The Vanishing of the Wronskian and the Problem of Linear Dependence.*)

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Although the identical vanishing of the Wronskian of n analytic functions of a single variable is a sufficient condition for their linear dependence, with non-analytic functions additional hypotheses are required.**) Peano was probably the first to call attention to this fact***), and to point out the importance of a further study of the subject. To the criteria for linear dependence given in Peano's papers Bôcher has since added others of a more general character, his results being summarized in an article published in the Transactions of the American Mathematical Society, vol. 2 (1901), p. 139.†) The present paper will show that in all the cases considered by Peano and Bôcher sufficient conditions can be given in terms of the rank of a functional matrix.

We shall consider functions, real or complex, of a real variable x. The theorems which follow are so stated as to apply to cases where the interval I, to which x is confined, is infinite, as well as to finite

*) An announcement of some of the results of this paper, without proof, has appeared in the Bulletin of the American Mathematical Society, ser. 2, vol. 12 (1906), p. 482.

**) The identical vanishing of the Wronskian is, however, a necessary condition for linear dependence if the functions have finite derivatives of the first $n-1$ orders.

***) In Mathesis, vol. 9 (1889), p. 75 and p. 110. In the latter note the case of the two functions x^2 and $x \cdot |x|$ is cited as an illustration. Further examples have been given by Bôcher in the articles hereafter cited. Peano has another paper on the same subject in the Rendiconti della R. Accademia dei Lincei, ser. 5, vol. 6, 1° sem. (1897), p. 413.

†) See also Bulletin of the American Mathematical Society, ser. 2, vol. 7 (1900), p. 120, and Annals of Mathematics, ser. 2, vol. 2 (1901), p 93. The properties of Wronskians of functions of a real variable have been further investigated by the same writer in the Bulletin of the American Mathematical Society, ser. 2, vol. 8 (1901), p. 53.

intervals; and I, if limited in either direction, may or may not include its endpoints.

The symbol $[P]$ will be used to designate any infinite set of points in I having a given limit point p which is included in the set. If p, though still a point of I, does not belong to the set, this will be indicated by the notation $[P^*]$. A function will be said to vanish in a point set $[P]$ or $[P^*]$ if it vanishes at each point of such a set.

§ 1.

The Vanishing of the Wronskian in Point sets and the Vanishing of Related Matrices.

The functional matrices to be considered are of the form:

$$\begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ u_1^{(k)} & u_2^{(k)} & \cdots & u_n^{(k)} \end{vmatrix},$$

where $u_1, u_2, \cdots, u_n$ are functions of x possessing finite derivatives of the first k orders $(k \geqq n-1)$ at each point of I. Such a matrix will be designated by the symbol $M_k(u_1, u_2, \cdots, u_n)$. The Wronskian $W(u_1, u_2, \cdots, u_n)$ is the n-rowed determinant whose matrix is $M_{n-1}(u_1, u_2 \cdots, u_n)$.

We now proceed to investigate relations which exist between the vanishing of the Wronskian and of other determinants of the matrix $M_k(u_1, u_2, \cdots, u_n)$, beginning with the simplest case, $n = 1$.

Theorem I.*) *Let $u(x)$ be a function of x which at every point of* I *has finite derivatives of the first k orders; then if u vanishes in a point set $[P]$, $u', u'', \cdots, u^{(k)}$ vanish, each in a set $[P]$.*

The vanishing of these successive derivatives in point sets $[P^*]$ is a consequence of Rolle's Theorem; and by continuity

$$u(p) = u'(p) = \cdots = u^{(k-1)}(p) = 0.$$

Though this argument does not apply to $u^{(k)}(p)$, we have directly

$$u^{(k)}(p) = \lim_{x=p} \frac{u^{(k-1)}(x)}{x-p} = 0.$$

Theorem II. *Let $u_1, u_2, \cdots, u_n$ be functions of x which at every point of* I *have finite derivatives of the first n orders; then if*

*) Cf. Bôcher, l. c., pp. 141, 142.

$$W(u_1, u_2, \cdots, u_n)$$

vanishes in a point set $[P]$, *at least one of the Wronskians*

$$W(u_1, u_2, \cdots, u_{n-1}), \quad W(u_1', u_2', \cdots, u_{n-1}'), \quad W(u_1', u_2', \cdots, u_n')$$

must vanish in a set $[P^*]$.

This theorem is a consequence of the relation

$$(1) \qquad W[W(u_1', u_2', \cdots, u_{n-1}'),\ W(u_1, u_2, \cdots, u_n)] = W(u_1, u_2, \cdots, u_{n-1}) \cdot W(u_1', u_2', \cdots, u_n'),$$

which we establish with the aid of the formula of Frobenius*)

$$(2) \qquad W(y_1, y_2, \cdots, y_n) \cdot [W(y_1, y_2, \cdots, y_m)]^{n-m-1} = W(w_{m,m+1}, w_{m,m+2}, \cdots, w_{m,n}) \qquad (1 \leqq m \leqq n-1),$$

where

$$w_{m,m+\varkappa} = W(y_1, y_2, \cdots, y_m, y_{m+\varkappa}) \qquad (\varkappa = 1, 2, \cdots, n-m).$$

The expansions

$$W(u_1, u_2, \cdots, u_n) = \sum_{i=1}^{n} (-1)^{i-1} u_i\, W(u_1', u_2', \cdots, u_{i-1}', u_{i+1}', \cdots, u_n'),$$

$$W'(u_1, u_2, \cdots, u_n) = \sum_{i=1}^{n} (-1)^{i-1} u_i\, W'(u_1', u_2', \cdots, u_{i-1}', u_{i+1}', \cdots, u_n')$$

enable us to write the left-hand member of (1) in the form

$$\sum_{i=1}^{n-1} (-1)^{i-1} u_i\, W[W(u_1', u_2', \cdots, u_{n-1}'),\ W(u_1', u_2', \cdots, u_{i-1}', u_{i+1}', \cdots, u_n')].$$

If to each term of this sum we apply formula (2) with the substitutions

$$m = n-2, \quad y_\nu = u_\nu' \ (\nu < i), \quad y_\nu = u_{\nu+1}' \ (i \gtreqless \nu < n-1), \quad y_{n-1} = u_i', \quad y_n = u_n',$$

we obtain the relation

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*) Cf. Crelle, vol. 77 (1874), p. 248. Although the memoir of Frobenius is concerned throughout with analytic functions, this formula is true for any functions having the requisite number of finite derivatives. It expresses an identity between polynomials in $y_i, y_i', \cdots, y_i^{(n-1)}$ $(i = 1, 2, \cdots, n)$, and Frobenius' methods of proof apply, with suitable changes in notation, when these n^2 symbols are considered as independent variables. This point of view renders it unnecessary to assume the continuity of $y_i^{(n-1)}$ $(i = 1, 2, \cdots, n)$ when we return to the interpretation of $y_i^{(\nu)}$ as the ν^{th} derivative of y_i.

$$W[W(u_1', u_2', \cdots, u_{n-1}'),\ W(u_1, u_2, \cdots, u_n)]$$
$$= \sum_{i=1}^{n-1} (-1)^{i-1} u_i W(u_1', u_2', \cdots, u_{i-1}', u_{i+1}', \cdots, u_{n-1}') \cdot W(u_1', u_2', \cdots, u_n'),$$

from which formula (1) immediately follows.

With the aid of (1) we obtain the formula

$$\frac{d}{dx}\left[\frac{W(u_1, u_2, \cdots, u_n)}{W(u_1', u_2', \cdots, u_{n-1}')}\right] = \frac{W(u_1, u_2, \cdots, u_{n-1}) \cdot W(u_1', u_2', \cdots, u_n')}{[W(u_1', u_2', \cdots, u_{n-1}')]^2}.$$

If $W(u_1, u_2, \cdots, u_n)$ vanishes in a set $[P]$, either $W(u_1', u_2', \cdots, u_{n-1}')$ vanishes in such a set, or else, as a consequence of Rolle's Theorem, at least one of the Wronskians in the numerator of the right-hand member of the above formula must vanish in a set $[P^*]$.*) Our theorem is thus established.

If $W(u_1, u_2, \cdots, u_{n-1})$ is the only one of the three Wronskians concerned in the conclusion of Theorem II which vanishes in a set $[P^*]$, we can apply the same theorem to this Wronskian. By repeating this process as often as necessary we obtain the result stated in the following theorem:

Theorem III. *Let $u_1, u_2, \cdots, u_n$ be functions of x which at every point of* I *have finite derivatives of the first n orders; then if $W(u_1, u_2, \cdots, u_n)$ vanishes in a point set $[P]$, at least one of the Wronskians*

$$W(u_1', u_2', \cdots, u_\nu') \qquad (\nu = 1, 2, \cdots, n)$$

vanishes in a set $[P^*]$.

Theorem IV. *Let $u_1, u_2, \cdots, u_n$ be functions of x which at every point of* I *have continuous derivatives of the first k orders $(k \geqq n)$; then if $W(u_1, u_2, \cdots, u_n)$ vanishes in a set $[P]$, all the n-rowed determinants of the matrix $M_k(u_1, u_2, \cdots, u_n)$ vanish at the point $x = p$.*

We first prove this theorem for the case $n = 2$, remarking that when $n = 1$ the above conclusion is a corollary of Theorem I. If, then, u_1, u_2 have continuous derivatives of the first k orders $(k \geqq 2)$, and their Wronskian vanishes in a set $[P]$, we have, as a consequence of Theorem I,

$$W'(u_1(p),\ u_2(p)) = 0;$$

and by Theorem III either $W(u_1', u_2')$ or u_1' vanishes in a set $[P^*]$. In the latter case u_1'' also vanishes in a set $[P^*]$. Since u_1'', u_2'' are by hypothesis continuous, we have in either alternative

*) We can, of course, replace $[P^*]$ by $[P]$ in this statement when $u_1^{(n)}, u_2^{(n)}, \cdots, u_n^{(n)}$ are continuous.

$$W(u_1'(p),\ u_2'(p)) = 0.$$

We have thus shown that all the two-rowed determinants of $M_2(u_1, u_2)$ vanish at p; we complete our proof for $n = 2$, $k > 2$ by the method of mathematical induction, assuming the truth of our theorem when $k = k_1 - 1$ and deducing as a consequence its validity when $k = k_1$.

Under this assumption every two-rowed determinant of the matrix $M_{k_1-1}(u_1, u_2)$ vanishes at p. By Theorem III, u_1'', u_2'' being continuous, either u_1' or $W(u_1', u_2')$ vanishes in a set $[P]$. But if u_1' vanishes in a set $[P]$ we have, by Theorem I,

$$u_1'(p) = u_1''(p) = \cdots = u_1^{(k_1)}(p) = 0,$$

while if the other alternative presents itself our assumption that the theorem is true when $k = k_1 - 1$ applies directly to the matrix $M_{k_1-1}(u_1', u_2')$. Hence in either case all the two-rowed determinants of this matrix vanish at p. It remains to show that the determinant $u_1 u_2^{(k_1)} - u_2 u_1^{(k_1)}$ vanishes at p. By Theorem I, $W^{(k_1-1)}(u_1, u_2) = 0$ at p; but this expression can be written as a sum whose first term is the determinant to be investigated and whose remaining terms are a linear combination of determinants of the matrix $M_{k_1-1}(u_1, u_2)$.*) Since all such determinants vanish at p, $u_1 u_2^{(k_1)} - u_2 u_1^{(k_1)}$ must also vanish at that point. We have thus shown that every two-rowed determinant of $M_{k_1}(u_1, u_2)$ vanishes at p. Our theorem has been proved for $k_1 = 2$, hence it is true for all values of k_1 up to and including k.

We will now establish our theorem for $n = r$ $(k \geqq r > 2)$, assuming its truth for $n = r - 1$. For this purpose we make use of formula (2), with the substitutions

$$n = r, \quad m = 1, \quad y_\varkappa = u_\varkappa \qquad (\varkappa = 1, 2, \cdots, r).$$

We thus obtain the relation

$$u_1^{r-2}\, W(u_1, u_2, \cdots, u_r) = W(w_{12}, w_{13}, \cdots, w_{1r}), \tag{3}$$

where

$$w_{1i} = W(u_1, u_i) \qquad (i = 2, 3, \cdots, r).$$

From formula (3) it follows that if $W(u_1, u_2, \cdots, u_r)$ vanishes in a set $[P]$, $W(w_{12}, w_{13}, \cdots, w_{1r})$ must also vanish in every point of that set. Since $w_{12}, w_{13}, \cdots, w_{1r}$ are functions satisfying all the conditions of our theorem, which has been assumed to be valid for $n = r - 1$, every $(r-1)$-rowed determinant of the matrix $M_{k-1}(w_{12}, w_{13}, \cdots, w_{1r})$ must vanish at p. A typical determinant of this matrix is

*) Cf. formula (4), p. 287.

$$\Omega_{k_1 k_2 \cdots k_{r-1}} \equiv \begin{vmatrix} w_{12}^{(k_1-1)} & w_{13}^{(k_1-1)} & \cdots & w_{1r}^{(k_1-1)} \\ w_{12}^{(k_2-1)} & w_{13}^{(k_2-1)} & \cdots & w_{1r}^{(k_2-1)} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ w_{12}^{(k_{r-1}-1)} & w_{13}^{(k_{r-1}-1)} & \cdots & w_{1r}^{(k_{r-1}-1)} \end{vmatrix} \; {}^{*)}$$

$$(0 < k_1 < k_2 < \cdots < k_{r-1} \leqq k).$$

We now proceed to express this determinant as a sum in each of whose terms there appears a determinant of the matrix $M_k(u_1, u_2, \cdots, u_r)$.

If we use the symbol $|u_1^{(\mu)} u_i^{(\nu)}|$ to designate the determinant $u_1^{(\mu)} u_i^{(\nu)} - u_1^{(\nu)} u_i^{(\mu)}$, we can express $w_{1i}^{(m)}$ in the form

$$(4) \qquad w_{1i}^{(m)} = \frac{d^m}{dx^m} |u_1 u_i'| = \sum_{\varrho=0}^{R} \varkappa_{m\varrho} |u_1^{(\varrho)} u_i^{(m-\varrho+1)}|$$

$$\begin{pmatrix} R = \frac{m-1}{2} \text{ if } m \text{ is odd,} \\ \;\;\;\, = \frac{m}{2} \text{ if } m \text{ is even} \end{pmatrix},$$

where the numbers $\varkappa_{m\varrho}$ are positive integers which do not depend on i, and $k_{m0} = 1$.

The formula

$$|u_1^{(\varrho)} u_i^{(m-\varrho+1)}| = \frac{u_1^{(\varrho)} |u_1 u_i^{(m-\varrho+1)}| - u_1^{(m-\varrho+1)} |u_1 u_i^{(\varrho)}|}{u_1}$$

when applied to (4) gives

$$(5) \qquad w_{1i}^{(m)} = \sum_{\varrho=0}^{m} K_{m\varrho} \frac{u_1^{(\varrho)}}{u_1} |u_1 u_i^{(m-\varrho+1)}|,$$

where the coefficients $K_{m\varrho}$ are numbers whose values in terms of the integers $\varkappa_{m\varrho}$ can be easily obtained. When each element of $\Omega_{k_1 k_2 \cdots k_{r-1}}$ has been replaced by its expression given by formula (5), the result can be expressed as a sum according to a well-known theorem on determinants whose elements are sums. We thus give to $\Omega_{k_1 k_2 \cdots k_{r-1}}$ the form

$$\sum_{\varrho_1=0}^{k_1-1} \sum_{\varrho_2=0}^{k_2-1} \cdots \sum_{\varrho_{r-1}=0}^{k_{r-1}-1} C_{\varrho_1 \varrho_2 \cdots \varrho_{r-1}} \frac{u_1^{(\varrho_1)} u_1^{(\varrho_2)} \cdots u_1^{(\varrho_{r-1})}}{u_1^{r-1}} V_{k_1-\varrho_1, k_2-\varrho_2, \cdots, k_{r-1}-\varrho_{r-1}},$$

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*) Throughout the present paper an upper index 0 is given a meaning by the convention $y^{(0)} \equiv y$.

where

$$V_{k_1-\varrho_1,\, k_2-\varrho_2,\, \cdots,\, k_{r-1}-\varrho_{r-1}} \equiv$$

$$\equiv \begin{vmatrix} |u_1 u_2^{(k_1-\varrho_1)}| & |u_1 u_3^{(k_1-\varrho_1)}| & \cdots & |u_1 u_r^{(k_1-\varrho_1)}| \\ |u_1 u_2^{(k_2-\varrho_2)}| & |u_1 u_3^{(k_2-\varrho_2)}| & \cdots & |u_1 u_r^{(k_2-\varrho_2)}| \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ |u_1 u_2^{(k_{r-1}-\varrho_{r-1})}| & |u_1 u_3^{(k_{r-1}-\varrho_{r-1})}| & \cdots & |u_1 u_r^{(k_{r-1}-\varrho_{r-1})}| \end{vmatrix},$$

and $C_{\varrho_1 \varrho_2 \cdots \varrho_{r-1}}$ is a positive or negative integer $(C_{00 \cdots 0} = 1)$ We can transform this expression by means of the formula *)

$$V_{\nu_1 \nu_2 \cdots \nu_{r-1}} = u_1^{r-2}\, U_{0 \nu_1 \cdots \nu_{r-1}}$$

where the meaning of the latter symbol is given by the identity

$$U_{\nu_0 \nu_1 \cdots \nu_{r-1}} \equiv \begin{vmatrix} u_1^{(\nu_0)} & u_2^{(\nu_0)} & \cdots & u_r^{(\nu_0)} \\ u_1^{(\nu_1)} & u_2^{(\nu_1)} & \cdots & u_r^{(\nu_1)} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ u_1^{(\nu_{r-1})} & u_2^{(\nu_{r-1})} & \cdots & u_r^{(\nu_{r-1})} \end{vmatrix}.$$

We thus obtain the desired expansion of $\Omega_{k_1 k_2 \cdots k_{r-1}}$ in terms of determinants of the matrix $M_k(u_1, u_2, \cdots, u_r)$:

$$(6) \qquad \Omega_{k_1 k_2 \cdots k_{r-1}} = \sum_{\varrho_1=0}^{k_1-1} \sum_{\varrho_2=0}^{k_2-1} \cdots \sum_{\varrho_{r-1}=0}^{k_{r-1}-1} C_{\varrho_1 \varrho_2 \cdots \varrho_{r-1}} \frac{u_1^{(\varrho_1)} u_1^{(\varrho_2)} \cdots u_1^{(\varrho_{r-1})}}{u_1} \times U_{0,\, k_1-\varrho_1,\, \cdots,\, k_{r-1}-\varrho_{r-1}}.$$

To facilitate the discussion at this point we introduce the idea of *precedence* among the r-rowed determinants of the matrix $M_k(u_1, u_2, \cdots, u_r)$ by means of the following definition: *The determinant* $U_{\mu_0 \mu_1 \cdots \mu_{r-1}}$ *precedes the determinant* $U_{\nu_0 \nu_1 \cdots \nu_{r-1}}$ *if*

$$\mu_i = \nu_i \ (i = 0, 1, \cdots, \lambda - 1), \quad \mu_\lambda < \nu_\lambda.$$

It can now be easily shown that every determinant of $M_k(u_1, u_2, \cdots, u_r)$ whose first row is the first row of that matrix vanishes at p. This is, of course, an obvious result in case $u_1, u_2, \cdots, u_r$ all vanish at p. If, however, one of these functions is different from zero at p, we take it as u_1 since this is only a matter of notation. Formula (6) is then valid for $x = p$, a substitution which reduces the left-hand member of (6) to

*) Cf. E. Pascal, Die Determinanten (translation by Leitzmann), p. 39.

zero. On the right-hand side the first term is $u_1^{r-2}\, U_{0\,k_1\cdots k_{r-1}}$, while in each of the other terms there appears as a factor a determinant that *precedes* $U_{0\,k_1\cdots k_{r-1}}$. This last determinant must therefore vanish at p if all the determinants which precede it vanish at p. But by hypothesis the Wronskian $W(u_1, u_2, \cdots, u_r)$ is zero at p, and this is, according to our rule for precedence, the first determinant of the matrix. Since we can give such values to $k_1, k_2, \cdots, k_{r-1}$ as to obtain successively, in order of precedence, all the remaining determinants $U_{0\,\nu_1\cdots\nu_{r-1}}$, it follows that every such determinant vanishes at p.

We have still to consider the r-rowed determinants of $M_k(u_1, u_2, \cdots, u_r)$ which belong to the matrix $M_{k-1}(u_1', u_2', \cdots, u_r')$. According to Theorem II one of the Wronskians

$$W(u_1, u_2, \cdots, u_{r-1}), \qquad W(u_1', u_2', \cdots, u_{r-1}'), \qquad W(u_1', u_2', \cdots, u_r')$$

must vanish in a set $[P^*]$, and therefore, by continuity, in a set $[P]$. But if either of the first two Wronskians vanishes in a set $[P]$, the assumption that our theorem is true for $n = r - 1$ necessitates the vanishing at p of all $(r-1)$-rowed determinants of

$$M_{k-1}(u_1', u_2', \cdots, u_{r-1}'),$$

and therefore of all r-rowed determinants of

$$M_{k-1}(u_1', u_2', \cdots, u_r').$$

If, on the other hand,

$$W(u_1', u_2', \cdots, u_r')$$

vanishes in a set $[P]$, we can show by the method of the preceding paragraph that every determinant of

$$M_{k-1}(u_1', u_2', \cdots, u_r')$$

whose first row is the first row of that matrix vanishes at p. To complete our proof for $n = r$ on the assumption that our theorem is true for $n = r - 1$, we have only to continue the process now sufficiently indicated. Since the theorem has already been established for $n = 2$, its validity follows for all values of n.

In Theorem IV we have required the continuity of $u_1^{(k)}, u_2^{(k)}, \cdots, u_n^{(k)}$; this being essential to the proof given.*) In the following theorem no such assumption is made:

Theorem V. *Let $u_1, u_2, \cdots, u_n$ be functions of x which at every point of* I *have finite derivatives of the first k orders ($k \geqq n$); then if $W(u_1, u_2, \cdots, u_n)$ and its first $k - n + 1$ derivatives vanish simultaneously*

*) The writer has not been able to determine whether this restriction is essential to the theorem itself.

in a point set $[P]$, *all the* n*-rowed determinants of the matrix* $M_k(u_1, u_2, \cdots, u_n)$ *vanish simultaneously either at* p *or in a set* $[P^*]$.

The former alternative of the above conclusion presents itself when each of the Wronskians

$$W_i \equiv W(u_1, u_2, \cdots, u_{i-1}, u_{i+1}, \cdots, u_n)$$
$$(i = 1, 2, \cdots, n)$$

vanishes in a set $[P]$, for by Theorem IV every $(n-1)$-rowed determinant of the matrix $M_{k-1}(u_1, u_2, \cdots, u_n)$ then vanishes at p, and therefore every n-rowed determinant of $M_k(u_1, u_2, \cdots, u_n)$ is zero at that point. If, however, one of the Wronskians W_i, which as a matter of notation we take as W_n, vanishes in no set $[P]$, there must exist a particular set $[P^*]_1$, at each point of which $W(u_1, u_2, \cdots, u_n)$ and its first $k-n+1$ derivatives vanish simultaneously, while $W_n \neq 0$.*) Let us now assume that every n-rowed determinant of $M_{r-1}(u_1, u_2, \cdots, u_n)$ $(k \geqq r > n+1)$ vanishes in the set $[P^*]_1$. Since $W(u_1, u_2, \cdots, u_n)$ is zero at each point of this set we shall have completed the proof of our theorem if we show, as a consequence of the hypothesis just made, that every n-rowed determinant of $M_r(u_1, u_2, \cdots, u_n)$ vanishes in $[P^*]_1$. This, however, follows at once, since $W_n \neq 0$ in $[P^*]_1$, provided the equations

$$u_1^{(\nu)} W_1 - u_2^{(\nu)} W_2 + \cdots + (-1)^{n-1} u_n^{(\nu)} W_n = 0 \tag{7}$$
$$(\nu = 0, 1, 2, \cdots, r),$$

hold simultaneously at each point of $[P^*]_1$. The first $n-1$ of these equations are identities; the left-hand members of the next $r-n+1$ are determinants of the matrix $M_{r-1}(u_1, u_2, \cdots, u_n)$ which by hypothesis vanish in $[P^*]_1$. As for the last equation, its left-hand member is a term of the sum of n-rowed determinants representing the value of $W^{(r-n+1)}(u_1, u_2, \cdots, u_n)$ and therefore vanishing in $[P^*]_1$. Since the other terms of this sum are constant multiples of determinants of the matrix $M_{r-1}(u_1, u_2, \cdots, u_n)$, the last equation of (7) must also be satisfied at each point of $[P^*]_1$. Our theorem is thus established.

§ 2.

Application to the Theory of Linear Dependence.

We shall now give a new sufficient condition for linear dependence deduced from the results of § 1 with the aid of the following theorem, due to Bôcher**):

*) Note that this statement is so worded as to include the possibility that all the Wronskians W_i vanish at p.

**) This theorem, as well as Theorem B in § 4, is taken directly from the

Theorem A. *Let $u_1, u_2, \cdots, u_n$ be functions of x which at every point of* I *have finite derivatives of the first k orders ($k \geqq n-1$) while $W(u_1, u_2, \cdots, u_{n-1})$ and its first $k-n+2$ derivatives do not all vanish at any one point of* I; *then if $W(u_1, u_2, \cdots, u_n)$ is identically zero $u_1, u_2, \cdots, u_n$ are linearly dependent, and in particular:*

$$u_n \equiv c_1 u_1 + c_2 u_2 + \cdots + c_{n-1} u_{n-1}.$$

On account of the importance of this preliminary theorem we give a brief demonstration, which differs in some details from Bôcher's.

In any finite and perfect subinterval I′ of I, W_n can vanish in only a finite number of points, by Theorem I; let these points, arranged in order from left to right of I′, be $x_0, x_1, \cdots, x_m$. From formula (2) we have

$$W_i' W_n - W_n' W_i \equiv 0 \qquad (i = 1, 2, \cdots, n-1),$$

so that in the interval between x_{j-1} and x_j

$$\frac{d}{dx}\frac{W_i}{W_n} \equiv 0,$$

and therefore

$$W_i \equiv (-1)^{n-i+1} c_{ij} W_n.$$

With this substitution in the identity

$$u_1 W_1 - u_2 W_2 + \cdots + (-1)^{n-1} u_n W_n \equiv 0$$

we have

$$u_n \equiv c_{1j} u_1 + c_{2j} u_2 + \cdots + c_{n-1j} u_{n-1} \tag{8}$$

from x_{j-1} to x_j, these points included since the functions u are continuous. But $c_{ij} = c_{i,j+1}$, for we can successively differentiate (8) and the corresponding identity where the index j is replaced by $j+1$, substitute x_j for x in the resulting system of equations, and by subtraction obtain the set

$$\left(c_{1j} - c_{1j+1}\right) u_1^{(\nu)}\left(x_j\right) + \left(c_{2j} - c_{2j+1}\right) u_2^{(\nu)}\left(x_j\right) + \cdots$$
$$\cdots + \left(c_{n-1j} - c_{n-1j+1}\right) u_{n-1}^{(\nu)}\left(x_j\right) = 0$$
$$(\nu = 0, 1, 2, \cdots, k).$$

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paper already cited (Transactions of the American Mathematical Society, vol. 2 (1901), p. 139, Theorems IV and VI). Relations between other theorems of the same paper and those of the remaining sections of the present article are as follows: The exceptions made in Bôcher's Lemma II (p. 147) are removed in my Theorem VII, so that in his Theorem VIII the assumption that the nth derivatives of the u's are continuous is superfluous; some properties established in his discussion of Peano's theorems are generalized by my Theorem VIII; and Theorem XII of the present paper may be compared with his Theorem VII. When $u_1^{(k)}, u_2^{(k)}, \cdots, u_n^{(k)}$ are continuous, Theorem XI is equivalent to Theorem X of his article in the Bulletin of the American Mathematical Society (ser. 2, vol. 8 (1901), p. 59).

If c_{ij} were not equal to $c_{i,j+1}$ for $i = 1, 2, \cdots, n-1$, every $(n-1)$-rowed determinant of $M_k(u_1, u_2, \cdots, u_{n-1})$ would vanish at x_j. Since the successive derivatives of W_n up to and including the $(k-n+2)^{\text{th}}$ can be expressed as linear combinations of such determinants, we thus arrive at a contradiction of our hypothesis that W_n and its first $k-n+2$ derivatives do not all vanish at any one point of I. Hence the coefficients in (8) do not depend on the index j, and our theorem is proved for every finite and perfect subinterval I′ of I. It is therefore true for I itself.

This theorem being established, we now give a new sufficient condition for linear dependence.

Theorem VI. *Let $u_1, u_2, \cdots, u_n$ be functions of x which at every point of* I *have finite derivatives of the first k orders $(k \geqq n-1)$ while the $(n-1)$-rowed determinants of $M_k(u_1, u_2, \cdots, u_{n-1})$ do not all vanish at any point of* I; *then if $W(u_1, u_2, \cdots, u_n)$ is identically zero, $u_1, u_2, \cdots, u_n$ are linearly dependent, and in particular:*

$$u_n \equiv c_1 u_1 + c_2 u_2 + \cdots + c_{n-1} u_{n-1}.$$

As before, we prove this theorem for any finite and perfect subinterval I′ of I, since its truth for I then follows.

We first observe that there can be only a finite number of points in I′ at which $W(u_1, u_2, \cdots, u_{n-1})$ and its first $k-n+2$ derivatives vanish simultaneously; otherwise, by Theorem V, all the $(n-1)$-rowed determinants of $M_k(u_1, u_2, \cdots, u_{n-1})$ would vanish in at least one point of I. By Theorem A there exists, throughout each of the m intervals into which I′ is divided by points x_j where $W_n = W_n' = \cdots = W_n^{(k-n+2)} = 0$, an identity

$$u_n \equiv c_{1j} u_1 + c_{2j} u_2 + \cdots + c_{(n-1)j}, u_{n-1}.$$

By continuity, this identity holds also at the points x_{j-1}, x_j. As in the proof of Theorem A, we can show that an inequality $c_{ij} \neq c_{ij+1}$ for any of the values which i and j may assume would necessitate the vanishing at x_j of all the $(n-1)$-rowed determinants of $M_k(u_1, u_2, \cdots, u_{n-1})$, contrary to our hypothesis. Hence the coefficients c_{ij} do not depend on the index j, and our theorem is proved.

It is obvious that n functions which satisfy the conditions of Theorem A must also satisfy the conditions of Theorem VI. An illustration will show that the latter theorem applies to cases where the hypotheses of Theorem A are not verified. We consider four functions u_1, u_2, u_3, u_4 defined as follows:

$$u_1 = -(x^4 + x^3 + x + 1), \qquad u_2 = x + 1,$$

$$u_3 = \begin{cases} x^2 + x + 1 & (x > 0), \\ x^4 + x^2 + x + 1 & (x \leqq 0), \end{cases} \qquad u_4 = \begin{cases} x^4 + x^3 - x^2 - x - 1 & (x > 0), \\ x^3 - x^2 - x - 1 & (x \leqq 0). \end{cases}$$

In any interval I which includes both positive and negative values of x the Wronskian of u_1, u_2, u_3, u_4 vanishes identically while the three-rowed determinants of $M_3(u_1, u_2, u_3)$ do not vanish simultaneously at any point. There is, however, no Wronskian of three functions u which does not vanish together with its first derivative at $x = 0$. Since u_3 and u_4 have no fourth derivatives at $x = 0$, this is a case where Theorem A fails, while the conditions of Theorem VI are met. We have, in fact, the linear relation

$$u_1 + u_2 + u_3 + u_4 \equiv 0.$$

§ 3.

The Identical Vanishing of the Wronskian and the Rank of the Matrix $M_k(u_1, u_2, \cdots, u_n)$.

Theorem VII. *Let $u_1, u_2, \cdots, u_n$ be functions of x which at every point of* I *have finite derivatives of the first k orders ($k \geqq n$); then if $W(u_1, u_2, \cdots, u_n)$ vanishes identically throughout* I *the matrix*

$$M_k(u_1, u_2, \cdots, u_n)$$

is of rank less than n at each point of I, *i. e. all the n-rowed determinants of this matrix vanish identically throughout* I.

If $u_1^{(k)}, u_2^{(k)}, \cdots, u_n^{(k)}$ are continuous throughout I this theorem is a direct consequence of Theorem IV. In case any of these derivatives are discontinuous it can still be shown that every n-rowed determinant of $M_k(u_1, u_2, \cdots, u_n)$ vanishes at any point p of I. For if each of the Wronskians $W_1, W_2, \cdots, W_n$ vanishes in a set $[P]$, then by Theorem IV every $(n-1)$-rowed determinant of $M_{k-1}(u_1, u_2, \cdots, u_n)$ must vanish at p, and consequently every n-rowed determinant of $M_k(u_1, u_2, \cdots, u_n)$ vanishes at that point. If, on the other hand, W_n does not vanish in any set $[P]$, there is a point q of I such that W_n does not vanish between p and q. By Theorem A we have, between p and q, an identity

$$u_n \equiv c_1 u_1 + c_2 u_2 + \cdots + c_{n-1} u_{n-1},$$

which holds by continuity at p. If we differentiate this relation k times and make the substitution $x = p$, we obtain a set of equations which can exist only when the rank of $M_k(u_1(p), u_2(p), \cdots, u_n(p))$ is less than n. Our theorem is thus verified in all cases.

If at each point of I at least one m-rowed determinant of

$$M_k(u_1, u_2, \cdots, u_n)$$

is different from zero while every $(m+1)$-rowed determinant of this matrix vanishes identically throughout I, we shall say that the matrix is *of constant rank m in* I.

An important property of matrices of constant rank is given in the following theorem:

Theorem VIII. *Let* $u_1, u_2, \cdots, u_n$ *be functions of* x *which at every point of* I *have continuous derivatives of the first* k *orders* $(k \geqq n-1)$; *then if* $M_k(u_1, u_2, \cdots, u_n)$ *is of constant rank* $m < n$ *in* I, *at least one matrix formed by suppressing* $n-m$ *columns of* $M_k(u_1, u_2, \cdots, u_n)$ *is of constant rank* m *in* I.

As a consequence of the hypothesis that $M_k(u_1, u_2, \cdots, u_n)$ is of constant rank m it follows that there must exist at least one set of m functions u whose Wronskian is not identically zero throughout I; otherwise, by Theorem VII, every m-rowed determinant of $M_k(u_1, u_2, \cdots, u_n)$ would vanish identically. Accordingly we so choose our notation that $W(u_1, u_2, \cdots, u_m)$ is different from zero at some point p of I. We shall now prove that $M_k(u_1, u_2, \cdots, u_m)$ is of constant rank m in I.

Since $m \leqq k$, $W(u_1, u_2, \cdots, u_m)$ is a continuous function of x; there is therefore a neighborhood of p throughout which this Wronskian does not vanish and where, consequently, $M_k(u_1, u_2, \cdots, u_m)$ is of constant rank m. If this matrix were not of constant rank in the whole interval I there would exist a finite and perfect subinterval I′ throughout which $M_k(u_1, u_2, \cdots, u_m)$ is of constant rank m, but at one of whose end-points, q, all the m-rowed determinants of the matrix vanish.*) By Theorem VI there exist throughout I′ identities of the form

$$(9) \qquad u_{m+i} \equiv c_{1i} u_1 + c_{2i} u_2 + \cdots + c_{mi} u_m \qquad (i = 1, 2, \cdots, n-m).$$

By forming the successive derivatives at the point q of these identities we obtain the equations

$$u_{m+i}^{(\nu)}(q) = c_{1i} u_1^{(\nu)}(q) + c_{2i} u_2^{(\nu)}(q) + \cdots + c_{mi} u_m^{(\nu)}(q) \qquad \begin{pmatrix} i = 1, 2, \cdots, n-m \\ \nu = 0, 1, \cdots, k \end{pmatrix}.$$

We can therefore express every m-rowed determinant of

$$M_k(u_1(q), u_2(q), \cdots, u_n(q))$$

as the product of some determinant of the matrix

$$M_k(u_1(q), u_2(q), \cdots, u_m(q))$$

and a polynomial in the coefficients c. Hence if every m-rowed determinant of the latter matrix vanishes, the same is true of every m-rowed determinant of the former. We are thus led to a contradiction of our

*) The argument here depends upon the hypothesis that $u_1^{(k)}, u_2^{(k)}, \cdots, u_n^{(k)}$ are continuous.

hypothesis that $M_k(u_1, u_2, \cdots, u_n)$ is of constant rank m in I, and the truth of our theorem is established.

In case any of the functions $u_1^{(k)}, u_2^{(k)}, \cdots, u_m^{(k)}$ are discontinuous the above argument does not, in general, apply. We may note, however, that the set of points in I in which $M_k(u_1, u_2, \cdots, u_m)$ is of rank less than m can include no point isolated from either side in that set; i. e. a point of that set can be at neither extremity of a subinterval of I in which $M_k(u_1, u_2, \cdots, u_m)$ is of constant rank m.

Theorem IX. *Let $u_1, u_2, \cdots, u_n$ be functions of x which at every point of* I *have finite derivatives of the first k orders ($k \geqq n-1$) while $W(u_1, u_2, \cdots, u_n) = 0$; then if no function (other than zero) of the form*

$$(10) \qquad g_1 u_1 + g_2 u_2 + \cdots + g_n u_n$$

(the g's being constants) vanishes together with its first k derivatives at any point of I, *both $M_k(u_1, u_2, \cdots, u_n)$ and at least one matrix formed by suppressing $n-m$ of its columns are of constant rank $m < n$ in* I. *Conversely, if $M_k(u_1, u_2, \cdots, u_n)$ and a matrix formed by suppressing $n-m$ of its columns are of constant rank $m < n$ in* I, *no function (other than zero) of form* (10) *vanishes together with its first k derivatives at any point of* I.

Before proceeding with the proof of this theorem let us note that if the functions $u_1^{(k)}, u_2^{(k)}, \cdots, u_n^{(k)}$ are continuous, then as a consequence of Theorem VIII it is superfluous in both parts of the above theorem to state or require that a matrix formed by suppressing $n-m$ columns of $M_k(u_1, u_2, \cdots, u_n)$ must be of constant rank $m < n$ in I.

To establish the first part of the above theorem we must show that if $u_1, u_2, \cdots, u_n$ satisfy the given conditions the matrix $M_k(u_1, u_2, \cdots, u_n)$ cannot be of different rank at different points of I. If it is of rank m at a point p there must, from the definition of the term rank, be a matrix composed of m of its columns (we take this matrix as $M_k(u_1, u_2, \cdots, u_m)$) which is of rank m at p. Accordingly there exist relations*):

$$(11) \qquad u_{m+i}^{(\nu)} = c_{1i} u_1^{(\nu)} + c_{2i} u_2^{(\nu)} + \cdots + c_{mi} u_m^{(\nu)} \qquad \begin{pmatrix} \nu = 0, 1, \cdots, k \\ i = 1, 2, \cdots, n-m \end{pmatrix},$$

which are valid at p. But by hypothesis such relations if true at one point, must be identities throughout I. All $(m+1)$-rowed determinants of $M_k(u_1, u_2, \cdots, u_n)$ must therefore vanish identically, so that this matrix is nowhere of higher rank than m. The same argument shows that the rank of the matrix $M_k(u_1, u_2, \cdots, u_m)$, and therefore the rank of

*) Cf. E. Pascal, Die Determinanten (translation by Leitzmann), pp. 193, 194.

$M_k(u_1, u_2, \cdots, u_n)$, is nowhere less than m, for if this were not true there would be a set of relations similar to (11), but involving only the functions $u_1, u_2, \cdots, u_m$ and their successive derivatives, which would by hypothesis be identities throughout I. The existence of such identities would contradict our assumption that $M_k(u_1, u_2, \cdots, u_m)$ is of rank m at p.

On the other hand, if $M_k(u_1, u_2, \cdots, u_n)$ and $M_k(u_1, u_2, \cdots, u_m)$ are both of constant rank $m < n$, it follows from Theorem VI that there exist $n - m$ identities of form (9) valid throughout I. Let us now consider any function of form (10) which vanishes together with its first k derivatives at any point p of I. This function would be reduced by the substitution of the identities (9) to the form

$$h_1 u_1 + h_2 u_2 + \cdots + h_m u_m.$$

Since this expression vanishes together with its first k derivatives at p, where $M_k(u_1, u_2, \cdots, u_m)$ is of rank m, it follows that

$$h_1 = h_2 = \cdots = h_m = 0,$$

so that the function considered vanishes identically.

§ 4.

New Form of Theorem VI and an equivalent Theorem. Applications to Linear Differential Equations.

Bôcher has given in the following theorem*) a test for linear dependence which we shall presently show to be equivalent to that contained in Theorem VI:

Theorem B. *Let $u_1, u_2, \cdots, u_n$ be functions of x which at every point of* I *have finite derivatives of the first k orders ($k \geqq n-1$) while no function (other than zero) of the form*

$$g_1 u_1 + g_2 u_2 + \cdots + g_n u_n$$

(the g's being constants) vanishes together with its first k derivatives at any point of I; *then if $W(u_1, u_2, \cdots, u_n) \equiv 0$ the functions u are linearly dependent.*

We now compare this result with the somewhat generalized form of Theorem VI contained in the following theorem:

Theorem X. *Let $u_1, u_2, \cdots, u_n$ be functions of x which at every point of* I *have finite derivatives of the first k orders ($k \geqq n-1$); then if both $M_k(u_1, u_2, \cdots, u_n)$ and at least one matrix formed by suppressing $n-m$ of its columns are of constant rank $m < n$ in* I**), *$u_1, u_2, \cdots, u_n$ are*

*) See foot-note **), pp. 290 and 291.

**) If $u_1^{(k)}, u_2^{(k)}, \cdots, u_n^{(k)}$ are *continuous* it is evident, from Theorem VIII, that we need here only require that $M_k(u_1, u_2, \cdots, u_n)$ be of constant rank $m < n$.

linearly dependent and the number of independent linear relations between these functions is $n-m$.

If $M_k(u_1, u_2, \cdots, u_m)$ and $M_k(u_1, u_2, \cdots, u_m, u_{m+i})$ $(i=1,2,\cdots,n-m)$ are of rank m the existence of $n-m$ independent relations of form (9) follows from Theorem VI. If there were an additional linear relation independent of the preceding we could substitute in the new relation the values of $u_{m+1}, u_{m+2}, \cdots, u_n$ given by identities (9) and thus obtain an identity

$$h_1 u_1 + h_2 u_2 + \cdots + h_m u_m \equiv 0$$

in which the coefficients do not all vanish. But the existence of such an identity together with its first k derivatives would contradict the assumption that $M_k(u_1, u_2, \cdots, u_m)$ is of rank m.

The equivalence of Theorem X and B (except for the specification of X as to the number of independent relations between the u's) is an immediate corollary of Theorem IX. We have shown that Theorem VI, and therefore Theorem X, is of wider application than Theorem A; hence of the two theorems A and B the latter is the more general and includes all cases which come under the former.

We can replace the requirement in Theorem B that the Wronskian vanish identically by any assumption which will make the rank of $M_k(u_1, u_2, \cdots, u_n)$ less than n at some point of I, since there will then be a function of form (9) which will vanish together with its first k derivatives at that point. Theorems IV and V furnish sufficient conditions for the vanishing at a point of I of all n-rowed determinants of

$$M_k(u_1, u_2, \cdots, u_n)$$

and enable us to give to Theorem B the following form:

Theorem XI. *Let* $u_1, u_2, \cdots, u_n$ *be functions of* x *which at every point of* I *have finite derivatives of the first* k *orders* $(k \geqq n-1)$ *while no function (other than zero) of the form*

$$g_1 u_1 + g_2 u_2 + \cdots + g_n u_n$$

(the g*'s being constants) vanishes together with its first* k *derivatives at any point of* I*; then if* $W(u_1, u_2, \cdots, u_n)$ *vanishes together with its first* $k-n+1$ *derivatives in a set* $[P]$ *it vanishes identically throughout* I *and the functions* u *are linearly dependent. If* $u_1^{(k)}, u_2^{(k)}, \cdots, u_n^{(k)}$ *are continuous the last condition may be simplified by requiring only that the Wronskian vanish in a set* $[P]$.

An immediate application of these results is given in the following theorems on linear differential equations.

Theorem XII. *Let* $p_1, p_2, \cdots, p_n$ *be functions of* x *which at every* *point of* I *are continuous, and let* $y_1, y_2, \cdots, y_m$ $(m \leqq n)$ *be functions of* x *which at every point of* I *satisfy the differential equation*

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = 0;$$

then if $W(y_1, y_2, \cdots, y_m)$ *vanishes in a set* $[P]$ *it vanishes identically throughout* I *and the functions* $y_1, y_2, \cdots, y_m$ *are linearly dependent.*

This theorem is a direct consequence of Theorem XI, since no solution other than zero of such a differential equation can vanish together with its first $n - 1$ derivatives at a point of I.

Theorem XIII. *Let* $p_1, p_2, \cdots, p_n$ *be functions of* x *which at every point of* I *have continuous derivatives of the first* $k - n$ *orders* $(k \geqq n)$*), *and let* $y_1, y_2, \cdots, y_r$ $(r \leqq k+1)$ *be functions of* x *which at every point of* I *satisfy the differential equation*

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = 0;$$

then the matrix $M_k(y_1, y_2, \cdots, y_r)$ *is of constant rank in* I.

From our hypothesis as to the coefficients of the differential equation it follows that $y_1, y_2, \cdots, y_r$ have continuous derivatives of the first k orders. Hence if the Wronskian of these functions vanishes identically (as is always the case when $r \geqq n + 1$) the above theorem follows from Theorem IX. On the other hand, if $W(y_1, y_2, \cdots, y_r)$ does not vanish identically (in which case $r \leqq n$) the matrix $M_k(y_1, y_2, \cdots, y_r)$ is of rank r at every point of I; otherwise there would exist a set of equations

$$g_1 y_1^{(\nu)} + g_2 y_2^{(\nu)} + \cdots + g_r y_r^{(\nu)} = 0 \qquad (\nu = 0, 1, \cdots, k)$$

satisfied at some point of I, and therefore valid throughout I. This is impossible when $W(y_1, y_2, \cdots, y_r)$ does not vanish identically.

*) If $k = n$ this means that the functions $p_1, p_2, \cdots, p_n$ are continuous.