The Power-Sum Formula and the Bernoullian Function
Author(s): W. F. Sheppard
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In these three results we touch what I believe to be the bed-rock of Euclid's ideas. Euclid indeed states them as propositions, and deduces them as consequences of his 5th and 7th definitions. But if we try to think about them and grasp their meaning, is it not clear that it is easier to see into their meaning than into that of the 5th and 7th definitions?

Suppose A, B and C represent segments of straight lines, and that A has the same length as B, then the relative magnitude of A when compared with C is the same as that of B when compared with C. And if A be a longer line than B, then the relative magnitude of A when compared with C is greater than the relative magnitude of B when compared with C.

(To be continued.)

THE POWER-SUM FORMULA AND THE BERNOULLIAN FUNCTION.

(IN reference to A. C. Dixon's note on $1^m+2^m+3^m+\ldots+n^m$, on pp. 283-4 of this volume, the following further notes may be of interest. They are taken, with some alterations, from a paper communicated to the London Mathematical Society in February, 1910.)

I. THE POWER-SUM FORMULA.

1. Writing $$Sn^r \equiv 1^r+2^r+3^r+\ldots+n^r,$$

the ordinary formula may be written either

$$Sn^r = 1/(r+1).\{n^{r+1}+\tfrac{1}{2}(r+1,\ 1)n^r+B_1(r+1,\ 2)\,n^{r-1} - B_2(r+1,\ 4)n^{r-3}+B_3(r+1,\ 6)n^{r-5}-\ldots\},\ \ldots\ldots\ldots\ldots(1)$$

or $$S(n-1)^r = 1/(r+1).\{n^{r+1}-\tfrac{1}{2}(r+1,\ 1)n^r+B_1(r+1,\ 2)n^{r-1}-\ldots\},\ \ldots(1\text{A})$$

where $(s,\ t)$ denotes $s!/\{t!(s-t)!\}$, and B_1, B_2, $B_3\ldots$ are Bernoulli's numbers; the series ending with the term in n^2 or in n according as r is odd or even. The expression in $\{\ \}$ presents two anomalies, viz. (i) the absence of a last term $(-)^{s-1}B_s$ for the case of $r=2s-1$, and (ii) the existence of the term in n^r, which spoils the regular progression of the series. The progression is otherwise quite regular if we introduce a Bernoulli's number $B_0\equiv -1$.

2. The first anomaly is explained by the fact that the sum is taken between limits, *i.e.* if $\Sigma n^r \equiv \ldots+1^r+2^r+\ldots+n^r$, then $Sn^r=\Sigma n^r-\Sigma 0^r$, so that, since Sn^r is equal to a polynomial in n, this polynomial contains $n-0$ as a factor. But it will be seen later that the complete series, which, when $r\equiv 2s-1$, is equal to the curious expression

$$1^{2s-1}+2^{2s-1}+3^{2s-1}+\ldots+n^{2s-1}+(-)^{s+1}B_s/(2s),$$

is of importance for certain purposes.

3. The reason of the second anomaly is that the formula we are considering is essentially a formula, not for $1^r+2^r+\ldots+n^r$, or for $0^r+1^r+2^r+\ldots+n^r$, but for $\tfrac{1}{2}.0^r+1^r+2^r+\ldots+(n-1)^r+\tfrac{1}{2}n^r$; just as the occurrence of $\sqrt{n}$ in the approximate formula for $n!$ (when n is large) is due to the fact that we are essentially using a formula for $\tfrac{1}{2}\log 1+\log 2+\log 3+\ldots+\log(n-1)+\tfrac{1}{2}\log n$. The expression $\tfrac{1}{2}.0^r+1^r+2^r+\ldots+(n-1)^r+\tfrac{1}{2}n^r$ is the ordinary trapezoidal approximation to the area, between limits $x=0$ and $x=n$, of a figure whose ordinate is x^r; and the series composed of the terms involving B_1, B_2, ... in (1) gives the difference between this approximate value and the true value $(n^{r+1}-0^{r+1})/(r+1)$. We should naturally expect that this difference would be represented by a consistent expression in terms of n; and the addition of $\tfrac{1}{2}n^r$ to both sides produces an inconsistency.

4. To obtain a consistent expression for

$$1^r+2^r+\ldots+n^r \quad \text{or} \quad 0^r+1^r+2^r+\ldots+n^r$$

as a single entity, we observe that it is an approximate expression for the area of a figure whose ordinate, as before, is x^r, but which extends from $x=\frac{1}{2}$ or $x=-\frac{1}{2}$ to $x=n+\frac{1}{2}$. It is therefore a function of $n+\frac{1}{2}$ rather than of n. The following are particular cases, writing $m\equiv 2n+1$.

$$Sn \;=\tfrac{1}{2}(\tfrac{1}{2})^2(m^2-1),$$
$$Sn^2=\tfrac{1}{3}(\tfrac{1}{2})^3 m(m^2-1),$$
$$Sn^3=\tfrac{1}{4}(\tfrac{1}{2})^4(m^2-1)^2,$$
$$Sn^4=\tfrac{1}{15}(\tfrac{1}{2})^5 m(m^2-1)(3m^2-7),$$
$$Sn^5=\tfrac{1}{6}(\tfrac{1}{2})^6(m^2-1)^2(m^2-3),$$
$$Sn^6=\tfrac{1}{21}(\tfrac{1}{2})^7 m(m^2-1)(3m^4-18m^2+31),$$
$$Sn^7=\tfrac{1}{24}(\tfrac{1}{2})^8(m^2-1)^2(3m^4-22m^2+51),$$
$$Sn^8=\tfrac{1}{45}(\tfrac{1}{2})^9 m(m^2-1)(5m^6-55m^4+239m^2-381),$$
$$Sn^9=\tfrac{1}{10}(\tfrac{1}{2})^{10}(m^2-1)^2(m^2-5)(m^4-8m^2+31),$$
$$Sn^{10}=\tfrac{1}{33}(\tfrac{1}{2})^{11}m(m^2-1)(m^2-5)(3m^6-37m^4+225m^2-511),$$
$$Sn^{11}=\tfrac{1}{12}(\tfrac{1}{2})^{12}(m^2-1)^2(m^8-20m^6+190m^4-964m^2+2073).$$

5. The general formulæ for Sn^r in terms of n and in terms of $\nu\equiv n+\frac{1}{2}$ are most simply obtained by using the central-difference notation. Writing

$$\delta f(x)\equiv f(x+\tfrac{1}{2}h)-f(x-\tfrac{1}{2}h), \qquad \mu f(x)\equiv\tfrac{1}{2}\{f(x+\tfrac{1}{2}h)+f(x-\tfrac{1}{2}h)\},$$
$$\sigma f(x)\equiv\ldots+f(x-\tfrac{3}{2}h)+f(x-\tfrac{1}{2}h),$$

so that $\delta\sigma f(x)=f(x)$, we have $\delta=2\sinh(\frac{1}{2}hD)$, $\mu=\cosh(\frac{1}{2}hD)$, where D is the operator which converts a polynomial in x into its first derivative; and therefore, taking $h=1$, $x=n$,

$$\ldots+(n-1)^r+\tfrac{1}{2}n^r$$
$$=\mu\sigma\,.\,n^r=\mu\delta^{-1}\,.\,n^r=\tfrac{1}{2}\coth\tfrac{1}{2}D\,.\,n^r$$
$$=D^{-1}(1+B_1D^2/2\,!-B^2D^4/4\,!+B_3D^6/6\,!-\ldots)n^r \quad\ldots\ldots(2)$$
$$=\frac{n^{r+1}}{r+1}+B_1\frac{r}{2\,!}n^{r-1}-B_2\frac{r(r-1)(r-2)}{4\,!}n^{r-3}+\ldots, \quad\ldots\ldots(2\text{A})$$

and

$$\ldots+(n-1)^r+n^r$$
$$=\sigma\,.\,\nu^r=\delta^{-1}\,.\,\nu^r=\tfrac{1}{2}\,\text{cosech}\,\tfrac{1}{2}D\,.\,\nu^r$$
$$=D^{-1}(1-p_1D^2/2\,!+p_2D^4/4\,!-p_3D^6/6\,!+\ldots)\nu^r \quad\ldots\ldots(3)$$
$$=\frac{1}{r+1}\{\nu^{r+1}-p_1(r+1,2)\nu^{r-1}+p_2(r+1,4)\nu^{r-3}-\ldots\} \quad\ldots\ldots(3\text{A})$$
$$=\frac{1}{r+1}(\tfrac{1}{2})^{r+1}\{m^{r+1}-P_1(r+1,2)m^{r-1}+P_2(r+1,4)m^{r-3}-\ldots\}, \quad(3\text{B})$$

where $\quad P_q\equiv(2^{2q}-2)B_q,\ \ p_q=(\frac{1}{2})^{2q}P_q=\{1-(\frac{1}{2})^{2q-1}\}B_q.$ ……(4)

6. For $\frac{1}{2}\,.\,0^r+1^r+2^r+\ldots+(n-1)^r+\frac{1}{2}n^r$, the series in (2A) will (§2) end with the term in n^2 or in n. To find the last term of the series in (3B), as a formula for Sn^r, we may proceed as follows.

(i) The term is found by taking the series between limits $\nu=\frac{1}{2}$ or $\nu=-\frac{1}{2}$ and $\nu=\nu$; *i.e.* it is found by the condition that the series in (3B) has m^2-1 as a factor. We have therefore to find the value of

$$1-P_1(r+1,2)+P_2(r+1,4)-\ldots,$$

continued up to the term in $(r+1, r-1)$ or in $(r+1, r)$ according as r is odd or even.

(ii) This involves equating coefficients in

$$\{1+\theta+\theta^2/2!+\theta^3/3!+\ldots\}\{1-P_1\theta^2/2!+P_2\theta^4/4!-P_3\theta^6/6!+\ldots\}$$
$$=e^\theta.\,\theta\operatorname{cosech}\theta=\theta+\theta\coth\theta$$
$$=1+\theta+2^2B_1\theta^2/2!-2^4B_2\theta^4/4!+2^6B_3\theta^6/6!-\ldots \qquad (5)$$

(*a*) Let $r=2s-1$. Then, equating coefficients of $\theta^{2s}/(2s)!$ in (5),

$$1-P_1(2s,\,2)+P_2(2s,\,4)-\ldots+(-)^sP_s=(-)^{s+1}2^{2s}B_s. \qquad (6)$$

Hence, if we write

$$\lambda_q\equiv P_q+2^{2q}B_q=2(P_q+B_q)=2(2^{2q}-1)B_q, \qquad (7)$$

we have

$$1-P_1(2s,\,2)+P_2(2s,\,4)-\ldots+(-)^{s-1}P_{s-1}(2s,\,2s-2)+(-)^s\lambda_s=0,$$

and therefore

$$Sn^{2s-1}=\frac{1}{2s}(\tfrac{1}{2})^{2s}\{m^{2s}-P_1(2s,\,2)m^{2s-2}+P_2(2s,\,4)m^{2s-4}-\ldots$$
$$+(-)^{s-1}P_{s-1}(2s,\,2s-2)m^2+(-)^s\lambda_s\} \quad (8)$$
$$=\frac{1}{2s}\{\nu^{2s}-p_1(2s,\,2)\nu^{2s-2}+p_2(2s,\,4)\nu^{2s-4}-\ldots$$
$$+(-)^{s-1}p_{s-1}(2s,\,2s-2)\nu^2+(-)^s\lambda_s/2^{2s}\}. \quad (8\text{A})$$

(*b*) Let $r=2s$. Then, equating coefficients of $\theta^{2s+1}/(2s+1)$ in (5),

$$1-P_1(2s+1,\,2)+P_2(2s+1,\,4)-\ldots+(-)^sP_s(2s+1,\,2s)=0\,; \qquad (9)$$

and therefore

$$Sn^{2s}=\frac{1}{2s+1}(\tfrac{1}{2})^{2s+1}\{m^{2s+1}-P_1(2s+1,\,2)m^{2s-1}+P_2(2s+1,\,4)m^{2s-3}-\ldots$$
$$+(-)^sP_s(2s+1,\,2s)m\} \quad (10)$$
$$=\frac{1}{2s+1}\{\nu^{2s+1}-p_1(2s+1,\,2)\nu^{2s-1}+p_2(2s+1,\,4)\nu^{2s-3}-\ldots$$
$$+(-)^sp_s(2s+1,\,2s)\nu\}. \quad (10\text{A})$$

(iii) We see that, if r is even, the expression for Sn^r is divisible by m, *i.e.* by $2n+1$; and that, whether r is even or odd, it is divisible by m^2-1, and therefore by $n(n+1)$.

7. Although the expression for Σn^r in terms of $n+\frac{1}{2}$ or $2n+1$ is simpler algebraically than the expression in terms of n, it is not so simple arithmetically, since the coefficients are unmanageably large. We know that $2B_q$ (except for $q=0$) is a fraction whose numerator and denominator are odd; and it may be shown that $\lambda_q\equiv 2B_q+2P_q$ is an odd integer. It follows from (4) that P_q is obtained from B_q by dividing its denominator by 2 and multiplying its numerator by $2^{2q-1}-1$; and this multiplier soon becomes large. The following are the values of λ_q and of P_q up to $q=10$.

$$\lambda_0=0,\ \lambda_1=1,\ \lambda_2=1,\ \lambda_3=3,\ \lambda_4=17,\ \lambda_5=155,\ \lambda_6=2073,\ \lambda_7=38227,$$
$$\lambda_8=929569,\ \lambda_9=28820619,\ \lambda_{10}=1109652905\,;$$
$$P_0=1,\ P_1=1/3,\ P_2=7/15,\ P_3=31/21,\ P_4=127/15,\ P_5=2555/33,$$
$$P_6=1414477/1365,\ P_7=57337/3,\ P_8=118518239/255,$$
$$P_9=5749691557/399,\ P_{10}=91546277357/165.$$

II. The Bernoullian Function.

8. The Bernoullian function of degree k is usually defined as being

$$\phi_k(x)\equiv x^k-\tfrac{1}{2}x^{k-1}+B_1(k,\,2)x^{k-2}-B_2(k,\,4)x^{k-4}+B_3(k,\,6)x^{k-6}-\ldots\,; \quad (11)$$

the series ending with the term in x^2 or in x according as k is even or odd. This is for $k>1$; for $k=1$, $\phi_1(x)$ is usually taken to be x, but it would be more consistent to take it to be $x-\frac{1}{2}$. Various properties of the function are given by Bromwich, *Theory of Infinite Series*, pp. 235-7. (In the last line on p. 237, $(-)^{m-1}$ should be $(-)^m$.)

9. The occurrence of $(-)^s\lambda_s$, in place of $(-)^sP_s$, in the last term of the series in (8), is due to the absence of the term in n^0 in the formula for Sn^{2s-1} in terms of n; and we should obtain neater results by introducing this latter term. We therefore write

$$\Phi_k(x) \equiv x^k - \tfrac{1}{2}kx^{k-1} + B_1(k,2)x^{k-2} - B_2(k,4)x^{k-4} + B_3(k,6)x^{k-6} - \ldots, \quad \ldots(12)$$

$$\Psi_k(x) \equiv x^k - p_1(k,2)x^{k-2} + p_2(k,4)x^{k-4} - p_3(k,6)x^{k-6} + \ldots; \quad \ldots(13)$$

the series in each case continuing until the coefficients become zero. Then we shall have

$$\Psi_k(x) = \Phi_k(x+\tfrac{1}{2}), \quad \ldots(14)$$

$$e^{xt}(\tfrac{1}{2}t\coth\tfrac{1}{2}t - \tfrac{1}{2}t) = \sum_{k=0}^{k=\infty}\Phi_k(x)\,.\,t^k/k!, \quad \ldots(15)$$

$$e^{xt}\,.\,\tfrac{1}{2}t\,\mathrm{cosech}\,\tfrac{1}{2}t = \sum_{k=0}^{k=\infty}\Psi_k(x)\,.\,t^k/k!. \quad \ldots(16)$$

For $k=0$ and $k=1$ the functions will be

$$\Phi_0(x)=1, \quad \Psi_0(x)=1, \quad \Phi_1(x)=x-\tfrac{1}{2}, \quad \Psi_1(x)=x.$$

It will be found that the following relations hold.

(i) $\Phi_k(x+1)-\Phi_k(x)=\Psi_k(x+\frac{1}{2})-\Psi_k(x-\frac{1}{2})=kx^{k-1}$.

(ii) $\Phi_k(1-x)=(-)^k\Phi(x)$; $\Psi_k(-x)=(-)^k\Psi_k(x)$.

(iii) $\Phi_k'(x)=k\Phi_{k-1}(x)$; $\Psi_k'(x)=k\Psi_{k-1}(x)$.

(iv) If $q>0$, $\Phi_{2q+1}(x)$ contains $x(x-\frac{1}{2})(x-1)$ as a factor, and $\Psi_{2q+1}(x)$ contains $x(x^2-\frac{1}{4})$ as a factor.

(v) If $q>1$, $\Phi_{2q}(x)+(-)^qB_q$ contains $x^2(x-1)^2$ as a factor, and $\Psi_{2q}(x)+(-)^qB_q$ contains $(x^2-\frac{1}{4})^2$ as a factor.

(vi) If $q>0$,

$$\Phi_{2q+1}(0)=\Phi_{2q+1}(\tfrac{1}{2})=\Phi_{2q+1}(1)=\Psi_{2q+1}(\pm\tfrac{1}{2})=\Psi_{2q+1}(0)=0.$$

(vii) $$\left.\begin{aligned}\Phi_{2q}(0)=\Phi_{2q}(1)=\Psi_{2q}(\pm\tfrac{1}{2})&=(-)^{q-1}B_q;\\ \Phi_{2q}(\tfrac{1}{2})=\Psi_{2q}(0)&=(-)^qp_q.\end{aligned}\right\}$$

(viii) $\Phi_{2q+1}(x)$ is of sign $(-)^{q+1}$ from $x=0$ to $x=\frac{1}{2}$, and of sign $(-)^q$ from $x=\frac{1}{2}$ to $x=1$, and has no zero value between $x=0$ and $x=1$ except at $x=\frac{1}{2}$; and $\Psi_{2q+1}(x)$ is of sign $(-)^{q+1}$ from $x=-\frac{1}{2}$ to $x=0$, and of sign $(-)^q$ from $x=0$ to $x=\frac{1}{2}$, and has no zero value between $x=-\frac{1}{2}$ and $x=\frac{1}{2}$ except at $x=0$.

(ix) $\Phi_{2q}(x)$ (for $q>0$) is zero once only between $x=0$ and $x=\frac{1}{2}$, and once only between $x=\frac{1}{2}$ and $x=1$, and has no stationary value between $x=0$ and $x=1$ except at $x=\frac{1}{2}$; and $\Psi_{2q}(x)$ (for $q>0$) is zero once only between $x=-\frac{1}{2}$ and $x=0$, and once only between $x=0$ and $x=\frac{1}{2}$, and has no stationary value between $x=-\frac{1}{2}$ and $x=\frac{1}{2}$ except at $x=0$.

(x) $\Phi_{2q}(x)+(-)^qB_q(=\Phi_{2q}(x)-\Phi_{2q}(0)=\Phi_{2q}(x)-\Phi_{2q}(1))$ is of sign $(-)^q$ from $x=0$ to $x=1$; and $\Psi_{2q}(x)+(-)^qB_q(=\Psi_{2q}(x)-\Psi_{2q}(\pm\frac{1}{2}))$ is of sign $(-)^q$ from $x=-\frac{1}{2}$ to $x=\frac{1}{2}$. Also $\Phi_{2q}(x)+(-)^{q-1}p_q(=\Phi_{2q}(x)-\Phi_{2q}(\frac{1}{2}))$ is of sign $(-)^{q-1}$ from $x=0$ to $x=\frac{1}{2}$, and from $x=\frac{1}{2}$ to $x=1$, and $\Psi_{2q}(x)+(-)^{q-1}p_q$ $(=\Psi_{2q}(x)-\Psi_{2q}(0))$ is of sign $(-)^{q-1}$ from $x=-\frac{1}{2}$ to $x=0$, and from $x=0$ to $x=\frac{1}{2}$.

(xi) If (so as to include the case of $r=0$) we define Sn^r as being $\frac{1}{2}.0^r+1^r+2^r+\ldots+n^r$, then

$$Sn^{2q-1}=1/(2q).\{\Phi_{2q}(n+1)+(-)^q B_q\}$$
$$=1/(2q).\{\Psi_{2q}(n+\tfrac{1}{2})+(-)^q B_q\},$$
$$Sn^{2q}=1/(2q+1).\Phi_{2q+1}(n+1)$$
$$=1/(2q+1).\Psi_{2q+1}(n+\tfrac{1}{2}).$$

W. F. SHEPPARD.

(*To be continued.*)

NOTICE.

WE have to thank the Officers of the Congress whose photographs are reproduced in this volume for their kind consent in allowing them to appear.

We are also indebted to Mr. John Murray for the electro of Roubiliac's statue of Newton. To Messrs. J. Palmer Clarke, Cambridge, Lafayette & Co., Ltd., London, and Elliott & Fry, Ltd., London, we beg to express our sense of the courtesy with which they accorded the permission it was necessary to obtain.

LOCAL BRANCHES.

THE N. WALES BRANCH.

THE N. Wales branch of the Mathematical Association recently organised a conference of mathematical teachers from the primary schools, secondary schools, training colleges, and the University College of N. Wales, which was held at Bangor on May 9. Prof. G. H. Bryan took the chair, and Principal Sir H. Reichel also attended.

The Chairman said that it was very important that such conferences should be held at the present time on account of the great changes which had recently taken place in the teaching of children and in the treatment of the different subjects. In one branch, indeed, teachers had continually to wage war against what he had previously characterised as "England's neglect of Mathematics." In some quarters it did not seem to be realised what an important part the teachers of the exact science played in our social and economic life. Statistics and political questions brought them face to face with a big problem, and the introduction of such questions would largely obtain in the schools of the future, inasmuch as in time to come political and economic problems would be resolved into questions of statistics. A comparison of the methods of teaching mathematics adopted now and twenty years ago would reveal the great improvement which had taken place. In regard to geometry, formerly the idea seemed to consist in the failing of the pupils to do riders, and in the learning by rote of theorems which were promptly forgotten when the examination day came round or when the distinguishing letters of the diagram were changed. A mistaken idea was abroad that mathematicians of high order made bad teachers. How absurd this was could be seen from the fact that Smith's prizemen and the products of Part II. have applied the same methods of research used in original papers to subjects taught out of elementary text-books. These men had cast a powerful searchlight on obscure and difficult parts of our elementary curricula, and had asked what was the good of pupils' studying things beyond their grasp. As a result, modern teaching tended more and more to give pupils a clear understanding of quantity and space instead of the mere power of juggling with collections of letters and of old and obscure systems. Nevertheless, it was not a destructive policy of teaching that they were inquiring after, but a rational one. Continuity of purpose did not imply uniformity of methods.