pi > 2$, $\varphi = \theta + 1 \leqq 2^{\theta} < \pi^{\theta} \leqq k$.

If $\pi = 2$, $\theta \geqq 3$, then $\varphi = \theta + 2 < 2^{\theta} \leqq k$.

If $\pi = 2$, $\theta = 0$, k is odd and $\varphi = 2 < 3 \leqq k$.

If $\pi = 2$, $\theta = 1$, then k is oddly even and $\varphi = 3 < 6 \leqq k$.

If $\pi = 2$, $\theta = 2$, then $\varphi = 4 < 6 \leqq k$, unless $k = 4$.

Thus $\varphi < k$ in every case except that in which $k=4$, $\pi = 2$, when $\varphi = k$.

Now let

$$n = \pi^{\varphi} m + n' \qquad (0 \leqq n' < \pi^{\varphi}).$$

If $n' \neq 0$ then $\beta = 0$ (since $\varphi < k$) and so, by Theorem 2,

$$\chi_{\pi}(n) = \pi^{\varphi(1-s)} N(\pi^{\varphi}, n) = \pi^{\varphi(1-s)} N(\pi^{\varphi}, n') = \chi_{\pi}(n').$$

But $\chi_{\pi}(n) > 0$ for large values of m, and therefore $\chi_{\pi}(n') > 0$. It follows that $\chi_{\pi} > 0$ for all values of n that are not divisible by π^{φ}.

Again, if $(m, \pi) = 1$, we have, by Theorem 2,

$$\chi_{\pi}(\pi^{\varphi} m) = \pi^{\varphi(1-s)} N(\pi^{\varphi}, 0),$$

since $\varphi < k$. The left hand side is positive if m is large, and so $N(\pi^{\varphi}, 0) > 0$. Hence, whatever be the value of m (prime to π),

$$\chi_{\pi}(\pi^{\varphi} m) \geqq \pi^{\varphi(1-s)} N(\pi^{\varphi}, 0) > 0.$$

It follows that $\chi_{\pi} > 0$ also when $n' = 0$, which proves the lemma.

5.2. Lemma 19. *The necessary and sufficient condition that*

$$N(\pi^{\varphi}, n) > 0, \tag{5.21}$$

for every n, is that $s \geqq \gamma_{\pi}$. Further,

$$\gamma'_{\pi} = \gamma_{\pi} \tag{5.22}$$

except when $k=4$, $\pi=2$, in which case

$$\gamma_2 = 16, \quad \gamma'_2 = 15. \tag{5.23}$$

Leaving aside the exceptional case, so that $\varphi < k$, let $s \geqq \gamma'_{\pi}$. Then $\chi_{\pi}(\pi^{\varphi}) > 0$. But $\beta = 0$ when $n = \pi^{\varphi}$ (since $\varphi < k$), and so

$$\chi_{\pi}(\pi^{\varphi}) = \pi^{\varphi(1-s)} N(\pi^{\varphi}, \pi^{\varphi}) = \pi^{\varphi(1-s)} N(\pi^{\varphi}, 0).$$

Hence

$$N(\pi^{\varphi}, 0) > 0.$$

If on the other hand $n \not\equiv 0 \pmod{\pi^{\varphi}}$, then $\beta = 0$ (since $\varphi \leqq k$). Hence

$$\chi_{\pi} = \pi^{\varphi(1-s)} N(\pi^{\varphi}, n)$$

and

$$N(\pi^{\varphi}, n) > 0.$$

Thus $s \geqq \gamma'_{\pi}$ is a sufficient condition that (5.21) should hold for every n.

Next, suppose that (5.21) holds for $s = s_1$ and every n. Then it holds, *a fortiori*, for $s \geqq s_1$ and every n, and the N's that occur in Theorem 2 are both positive. Hence

$$\chi_{\pi} \geqq \pi^{\varphi(1-s)} \qquad (s \geqq s_1)$$

and so

$$s_1 \geqq \gamma_{\pi} \geqq \gamma'_{\pi}.$$

It follows, first that $s \geqq \gamma'_{\pi}$ is both necessary and sufficient for (5.21), and secondly that $s \geqq \gamma'_{\pi}$ involves $s \geqq \gamma_{\pi}$, *i. e.* that $\gamma'_{\pi} = \gamma_{\pi}$.

If $k = 4$, $\pi = 2$, then $2^{\varphi} = 16$. Now x^4 is congruent to 0 or to 1 to modulus 16, according as x is even or odd. It follows that $N(16, n) > 0$ for $s \geqq 16$ and every n; that

$$N(16, n) > 0 \quad (16 \nmid n), \qquad N(16, 0) = 0$$

when $s = 15$; and that $N(16, 15) = N(16, 0) = 0$ when $s < 15$. Finally it follows, from Theorem 2, that

$$\chi_2 > h_2 \quad (s \geqq 16), \qquad \chi_2 > 0 \quad (s = 15),$$

$$\chi_2(16^{\beta}.\,15) = 2^{\beta(4-15)+4(1-15)} N(16, 15) = 2^{-11(\beta+1)} \qquad (s = 15),\ ^{27})$$

and

$$\chi_2(16^{\beta}.\,15) = 0 \qquad (s < 15).$$

Since $2^{-11(\beta+1)} \to 0$ when $\beta \to \infty$, these results embody (5.23). Incidentally we see that Lemma 18 is still true in the exceptional case.

5.3. Theorem 4: $G(k) \geqq \Gamma(k)$.

Leaving aside for the moment the exceptional case $k = 4$, $\pi = 2$, suppose that $s \geqq G(k)$. Then any sufficiently large n is the sum of s k-th powers, so that $\chi_{\pi} > 0$ for every π and all sufficiently large values of n. Hence, by Lemma 18, $\chi_{\pi} > 0$ for every π and every n, so that $s \geqq \gamma_{\pi}$. It follows that $G(k) \geqq \gamma_{\pi}$ for every π, which proves the theorem, apart from the exceptional case. In this case $\gamma_2 = 16$, and the result is still true, since $G(4) \geqq 16$ [28]).

---

[27]) $N(16, 15) = 8^{15}$ when $s = 15$, since each x may have any one of the values 1, 3, 5, ..., 15.

[28]) The lower bound Γ for G is associated with the vanishing of the singular series S for $s \leqq \Gamma - 1$, *except when* $k = 4$. When $k = 4$, $\Gamma = 16$, and the series is positive for $s = 15$, but assumes arbitrarily small values for suitable values of n.

It should be observed that our proof (see § 5.5 below) that

$$G(\pi^{\theta}(\pi - 1)) \geqq \gamma_{\pi} = \pi^{\varphi} \qquad (\pi > 2)$$

(Fortsetzung der Fußnote 28 auf nächster Seite)

5.4. Lemma 20. *Suppose that* $\pi^\theta | k$, *and that* φ *is defined as in Theorem 2. Further, suppose that*

(5.41) $$k = \pi^\theta \varepsilon k_0,$$

where

(5.42) $$\varepsilon = (\pi^{-\theta} k, \pi - 1),$$

and

(5.43) $$d = \frac{\pi - 1}{\varepsilon};$$

so that $\varepsilon | \pi - 1$ *and* $(k_0, d) = 1$. *Then*

(5.44) $$\gamma_\pi \leqq c = c_\pi = c(k, \pi) = \frac{\pi^\varphi - 1}{\pi - 1} \varepsilon + 1.$$

We write $\varrho = \pi^\varphi$. We must distinguish the cases $\pi > 2$ and $\pi = 2$.

(i) If $\pi > 2$, $\varphi = \theta + 1$. We suppose that G is a primitive root (mod ϱ). We divide the residues to modulus ϱ into classes as follows. Consider first the residues n_0 prime to ϱ. If v is the index of n_0, we have

$$n_0 \equiv G^v \equiv G^{m_0 \psi_0 + e} \qquad (\text{mod } \varrho),$$

where

$$\psi_0 = \frac{\Phi(\varrho)}{d} = \frac{\Phi(\pi^\varphi)}{d} = \frac{\pi^{\varphi - 1}(\pi - 1)}{d} = \pi^{\varphi - 1} \varepsilon,$$ [29])

m_0 has one or other of the d values $0, 1, \ldots, d - 1$, and e one or other of the ψ_0 values $0, 1, \ldots, \psi_0 - 1$. The d values of n_0 with a common e we class together and call the numbers

$$a_e^0 \qquad (e = 0, 1, \ldots, \psi_0 - 1);$$

the class of numbers a_e^0 with a fixed e we call C_e^0.

Next, consider the residues n_i for which $\pi^i | n_i$, where $0 < i < \varphi$. We have

$$n_i \equiv \pi^i N_i,$$

where the N_i's are the $\Phi(\pi^{\varphi - i})$ numbers less than and prime to $\pi^{\varphi - i}$. As G is also a primitive root to modulus $\pi^{\varphi - i}$, we can write

$$N_i \equiv G^{m_i \psi_i + e} \qquad (\text{mod } \pi^{\varphi - i}),$$

$$n_i = \pi^i N_i \equiv \pi^i G^{m_i \psi_i + e} \qquad (\text{mod } \pi^\varphi),$$

---

is essentially the same as Kempner's proof (see pp. 45—46 of his Inaugural-Dissertation) that

$$G(2^\theta) \geq 2^\varphi = 2^{\theta + 2}.$$

His proof too fails when $k = 4$, and he has to appeal to the structure of the particular number 31.

[29]) We write $\Phi(\varrho)$ for Euler's function usually denoted by $\varphi(\varrho)$, as φ is used here in a different sense.

where

$$\psi_i = \frac{\Phi(\pi^{\varphi-i})}{d} = \frac{\pi^{\varphi-i-1}(\pi-1)}{d} = \pi^{\varphi-i-1}\varepsilon,$$

m_i has again one or other of the values $0, 1, \ldots, d-1$, and e one or other of the values $0, 1, \ldots, \psi_i - 1$. The ψ_i new classes obtained in this manner we denote by

$$C_e^i \qquad (e = 0, 1, \ldots, \psi_i - 1),$$

and a typical member of C_e^i by α_e^i.

Finally, the single number 0 is the sole member α_0^φ of a class C_0^φ. The total number of classes into which the residues are divided is

$$\psi_0 + \psi_1 + \ldots + \psi_{\varphi-1} + 1 = \frac{\pi^\varphi - 1}{d} + 1 = c_\pi = c.$$

We may denote the whole system of classes, in the order in which they have been defined, by $C_0, C_1, \ldots, C_r, \ldots, C_c$, and a typical member of C_r by α_r.

The class C_0 consists of the residues of k-th powers of numbers x prime to π. For

$$k = \pi^\theta \varepsilon k_0 = k_0 \frac{\pi^\theta(\pi-1)}{d} = k_0 \psi_0.$$

Also $x \quad G^t$ for some t (since $(x, \pi) = 1$), and

$$x^k \equiv G^{t k_0 \psi_0} = G^{m_0 \psi_0},$$

so that x^k is an α_0. Moreover we can choose t so that $t k_0$ has an arbitrary residue m_0 to modulus d, since $(k_0, d) = 1$, so that every α_0 is an x^k.

Finally, to complete the properties of the classes which are immediately relevant, (1) 1 belongs to C_0, (2) $\alpha_0 \alpha_r$, where α_0 and α_r are any members of C_0 and C_r respectively, belongs to C_r, and (3) $\alpha_0 \alpha_r$, where α_r is a given member of C_r, can be identified with any member of C_r by choice of α_0.

Of these properties (1) is obvious. To prove (2) we observe that, if

$$\alpha_0 \equiv n_0 \equiv G^{m_0 \psi_0}, \quad \alpha_r \equiv n_i \equiv \pi^i G^{m_i \psi_i + e},$$

then

$$\alpha_0 \alpha_r \equiv \pi^i G^{m_0 \psi_0 + m_i \psi_i + e}$$

is an α_r, since $\psi_i \mid \psi_0$. Finally

$$m_0 \psi_0 + m_i \psi_i = (\pi^i m_0 + m_i) \psi_i,$$

and we can choose m_0 so that $\pi^i m_0 + m_i$ shall have an arbitrary residue (mod d), since $(\pi, d) = 1$; hence $\alpha_0 \alpha_r$ can be identified with any member of C_r.

5. 5. To prove Lemma 20 it is enough, by Lemma 19, to show that

$$N(\pi^\varphi, n) > 0 \tag{5.51}$$

for $s \geqq c$ and every n. And the necessary and sufficient condition for (5. 51) is that every n should be congruent $(\text{mod}\ \pi^\varphi)$ to the sum of at most c numbers α_0. If any α_r is the sum of not more than c α_0's, then so, by (2) and (3) of the last paragraph, is every α_r. In these circumstances we shall say that C_r is representable, and what we have to prove is that this is so for all the c values of r.

Suppose that $1 \leqq c' \leqq c$. *Then there are at least c' different classes representable by not more than c' α_0's.* For, in the first place, this is true when $c' = 1$. Suppose that it is true for $c' = \bar{c} < c$ but false for $c' = \bar{c} + 1$, and let $\bar{C}$ be a typical class representable by $\bar{c}$ α_0's, and C_r a $\bar{C}$. Then α_r belongs to a $\bar{C}$, and therefore, since no new classes become representable when $\bar{c}$ is changed to $\bar{c}+1$, $\alpha_r + 1$ belongs to a $\bar{C}$. Similarly $\alpha_r + 1 + 1 = \alpha_r + 2$ belongs to a $\bar{C}$, and, repeating the argument, *every* residue $(\text{mod}\ \varrho)$ belongs to a $\bar{C}$, which is a contradiction.

Taking $c' = c$ we see that c distinct classes, and therefore all residues $(\text{mod}\ \varrho)$, are representable by c α_0's, which proves the lemma, when $\pi > 2$.

(ii) There remains the case $\pi = 2$, in which $\varphi = \theta + 2$, $\varepsilon = d = 1$, $c = \pi^\varphi = \varrho$. In this case there is nothing to prove, for any residue $(\text{mod}\ \varrho)$ is representable by at most ϱ 1's.

A particularly interesting case is that in which $d = 1$, $\varepsilon = \pi - 1$. In this case

$$k = \pi^\theta (\pi - 1) k_0,$$

where k_0 is prime to π. Here

$$\gamma_\pi \leqq \pi^\varphi = \pi^{\theta+1} \quad (\pi > 2), \qquad \gamma_2 \leqq 2^\varphi = 2^{\theta+2} \qquad (\pi = 2).$$

If $\pi > 2$, $\gamma_\pi = \pi^\varphi$. For

$$x^k = x^{\pi^\theta (\pi-1) k_0} \equiv 1 \qquad (\text{mod}\ \pi^\varphi),$$

so that 1 is the only α_0. Hence $N(\pi^\varphi, 0) = 0$ if $s < \pi^\varphi$, and $\gamma_\pi \geqq \pi^\varphi$, by Lemma 19. In particular

$$\gamma_\pi = \pi = k + 1$$

if $k = \pi - 1$. Thus $\gamma_5 = 5$ if $k = 4$, $\gamma_7 = 7$ if $k = 6$.

If $\pi = 2$, $k = 2^\theta k_0$. Suppose first that $\theta > 0$. Then

$$x^{2^\theta} \equiv 1 \qquad (\text{mod}\ 2^{\theta+2}),$$

and so $x^k \equiv 1 \ (\text{mod}\ 2^\varphi)$. Except when $k = 4$ our argument above applies, and we obtain

$$\gamma_2 = 2^\varphi = 2^{\theta+2} \qquad (\theta > 0).$$

The result still holds when $k = 4$, since then $\gamma_2 = 16 = 2^4$.

The argument fails if $\theta = 0$ (so that k is odd). Here $\varrho = 2^2 = 4$; -1 is a k-ic residue (mod 4); and 0, 1, 2, 3 are all representable by at most two of the numbers ± 1. Thus

$$\gamma_2 = 2 = 2^{\theta+1} \qquad (\theta = 0).$$

5.6. In general it is possible to go a little further than in Lemma 20.

Lemma 21. *Suppose that* $d_1 | d$, *where* $d_1 > 1$. *Then*

(5.61) $$\gamma_\pi \leqq \operatorname{Max}(d_1, c-1).$$

Since $d_1 | \pi - 1$, (5.61) gives in particular

$$\gamma_\pi \leqq \operatorname{Max}(\pi - 1, c - 1)$$

in all cases and,

$$\gamma_\pi \leqq \operatorname{Max}(k - 1, c - 1)$$

if $\theta > 0$.

To prove Lemma 21, suppose that $1 \leqq c' \leqq c$, and let $\nu(c')$ be the number of classes, other than the class C_c (containing the residue 0 only), that are representable by not more than c' α_0's. Then

(5.62) $$\nu(c'+1) \geqq \operatorname{Min}(\nu(c')+1, c-1).$$

For, if (5.62) is false $\nu(c'+1) = \nu(c') < c - 1$. Let $\bar{C}$ be a typical class of the $\nu(c')$ classes, and C_r a $\bar{C}$. Then, if α_r belongs to C, $\alpha_r + 1$ must belong to a $\bar{C}$ or to C_c, since no new classes, other than perhaps C_c, are representable by $c' + 1$ α_0's. If $\alpha_r + 1 \equiv 0$, $\alpha_r + 2$ belongs to C_0, and therefore to a $\bar{C}$. If $\alpha_r + 1$ belongs to a $\bar{C}$, $\alpha_\kappa + 2$ must belong to a $\bar{C}$ or to C_c. Repeating the argument, we see that every residue, other than 0, belongs to a $\bar{C}$, which is a contradiction.

From (5.62) it follows that

$$\nu(c-1) \geqq c - 1,$$

so that all residues, 0 perhaps excepted, are representable by at most $c - 1$ α_0's. It remains to consider the residue 0. Let $d = \eta d_1$ and

$$\alpha_0' \equiv G^{\eta\psi_0} \qquad (\operatorname{mod} \varrho).$$

Then $\alpha_0' \not\equiv 1$, since $\eta\psi_0 \not< \varphi(\varrho)$ and G is a primitive root (mod ϱ),

$$(\alpha_0')^{d_1} \equiv G^{d\psi_0} = G^{\Phi(\varrho)} \equiv 1 \qquad (\operatorname{mod} \varrho),$$

$$1 + \alpha_0 + (\alpha_0')^2 + \ldots + (\alpha_0')^{d_1 - 1} = \frac{1 - (\alpha_0')^{d_1}}{1 - \alpha_0'} = 0 \qquad (\operatorname{mod} \varrho),$$

and 0 is representable by d_1 α_0's, which completes the proof of the lemma.

Suppose in particular that $d_1 = d = 2$, so that $\pi > 2$ and

$$k = \frac{1}{2}\pi^\theta(\pi - 1)k_0.$$

In this case the α_0's are the two numbers ± 1, and

$$\gamma_\pi \geqq \frac{1}{2}(\pi^\varphi - 1).$$

But

$$c - 1 = \frac{1}{2}(\pi^{\theta+1} - 1) = \frac{1}{2}(\pi^\varphi - 1),$$

so that

$$\gamma_\pi = \frac{1}{2}(\pi^\varphi - 1) = c - 1.$$

Thus in this case also we can determine γ_π exactly.

5. 7. It is convenient to sum up our results concerning the cases $d = 1$ and $d = 2$ in a separate lemma.

Lemma 22. *If $k = \pi^\theta(\pi - 1)k_0$, where $\pi > 2$ and k_0 is prime to π, then*

$$(5.71) \qquad \gamma_\pi = \pi^{\theta+1}.$$

If $k = 2^\theta k_0$, where $\theta > 0$ and k_0 is odd, then

$$(5.72) \qquad \gamma_2 = 2^{\theta+2}.$$

If k is odd, then $\gamma_2 = 2$.

If $k = \frac{1}{2}\pi^\theta(\pi - 1)k_0$, where $\pi > 2$ and k_0 is prime to π, then

$$(5.73) \qquad \gamma_\pi = \frac{1}{2}(\pi^{\theta+1} - 1).$$

5. 8. We know that $G(k) \geqq \Gamma(k) = \operatorname{Max} \gamma_\pi$. Thus, when k is given, every value of γ_π gives a lower bound for $G(k)$. These, when less than $k + 2$, add nothing to our knowledge of $G(k)$, since $G(k)$ is always greater than k. There is therefore a special interest in determining as systematically as possible all cases in which

$$\gamma_\pi > k + 1.$$

Lemma 23. *We have*

$$(5.81) \qquad \gamma_\pi \leqq k + 1$$

unless (α) $k = 2^\theta$ $(\theta > 0)$, $\pi = 2$, *when* $\gamma_2 = 2^{\theta+2} = 4k$,

(β) $k = 2^\theta 3$ $(\theta > 0)$, $\pi = 2$, *when* $\gamma_2 = 2^{\theta+2} = \frac{4}{3}k$,

or (γ) $k = \pi^\theta \varepsilon$ $(\theta > 0)$, *where* $\pi > 2$ *and* $\varepsilon \mid \pi - 1$.

In cases (α) and (β) (5.81) is false; in case (γ) it may be true or false.

We write $k = \pi^\theta \varepsilon k_0$, as in Lemma 20. If $\theta = 0$, $\pi > 2$, then

$$\gamma_\pi \leqq c = \varepsilon + 1 \leqq k + 1,$$

by Lemma 20. If $\theta = 0$, $\pi = 2$, then $\gamma_2 = 2$ by Lemma 22. Thus we need only consider cases in which $\theta > 0$.

Suppose first $\pi > 2$. If $k_0 > 1$, we have

$$\gamma_\pi \leqq c = \frac{\pi^{\theta+1}-1}{\pi-1}\varepsilon + 1 < \frac{2(\pi^{\theta+1}-\pi^\theta)}{\pi-1}\varepsilon + 1 \leqq \pi^\theta \varepsilon k_0 + 1 = k + 1.$$

Thus (5. 81) is true unless $k_0 = 1$, $k = \pi^\theta \varepsilon$, which is case (γ).

Next suppose $\pi = 2$, $k = 2^\theta k_0$. If $k_0 > 3$, we have

$$\gamma_2 = 2^{\theta+2} = 4\frac{k}{k_0} < k + 1.$$

Thus (5. 81) is true unless $k_0 = 1$ or 3, cases (α) and (β).

The case in which $k = 6$ is interesting as falling under both (β) and (γ). If $\pi = 3$, $k = 3.2 = \pi(\pi-1)$, $\varepsilon = \pi - 1$, $d = 1$, and $\gamma_3 = 3^2 = 9$. And $\gamma_2 = 2^3 = 8$.

In case (γ), (5. 81) may be true or false. Thus it is true when $k = 3$, $\pi = 3$, for then $\gamma_3 = 4$. But it is false when $k = 6$, $\pi = 3$.

5. 9. We must now collect our results and state them as theorems concerning $\Gamma(k)$. We shall say that k is exceptional if it has one of the forms in (α), (β), or (γ) of Lemma 23.

Theorem 5. *If k is not exceptional, then*

$$\Gamma(k) \leqq k + 1.$$

This is an immediate corollary of Lemma 23.

Theorem 6. *If $\theta > 1$ then $\Gamma(2^\theta) = 2^{\theta+2}$.*

Theorem 7. *If $\theta > 1$ then $\Gamma(2^\theta 3) = 2^{\theta+2}$.*

Theorem 8. $\Gamma(6) = 9$.

These theorems follow from Lemma 23, when we observe that the numbers in question in each case exceed $k+1$.

Theorem 9. *If $\pi > 2$, $\theta > 0$, then $\Gamma(\pi^\theta(\pi-1)) = \pi^{\theta+1}$. This equality holds also when $\theta = 0$, provided that $k = \pi - 1$ is not exceptional.*

The second part follows from Theorem 5 and Lemma 22. We may therefore suppose $\theta > 0$. We have already seen that $\gamma_\pi = \pi^{\theta+1}$, which is greater than $k+1$. If π_1 is a prime other than π, $\gamma_{\pi_1} \leqq k+1$ unless $\pi_1 = 2$, $\pi^\theta(\pi-1) = 2^{\theta_1}$, or $\pi_1 = 2$, $\pi^\theta(\pi-1) = 2^{\theta_1}3$, or $\pi_1 > 2$, $\pi^\theta(\pi-1) = \pi_1^{\theta_1}\varepsilon_1$, where $\varepsilon_1 \mid \pi_1 - 1$.

It is easy to see that the first and third alternatives are impossible, and that the second can occur only when $\pi = 3$, $\theta = 1$, $k = 6$. In this case the result has been proved already; in all other cases we have $\gamma_{\pi_1} < \gamma_\pi$ and $\Gamma(k) = \gamma_\pi = \pi^{\theta+1}$.

Theorem 10. *If $\pi > 2$, $\theta > 0$, then*

$$\Gamma\left(\frac{1}{2}\pi^\theta(\pi-1)\right) = \frac{1}{2}(\pi^{\theta+1}-1).$$

Here $\gamma_\pi = \frac{1}{2}(\pi^{\theta+1} - 1)$, since $d = 2$. This is greater than $k + 1$ except when $\pi = 3$, $\theta = 1$, $k = 3$, when the two numbers are equal. Moreover $\frac{1}{2}\pi^\theta(\pi - 1)$ cannot be equal to 2^{θ_1}, $2^{\theta_1}3$, or $\pi_1^{\theta_1}\varepsilon_1$, where $\pi_1 \neq \pi$, $\theta_1 > 0$, $\varepsilon_1 \mid \pi_1 - 1$. Hence $\gamma_{\pi_1} \leqq \gamma_\pi$ and $\Gamma(k) = \gamma_\pi$.

Theorem 11. *If $\pi > 2$ and $k = \pi^\theta \varepsilon$, where $\theta > 0$, $\varepsilon \mid \pi - 1$, then*

$$\Gamma(k) \leqq \mathrm{Max}(\gamma_\pi, k+1).$$

It may be verified at once that $\pi^\theta \varepsilon$ cannot be of any of the forms 2^{θ_1}, $2^{\theta_1}3$, $\pi_1^{\theta_1}\varepsilon_1$, except when $\pi = 3, \theta = 1, \varepsilon = 2, k = 6$. In this case $\Gamma(k) = \gamma_3 = 9$. The result follows from Lemma 23.

Theorem 12. *In all cases*

$$\Gamma(k) \leqq 4k.$$

The sign of equality occurs if and only if $k = 2^\theta$ $(\theta \geqq 2)$.

Theorem 13. *In all cases*

$$\Gamma(k) < (k-2)2^{k-1} + 5.$$

This theorem, which is included in Theorem 12 except when $k = 3$, is inserted only because it is what we require for the proof of Theorem 1. Our actual bounds for $\Gamma(k)$ are much better.

When $k = 3$, $\Gamma(3) = 4 < 9 = 1 \cdot 4 + 5$.

It may help to elucidate the results which we have obtained if we show in tabular form the actual values of $\Gamma(k)$ for a number of values of k.

$k =$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\Gamma(k) =$	4	16	5	9	4	32	13	12	11	16	6	14	15	64	6	27

$k =$	19	20	21	22	23	24	25	26	27	28	29	30	31	32
$\Gamma(k) =$	4	25	24	23	23	32	10	26	40	29	29	30	5	128

The values of $\Gamma(k)$ for $k = 3, 4, 6, 8, 9, 10, 12, 16, 18, 20, 21, 24, 27$ and 28 are given by the actual theorems and lemmas which we have proved; the determination of the remaining values demands further calculations into which we cannot enter here.

6. The behaviour of the singular series when $s \geqq \Gamma(k)$.

6.1. Theorem 15. *Suppose that $k > 2$ and $s_1 = \mathrm{Max}(\Gamma(k), 4)$. Then*

$$S > \sigma \tag{6.11}$$

for $s \geqq s_1$ and all values of n.

By Lemma 16, we have

$$\chi_\pi > 1 - \sigma\pi^{\frac{1}{2} - \frac{1}{2}s} \qquad (s \geqq s_1).$$

Hence there is a $\pi_0 = \pi_0(k, s)$ such that

$$\chi_\pi > 1 - \sigma\pi^{\frac{1}{2}-\frac{1}{2}s} > 1 - \sigma\pi^{-\frac{3}{2}} \qquad (s \geqq s_1, \pi \geqq \pi_0);$$

and so

$$\prod_{\pi \geqq \pi_0} \chi_\pi > \sigma \qquad (s \geqq s_1). \tag{6.12}$$

But $\chi_\pi > \sigma$ if $\pi < \pi_0$ and $s \geqq \Gamma(k)$, and so

$$\prod_{\pi < \pi_0} \chi_\pi > \sigma \qquad (s \geqq s_1); \tag{6.13}$$

and (6.11) follows from (6.12) and (6.13).

It is plain that our main purpose is now accomplished; with Theorems 13 and 15, the proof of Theorem 1 is completed.

6.2. It is of some interest also to obtain an upper bound for S.

Theorem 16. *If* $s \geqq k + 2$ *then*

$$S < \sigma. \tag{6.21}$$

For, by Lemma 16,

$$\chi_\pi < (1 + \sigma\pi^{-2})(1 + \sigma\pi^{-\frac{3}{2}}) < 1 + \sigma\pi^{-\frac{3}{2}};$$

and the result follows immediately.

Theorem 17. *If* $s \geqq k > 3$, *then*

$$S < n^\epsilon \tag{6.22}$$

for all sufficiently large values of n.

By Lemma 16

$$\chi_\pi < \left(1 + \sigma\pi^{-\frac{3}{2}}\right)\varrho_\pi,$$

where $\varrho_\pi = 1$ unless $\pi^k \mid n$, and then $\varrho_\pi = 1 + \beta$. It is plain that

$$\prod \varrho_\pi \leqq \prod_{\pi \mid n} (1 + a) = d(n),$$

where $\pi^a \mid n$. As $d(n) = O(n^\epsilon)$, the theorem follows.

The interest of this theorem lies in the resulting equation

$$\varrho_{k,k}(n) = O(n^\epsilon). \tag{6.23}$$

There is some reason for supposing that

$$r_{k,k}(n) = O(n^\epsilon), \tag{6.24}$$

an equation from which very important consequences would follow. This equation would cease to be plausible if (6.23) at any rate were not true.

6.3. In conclusion, we return for a moment the equations (1.15) and (1.151). As we remarked before, the equation (1.15) is sufficient for our present purpose; but it is interesting to bring the remark of Ostrowski into relation with our analysis.

Suppose that

$$N(\pi^{\varphi}, n) \geqq 1$$

for every n and for $s = s_0$. There is then a primitive solution of

$$x_1^k + x_2^k + \dots + x_{s_0}^k \equiv n \pmod{\pi^{\varphi}}$$

for every n. Consider now the similar congruence in which s_0 is replaced by $s > s_0$. Of the x's, the last $s - s_0$ may then be selected arbitrarily, and there will be at least one primitive solution of the ensuing congruence in the first s_0. Hence

$$N_s(\pi^{\varphi}, n) \geqq \pi^{\varphi(s-s_0)}.$$

It follows that the inequalities which we have used, of the type

$$\chi_\pi \geqq \pi^{\varphi(1-s)};$$

may be replaced by inequalities of the type

$$\chi_\pi \geqq \pi^{\varphi(1-s)} \pi^{\varphi(s-s_0)} = \pi^{\varphi(1-s_0)};$$

and our numbers $h_\pi = h(k, \pi, s)$ and $\sigma = \sigma(k, s)$ by numbers of the type $h_\pi = h(k, \pi, s_0) = h(k, \pi)$, and $\sigma = \sigma(k, s_0) = \sigma(k)$. It is however unnecessary to develop this remark further at the moment.

We add, finally, that the number $\Gamma(k)$ has a simple and interesting arithmetical interpretation. In fact $\Gamma(k)$ is: *the least number m such that every arithmetical progression contains an infinity of numbers which are sums of m k-th powers.*

(Eingegangen am 31. Oktober 1921.)