



XXIV. The third-order aberrations of a symmetrical optical instrument

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ZnS above, as well as in NaCl in the previous paper, are due to this cause. The diamond appears to be unique in that there is only one kind of atom and that the inner ring does not contain one or two electrons. I am not aware that any other crystal has been studied experimentally where none of the atoms have, according to the theory, one or two electrons at the centre.

Fig. 5 is a photograph of a model of the diamond, and may be found of assistance in supplementing figs. 3 and 4.

Note.

Since this paper was communicated a considerable number of crystals on the isometric system has been worked out, and experiment found to agree with theory. These include zinc and copper.

XXIV. *The Third-Order Aberrations of a Symmetrical Optical Instrument.* By E. HOWARD SMART, M.A., *Head of the Mathematical Department, Birkbeck College, London*.*

THE five third-order aberrations of a symmetrical optical instrument, commonly associated with the name of Von Seidel, have been frequently discussed. But the mode of presentation often leaves something to be desired from the practical optician's point of view, direction cosines of rays and the like being to him of inferior importance compared with angular aperture and field of view. Sometimes even the whole subject is treated in general terms, the constants of the instrument not being considered; and occasionally there is some obscurity regarding the relations between the several errors.

In this paper an attempt has been made to effect some improvement in these respects, and at the same time to indicate a method by which the investigations could be extended so as to deal with the fifth-order corrections.

Let C_i (fig. 1) be the centre and O_i the vertex of the i th of a system of coaxial spherical surfaces, and let this surface separate media of refractive indices μ_{i-1} and μ_i . Let r_i be the radius of curvature of the surface considered positive when the surface is convex to the incident light. Take O_i as origin and a pair of rectangular tangents at O_i , and the axis of symmetry as coordinate axes.

* Communicated by the Author.

Also
$$C_i P_i / C_i P_{i+1} = (x_i + \delta x_i) / (x_{i+1} + \delta x_{i+1}). \quad \dots \quad (3)$$

Hence (2) gives

$$\frac{\mu_{i-1}(x_i + \delta x_i)}{\text{above expression for } Q_i P_i} = \frac{\mu_i(x_{i+1} + \delta x_{i+1})}{\text{corresponding expression for } Q_i P_{i+1}}. \quad (4)$$

Let (4) be expanded by Taylor's theorem. Denoting

$$\frac{\mu_{i-1} x_i}{\{x_i^2 + y_i^2 + s_i^2 - 2(\xi_i x_i + \eta_i y_i) - 2\zeta_i(s_i - r_i)\}^{\frac{1}{2}}} \text{ by } F_i,$$

we get

$$\begin{aligned} & F_i + \delta x_i \frac{\partial F_i}{\partial x_i} + \delta y_i \frac{\partial F_i}{\partial y_i} + \Delta s_i \frac{\partial F_i}{\partial s_i} + \dots \\ &= F_{i+1} + \delta x_{i+1} \frac{\partial F_{i+1}}{\partial x_{i+1}} + \delta y_{i+1} \frac{\partial F_{i+1}}{\partial y_{i+1}} + \Delta s_i' \frac{\partial F_{i+1}}{\partial s_i'} + \dots \quad (5) \end{aligned}$$

We develop this expansion in terms of

$$\frac{x_i}{s_i}, \quad \frac{x_{i+1}}{s_i}, \quad \frac{\xi_i}{s_i}, \quad \frac{\eta_i}{s_i}, \quad \&c.,$$

supposed of the same order of magnitude.

F_i (up to terms of the third order)

$$= \frac{\mu_{i-1} x_i}{s_i} \left(1 + \frac{\xi_i^2 + \eta_i^2}{2s_i} \left(\frac{1}{r_i} - \frac{1}{s_i} \right) + \frac{\xi_i x_i + \eta_i y_i}{s_i^2} - \frac{x_i^2 + y_i^2}{2s_i^2} \right),$$

making use of (1), and F_{i+1} = a similar expression with

$$\mu_i, \quad x_{i+1}, \quad y_{i+1}, \quad s_i' \text{ for } \mu_{i-1}, \quad x_i, \quad y_i, \quad s_i.$$

Also

$$\delta x_i / x_i = \delta y_i / y_i = \Delta s_i / (s_i - r_i)$$

and

$$\delta x_{i+1} / x_{i+1} = \delta y_{i+1} / y_{i+1} = \Delta s_i' / (s_i' - r_i)$$

from the figure $\dots \dots \dots (5')$

Eliminating δx_i , δy_i , δx_{i+1} , δy_{i+1} with the help of these from (5), we get, after a little reduction, as the coefficient of

$\frac{\Delta s_i}{s_i - r_i}$ on the left-hand side of (5),

$$- \frac{\mu_{i-1} x_i (\xi_i x_i + \eta_i y_i + \zeta_i (s_i - r_i) - r_i s_i)}{\{x_i^2 + y_i^2 + s_i^2 - 2(\xi_i x_i + \eta_i y_i) - 2\zeta_i (s_i - r_i)\}^{\frac{1}{2}}}$$

with a similar expression on the right-hand side.

Approximating in (5) we have, rejecting 3rd order terms,

$$\frac{\mu_{i-1}x_i}{s_i} = \frac{\mu_i x_{i+1}}{s_i'} \quad \dots \quad (6)$$

This is obviously true, for by considering the points K_i , K_{i+1} in the figure, we have

$$\frac{x_i}{s_i - r_i} = \frac{x_{i+1}}{s_i' - r_i},$$

and the positions of the Gaussian image-planes are connected by the relation

$$\mu_{i-1} \left(\frac{1}{r_i} - \frac{1}{s_i} \right) = \mu_i \left(\frac{1}{r_i} - \frac{1}{s_i'} \right) \quad \dots \quad (7)$$

To the next approximation including 3rd order terms we have

$$\begin{aligned} \frac{\mu_{i-1}x_i}{s_i} \left\{ 1 + \frac{1}{2} \left(\frac{1}{r_i} - \frac{1}{s_i} \right) \frac{\xi_i^2 + \eta_i^2}{s_i} + \frac{\xi_i x_i + \eta_i y_i}{s_i^2} \right. \\ \left. - \frac{1}{2} \frac{x_i^2 + y_i^2}{s_i^2} \right\} + \frac{\mu_{i-1} \Delta s_i x_i}{s_i^3} \cdot \frac{r_i s_i}{s_i - r_i} \\ = \frac{\mu_i x_{i+1}}{s_i'} \left\{ 1 + \frac{1}{2} \left(\frac{1}{r_i} - \frac{1}{s_i'} \right) \frac{\xi_i^2 + \eta_i^2}{s_i'} + \frac{\xi_i x_{i+1} + \eta_i y_{i+1}}{s_i'^2} \right. \\ \left. - \frac{1}{2} \frac{x_{i+1}^2 + y_{i+1}^2}{s_i'^2} \right\} + \frac{\mu_i \Delta s_i' x_{i+1}}{s_i'^3} \cdot \frac{r_i s_i'}{s_i' - r_i}. \end{aligned}$$

Making use of (6) and denoting the equal expressions in (7) by the symbol Q_i , we have as the simplified form of the last equation,

$$\begin{aligned} \frac{\mu_i \Delta s_i'}{s_i'^2} - \frac{\mu_{i-1} \Delta s_i}{s_i^2} = -\frac{1}{2} Q_i (\xi_i^2 + \eta_i^2) \left(\frac{1}{\mu_i s_i'} - \frac{1}{\mu_{i-1} s_i} \right) \\ - \left(\frac{\mu_{i-1}}{s_i} \right) Q_i (\xi_i x_i + \eta_i y_i) \left(\frac{1}{\mu_i s_i'} - \frac{1}{\mu_{i-1} s_i} \right) \\ + \frac{1}{2} \left(\frac{\mu_{i-1}}{s_i} \right)^2 (x_i^2 + y_i^2) Q_i \left(\frac{1}{\mu_i^2} - \frac{1}{\mu_{i-1}^2} \right). \end{aligned}$$

Now

$$\begin{aligned} Q_i \left(\frac{1}{\mu_i^2} - \frac{1}{\mu_{i-1}^2} \right) &= \frac{1}{\mu_i} \left(\frac{1}{r_i} - \frac{1}{s_i'} \right) - \frac{1}{\mu_{i-1}} \left(\frac{1}{r_i} - \frac{1}{s_i} \right) \\ &= \frac{1}{r_i} \left(\frac{1}{\mu_i} - \frac{1}{\mu_{i-1}} \right) - \left(\frac{1}{\mu_i s_i'} - \frac{1}{\mu_{i-1} s_i} \right), \quad (8) \end{aligned}$$

So the last equation takes the final form

$$\begin{aligned} \frac{\mu_i \Delta s'_i}{s_i'^2} - \frac{\mu_{i-1} \Delta s_i}{s_i^2} = & -\frac{1}{2} Q_i^2 (\xi_i^2 + \eta_i^2) \left(\frac{1}{\mu_i s'_i} - \frac{1}{\mu_{i-1} s_i} \right) \\ & - \left(\frac{\mu_{i-1}}{s_i} \right) Q_i (\xi_i x_i + \eta_i y_i) \left(\frac{1}{\mu_i s'_i} - \frac{1}{\mu_{i-1} s_i} \right) \\ & - \frac{1}{2} \left(\frac{\mu_{i-1}}{s_i} \right)^2 (x_i^2 + y_i^2) \left(\frac{1}{\mu_i s'_i} - \frac{1}{\mu_{i-1} s_i} \right) \\ & + \frac{1}{2} \left(\frac{\mu_{i-1}}{s_i} \right)^2 (x_i^2 + y_i^2) \frac{1}{r_i} \left(-\frac{1}{\mu_i} \frac{1}{\mu_{i-1}} \right). \quad \dots \quad (9) \end{aligned}$$

This is the recurrence formula for the longitudinal aberration of the successive images P_i, P_{i+1} .

To sum this throughout the system we require first relations between successive ξ 's and η 's, the coordinates of the points at which the ray strikes the successive surfaces.

The condition that the refracted ray passes through the point $(\xi_{i+1}, \eta_{i+1}, d_i + \zeta_{i+1})$, where d_i is the distance between the i th and $(i+1)$ th surfaces, is

$$\frac{\xi_{i+1} - \xi_i}{x_{i+1} + \delta x_{i+1} - \xi_i} = \frac{\eta_{i+1} - \eta_i}{y_{i+1} + \delta y_{i+1} - \eta_i} = \frac{d_i + \zeta_{i+1} - \zeta_i}{s'_i + \Delta s'_i - \zeta_i}.$$

In approximating to this we need only retain first-order quantities; so remembering that

$$\frac{\delta x_{i+1}}{x_{i+1}}, \quad \frac{\delta y_{i+1}}{y_{i+1}}, \quad \frac{\Delta s'_i}{s'_i}, \quad \frac{\zeta_i}{s'_i}, \quad \text{and} \quad \frac{\zeta_{i+1}}{s'_i},$$

are all of the second order, we get from the equality of the first and third

$$\xi_{i+1} - \xi_i \left(1 - \frac{d_i}{s'_i} \right) = \frac{x_{i+1} d_i}{s'_i}.$$

Making use of (6) this may be written symmetrically

$$\frac{\mu_i x_{i+1} \xi_{i+1}}{s_{i+1}} = \frac{\mu_{i-1} x_i \xi_i}{s_i} + \frac{\mu_{i-1} d_i x_i x_{i+1}}{s_i s_{i+1}},$$

since

$$s'_i = s_{i+1} + d_i. \quad \dots \quad (10)$$

Similarly

$$\frac{\mu_i y_{i+1} \eta_{i+1}}{s_{i+1}} + \frac{\mu_{i-1} y_i \eta_i}{s_i} + \frac{\mu_{i-1} d_i y_i y_{i+1}}{s_i s_{i+1}}. \quad \dots \quad (11)$$

For purposes of summation we shall also need the quantities $h_1, h_2, \dots, h_i, \dots$ which are the heights above the axis at which a paraxial ray, starting from the axial point of the object and passing through the system, cuts the successive surfaces. It should be pointed out, however, that these are introduced merely as a convenient device for summation, as used by Seidel; the quantities h must not be confused with the ξ 's and η 's which indicate the true order of approximation as regards aperture.

We have

$$\frac{s_{i+1}}{h_{i+1}} = \frac{s'_i}{h_i}; \quad \frac{s_i}{h_i} = \frac{s'_{i-1}}{h_{i-1}}; \text{ and so on } \dots \quad (12)$$

Hence the relation (6) may be written

$$\begin{aligned} \frac{\mu_i x_{i+1} h_{i+1}}{s_{i+1}} &= \frac{\mu_{i-1} x_i h_i}{s_i} = \text{similarly } \frac{\mu_{i-2} x_{i-1} h_{i-1}}{s_{i-1}} = \&c. \dots \\ &= \frac{\mu_0 x_1 h_1}{s_1} = \lambda \text{ (say)}, \quad \dots \quad (13) \end{aligned}$$

where λ is a constant for the ray throughout its passage through the instrument.

Similarly

$$\frac{\mu_i y_{i+1} h_{i+1}}{s_{i+1}} = \&c. \dots = \frac{\mu_0 y_1 h_1}{s_1} = \text{another constant } \mu. \quad (14)$$

Summing the equations (10) and (11) we have

$$\begin{aligned} \frac{\mu_i x_{i+1} \xi_{i+1}}{s_{i+1}} &= \frac{\mu_0 x_1 \xi_1}{s_1} + \sum_{p=1}^{p=i} \frac{\mu_{p-1} d_p x_p x_{p+1}}{s_p s_{p+1}}; \\ \frac{\mu_i y_{i+1} \eta_{i+1}}{s_{i+1}} &= \frac{\mu_0 y_1 \eta_1}{s_1} + \sum_{p=1}^{p=i} \frac{\mu_{p-1} d_p y_p y_{p+1}}{s_p s_{p+1}}; \end{aligned}$$

which by (13) and (14) may be written compactly

$$\frac{\xi_{i+1}}{h_{i+1}} = \frac{\xi_1}{h_1} + \lambda \sum_{p=1}^{p=i} \frac{d_p}{\mu_p h_p h_{p+1}}; \quad \frac{\eta_{i+1}}{h_{i+1}} = \frac{\eta_1}{h_1} + \mu \sum_{p=1}^{p=i} \frac{d_p}{\mu_p h_p h_{p+1}}. \quad (15)$$

We can now effect the summation of (9) for all the refracting surfaces to the i th. For brevity we shall denote

$$\sum_{p=1}^{p=i-1} \frac{d_p}{\mu_p h_p h_{p+1}} \text{ by } \Sigma \text{ and } \frac{1}{\mu_i s'_i} - \frac{1}{\mu_{i-1} s_i} \text{ by } \Delta \left(\frac{1}{\mu s} \right).$$

Then multiplying equation (9) through by h_i^2 , and since $\Delta s_i' = \Delta s_{i+1}$,

$$\begin{aligned} \frac{\mu_i \Delta s_{i+1} h_{i+1}^2}{s_{i+1}^2} - \frac{\mu_{i-1} \Delta s_i h_i^2}{s_i^2} = & -\frac{1}{2} Q_i^2 h_i^4 \left\{ \frac{\xi_1^2 + \eta_1^2}{h_1^2} + \frac{2(\xi_1 \lambda + \eta_1 \mu)}{h_1} \Sigma \right. \\ & \left. + (\lambda^2 + \mu^2) \Sigma^2 \right\} \Delta \left(\frac{1}{\mu s} \right) - Q_i h_i^2 \Delta \left(\frac{1}{\mu s} \right) \left\{ \frac{\xi_1 \lambda + \eta_1 \mu}{h_1} + (\lambda^2 + \mu^2) \Sigma \right\} \\ & - \frac{1}{2} (\lambda^2 + \mu^2) \Delta \left(\frac{1}{\mu s} \right) + \frac{1}{2} (\lambda^2 + \mu^2) \frac{1}{r_i} \left(\frac{1}{\mu_i} - \frac{1}{\mu_{i-1}} \right). \end{aligned}$$

By summation, since $\Delta s_1 = 0$,

$$\begin{aligned} \frac{\mu_i \Delta s_{i+1} h_{i+1}^2}{s_{i+1}^2} = & -\frac{1}{2} \left(\frac{\xi_1^2 + \eta_1^2}{h_1^2} \right) \sum_1^i Q_i^2 h_i^4 \Delta \left(\frac{1}{\mu s} \right) - \left(\frac{\xi_1 \lambda + \eta_1 \mu}{h_1} \right) \sum_1^i Q_i h_i^2 \Delta \left(\frac{1}{\mu s} \right) (1 + Q_i h_i^2 \Sigma) \\ & - \frac{1}{2} (\lambda^2 + \mu^2) \sum_1^i \Delta \left(\frac{1}{\mu s} \right) (1 + Q_i h_i^2 \Sigma)^2 + \frac{1}{2} (\lambda^2 + \mu^2) \sum_1^i \frac{1}{r_i} \left(\frac{1}{\mu_i} - \frac{1}{\mu_{i-1}} \right). \quad (16) \end{aligned}$$

When λ and μ are each zero the object and image are on the axis. The aberration is then the central spherical aberration only and is given by the first term of the right-hand side of (16).

The condition for no central aberration is thus

$$\sum_1^i Q_i^2 h_i^4 \left(\frac{1}{\mu_i s_i} - \frac{1}{\mu_{i-1} s_i} \right) = 0 \quad . \quad . \quad . \quad (17)$$

Supposing this satisfied, let λ and μ be now small quantities whose square may be neglected.

Then the expression for the longitudinal aberration is

$$\left(\frac{\xi_1 \lambda + \eta_1 \mu}{h_1} \right) \sum_1^i Q_i h_i^2 \left(\frac{1}{\mu_i s_i} - \frac{1}{\mu_{i-1} s_i} \right) \left(1 + Q_i h_i^2 \sum_{p=1}^{i-1} \frac{d_p}{\mu_p h_p h_{p+1}} \right).$$

The evanescence of the coefficient of $\frac{\xi_1 \lambda + \eta_1 \mu}{h_1}$ is the condition for the absence of coma, the balloon-shaped flare produced in the image-plane owing to the images for different values of ξ_1 and η_1 being distributed along the line $C_i K_i K_{i+1}$ in fig. 1. As in Whittaker's tract ('Theory of Optical Instruments,' 1907, § 25) it may be shown that to this order of approximation this is identical with Abbe's sine condition.

We shall now consider the union of rays in the primary and secondary planes respectively.

Since in equation (16) $\lambda/x_1 = \mu/y_1$ we must have in primary planes $\xi_1/\lambda = \eta_1/\mu$ and for secondary planes $\xi_1\lambda + \eta_1\mu = 0$. Let ${}_p\Delta s'_i$ denote the aberration from the image-plane of the primary focus of a small pencil proceeding from a point distant x_i from the axis of the instrument and s_i from the i th surface, and whose chief ray is incident on the latter at a distance H_i from the axis (where now $H_i^2 = \xi_i^2 + \eta_i^2$).

Then, taking for convenience the primary plane as the plane of xz , the equation of the refracted ray being

$$\frac{Z - \frac{H_i^2}{2r_i}}{s'_i + \Delta s'_i - \frac{H_i^2}{2r_i}} = \frac{X - H_i}{x'_i + \delta x'_i - H_i},$$

we must have for primary foci

$$\frac{\left(s'_i + {}_p\Delta s'_i - \frac{H_i^2}{2r_i}\right)(x'_i + \delta x'_i - H_i)}{s'_i + \Delta s'_i - \frac{H_i^2}{2r_i}} + H_i$$

stationary for small changes in H_i .

This expression simplifies to

$$x'_i \left(1 + \frac{\Delta s'_i}{s'_i - r_i} + \frac{{}_p\Delta s'_i}{s'_i} - \frac{\Delta s'_i}{s'_i}\right) + H_i \left(\frac{\Delta s'_i}{s'_i} - \frac{{}_p\Delta s'_i}{s'_i}\right).$$

Differentiating, we get

$${}_p\Delta s'_i - \Delta s'_i = \left(\frac{r_i x'_i}{s'_i - r_i} + H_i\right) \frac{d\Delta s'_i}{dH_i} = \left(\frac{\lambda'}{Q_i} + H_i\right) \frac{ds'_i}{dH_i},$$

where $\lambda' = \lambda/h_i$.

Taking y_i and η_i zero in equation (9) and writing H_i for ξ_i , (9) may be written

$$\frac{\mu_i \Delta s'_i}{s_i'^2} - \frac{\mu_{i-1} \Delta s_i}{s_i^2} = \frac{1}{2} (Q_i H_i + \lambda')^2 \Delta \left(\frac{1}{\mu s}\right) + \frac{1}{2} \lambda'^2 \frac{1}{r_i} \left(\frac{1}{\mu_i} - \frac{1}{\mu_{i-1}}\right).$$

Hence we easily find that

$$\frac{\mu_i {}_p\Delta s'_i}{s_i'^2} - \frac{\mu_{i-1} {}_p\Delta s_i}{s_i^2} = -\frac{3}{2} (Q_i H_i + \lambda')^2 \Delta \left(\frac{1}{\mu s}\right) + \frac{1}{2} \lambda'^2 \frac{1}{r_i} \left(\frac{1}{\mu_i} - \frac{1}{\mu_{i-1}}\right). \quad (18)$$

So that the separation of the primary focus from the Gauss-image plane after refraction at the i th surface is given by

$$\begin{aligned} \frac{\mu_i p \Delta s_i' h_i^2}{s_i'^2} = & -\frac{3}{2} \left(\frac{H_1}{h_1} \right)^2 \sum_1^i Q_i^2 h_i^4 \Delta \left(\frac{1}{\mu s} \right) \\ & - 3 \left(\frac{H_1}{h_1} \right) \lambda \sum_1^i Q_i h_i^2 \Delta \left(\frac{1}{\mu s} \right) (1 + Q_i h_i^2 \Sigma) - \frac{3}{2} \lambda^2 \sum_1^i \Delta \left(\frac{1}{\mu s} \right) (1 + Q_i h_i^2 \Sigma)^2 \\ & + \frac{1}{2} \lambda^2 \sum_1^i \frac{1}{r_i} \left(\frac{1}{\mu_i} - \frac{1}{\mu_{i-1}} \right). \quad \dots \quad (19) \end{aligned}$$

The separation of the secondary focus is easily written down from (16). Denoting this by ${}_s \Delta s_i'$ we have

$$\begin{aligned} \frac{\mu_i {}_s \Delta s_i' h_i^2}{s_i'^2} = & -\frac{1}{2} \left(\frac{H_1}{h_1} \right)^2 \sum_1^i Q_i^2 h_i^4 \Delta \left(\frac{1}{\mu s} \right) - \frac{1}{2} \lambda^2 \sum_1^i \Delta \left(\frac{1}{\mu s} \right) (1 + Q_i h_i^2 \Sigma)^2 \\ & + \frac{1}{2} \lambda^2 \sum_1^i \frac{1}{r_i} \left(\frac{1}{\mu_i} - \frac{1}{\mu_{i-1}} \right). \quad \dots \quad (20) \end{aligned}$$

Assuming, then, the errors of spherical aberration and coma to have been eliminated, it is clear from equations (19) and (20) that the primary and secondary foci can only be brought into complete coincidence if also

$$\sum_1^i \Delta \left(\frac{1}{\mu s} \right) (1 + Q_i h_i^2 \Sigma)^2 = 0.$$

This term occurring as the coefficient of λ^2 , and therefore depending on x^2 , indicates that the corresponding astigmatism, which the evanescence of this expression removes, is due to the outer parts of the field.

It may be further noted, from equations (16), (19), and (20), that when central spherical aberration, coma, and astigmatism have been corrected the image still suffers a displacement from the focal plane given by

$$\frac{\mu_i \Delta s_{i+1} h_{i+1}^2}{s_{i+1}^2} = \frac{1}{2} \lambda^2 \sum_1^i \frac{1}{r_i} \left(\frac{1}{\mu_i} - \frac{1}{\mu_{i-1}} \right).$$

Hence

$$\frac{2 \Delta s_{i+1}}{\mu_i x_{i+1}^2} = \sum_1^i \frac{1}{r_i} \left(\frac{1}{\mu_i} - \frac{1}{\mu_{i-1}} \right). \quad \dots \quad (21)$$

The left-hand side of equation (21) is $\frac{1}{\mu_i}$ times the curvature of the image after the i th refraction. Hence for a flat image we must have

$$\sum_1^i \frac{1}{r_i} \left(\frac{1}{\mu_i} - \frac{1}{\mu_{i-1}} \right) = 0,$$

the well-known Petzval condition.

It remains to consider the defect of distortion due to unequal magnification in the inner and outer parts of the field. It is assumed that the image has been corrected for central spherical aberration, coma, astigmatism, and curvature, so that it now lies in the focal plane. We proceed to obtain the condition that it shall be a faithful copy to scale of the object.

The refracted ray at the i th surface is

$$\frac{X - \xi_i}{x_i' + \delta x_i' - \xi_i} = \frac{Y - \eta_i}{y_i' + \delta y_i' - \eta_i} = \frac{Z - \frac{1}{2r_i}(\xi_i^2 + \eta_i^2)}{s_i' + \Delta s_i' - \frac{1}{2r_i}(\xi_i^2 + \eta_i^2)}.$$

This cuts the image plane $Z = s_i'$ where $X = x_i' + \Delta x_i$, $Y = y_i' + \Delta y_i'$ (say).

Then we get

$$\begin{aligned} & \left(1 + \frac{\Delta x_i'}{x_i'} - \frac{\xi_i}{x_i'} \right) \left(1 + \frac{\Delta s_i'}{s_i'} - \frac{1}{2r_i s_i'} (\xi_i^2 + \eta_i^2) \right) \\ &= \left(1 + \frac{\delta x_i'}{x_i'} - \frac{\xi_i}{x_i'} \right) \left\{ 1 - \frac{1}{2r_i s_i'} (\xi_i^2 + \eta_i^2) \right\}, \end{aligned}$$

and similarly for the y 's.

Rejecting terms of the fourth order this gives

$$\frac{\Delta x_i'}{x_i'} - \frac{\delta x_i'}{x_i'} = -\frac{\Delta s_i'}{s_i'} \left(1 - \frac{\xi_i}{x_i'} \right).$$

By (5') we have

$$\begin{aligned} \frac{\Delta x_i'}{x_i'} &= \frac{\Delta s_i'}{s_i' - r_i} - \frac{\Delta s_i'}{s_i'} \left(1 - \frac{\xi_i}{x_i'} \right) = \frac{\mu_i \Delta s_i'}{s_i'^2} \cdot \frac{1}{Q_i} + \frac{h_i \xi_i}{x_i' s_i'} \Delta s_i' \\ &= \frac{\mu_i \Delta s_i'}{s_i'^2} \left\{ \frac{1}{Q_i} + \frac{h_i \xi_i}{\lambda} \right\} \text{ from (13)} \end{aligned}$$

A similar equation follows for the incident ray.

Therefore by (15) and (16) putting, as in Whittaker's tract,

$$\Theta_i = Q_i^2 h_i^4 \Delta \left(\frac{1}{\mu s} \right), \quad U_i = \frac{1}{h_i^2 Q_i} (1 + Q_i h_i^2 \Sigma),$$

$$\begin{aligned} \frac{\Delta x_i'}{x_i'} - \frac{\Delta x_i}{x_i} &= -\frac{1}{2} \left\{ \frac{1}{\lambda} \left(\frac{\xi_1}{h_1} \right) + \frac{1}{Q_i h_i^2} + \Sigma \right\} \left\{ \Theta_i \left(\frac{\xi_1^2 + \eta_1^2}{h_1^2} \right) \right. \\ &\quad \left. + 2\Theta_i U_i \left(\frac{\xi_1 \lambda + \eta_1 \mu}{h_1} \right) + \left(\Theta_i U_i^2 - \frac{1}{r_i} \left(\frac{1}{\mu_i} - \frac{1}{\mu_{i-1}} \right) \right) (\lambda^2 + \mu^2) \right\} \\ &= -\frac{1}{2\lambda} \left\{ \Theta_i \left(\frac{\xi_1}{h_1} \right) \left(\frac{\xi_1^2 + \eta_1^2}{h_1^2} \right) + \Theta_i U_i \left(\lambda \frac{\xi_1^2 + \eta_1^2}{h_1^2} + 2 \frac{\xi_1}{h_1} \left(\frac{\xi_1 \lambda + \eta_1 \mu}{h_1} \right) \right) \right. \\ &\quad \left. + \Theta_i U_i^2 \left\{ (\lambda^2 + \mu^2) \frac{\xi_1}{h_1} + 2\lambda \frac{(\xi_1 \lambda + \eta_1 \mu)}{h_1} \right\} \right. \\ &\quad \left. - \frac{\xi_1}{h_1} (\lambda^2 + \mu^2) \frac{1}{r_i} \left(\frac{1}{\mu_i} - \frac{1}{\mu_{i-1}} \right) + \lambda (\lambda^2 + \mu^2) U_i \left\{ \Theta_i U_i^2 - \frac{1}{r_i} \left(\frac{1}{\mu_i} - \frac{1}{\mu_{i-1}} \right) \right\} \right\}. \end{aligned}$$

Now the quantities ξ_1 , η_1 , h_1 , λ , and μ are the same throughout the system: hence by summation

$$\begin{aligned} \frac{\Delta x_i'}{x_i'} &= -\frac{1}{2\lambda} \left\{ \frac{\xi_1 (\xi_1^2 + \eta_1^2)}{h_1^3} \Sigma \Theta_i + \frac{\lambda (3\xi_1^2 + \eta_1^2) + 2\xi_1 \eta_1 \mu}{h_1^2} \Sigma \Theta_i U_i^2 \right. \\ &\quad \left. + \frac{\lambda^2 3\xi_1 + 2\lambda \mu \eta_1 + \mu^2 \xi_1}{h_1} \Sigma \Theta_i U_i^2 - \frac{\xi_1 (\lambda^2 + \mu^2)}{h_1} \Sigma \frac{1}{r_i} \left(\frac{1}{\mu_i} - \frac{1}{\mu_{i-1}} \right) \right. \\ &\quad \left. + \lambda (\lambda^2 + \mu^2) \Sigma U_i \left\{ \Theta_i U_i^2 - \frac{1}{r_i} \left(\frac{1}{\mu_i} - \frac{1}{\mu_{i-1}} \right) \right\} \right\} \quad \dots \quad (22) \end{aligned}$$

Now correcting for the four errors above mentioned we have in order

$$\Sigma \Theta_i = 0, \quad \Sigma \Theta_i U_i = 0, \quad \Sigma \Theta_i U_i^2 = 0, \quad \Sigma \frac{1}{r_i} \left(\frac{1}{\mu_i} - \frac{1}{\mu_{i-1}} \right) = 0.$$

But $\frac{\Delta x_i'}{x_i'}$ must be independent of λ^2 and μ^2 , since these depend on the square of the distance of the image-point from the axis.

$$\therefore \Sigma U_i \left\{ \Theta_i U_i^2 - \frac{1}{r_i} \left(\frac{1}{\mu_i} - \frac{1}{\mu_{i-1}} \right) \right\} = 0$$

is the condition for freedom from distortion for full pencils to this order of approximation. The same result is obtained by considering $\Delta y_i'/y_i'$. This of course necessitates that

$\Delta x'_i = 0$ $\Delta y'_i = 0$, as might be expected; since the image-point must now coincide with the normal magnified position given by (x'_i, y'_i) in the final image plane.

The expressions for the coordinates of the point in which the ray, after refraction at the i th surface, cuts the image plane, in the general case when the errors are uncorrected, may be easily written down.

Denoting

$$\Sigma \Theta_i, \Sigma \Theta_i U_i, \Sigma \Theta_i U_i^2, \Sigma \frac{1}{r_i} \left(\frac{1}{\mu_i} - \frac{1}{\mu_{i-1}} \right),$$

$$\text{and} \quad \Sigma U_i^2 \left\{ \Theta_i U_i^2 - \frac{1}{r_i} \left(\frac{1}{\mu_i} - \frac{1}{\mu_{i-1}} \right) \right\}$$

by A, B, C, D, E respectively, and putting $y_i = 0$ so that μ also vanishes, we have, if M be the linear magnification, $x'_i = Mx_1$ and the x -coordinate of the required point is

$$Mx_1 \left[1 - \frac{1}{2\lambda h_1^3} (A\xi_1(\xi_1^2 + \eta_1^2) + B(\lambda h_1)(3\xi_1^2 + \eta_1^2) + (3C - D)(\lambda h_1)^2 \xi_1 + E(\lambda h_1)^3) \right],$$

where λ is given by (13).

The y -coordinate is similarly

$$My_1 \left[1 - \frac{1}{2\lambda h_1^3} (A\eta_1(\xi_1^2 + \eta_1^2) + 2B\xi_1\eta_1(\lambda h_1) + (C - D)(\lambda h_1)^2 \eta_1) \right].$$

We shall conclude by deducing as illustrations of formulæ (18) and (20) the expressions for the deflexion of the primary and secondary foci from the focal plane in the case of a small parallel pencil incident centrally on a lens of index μ at an angle ϕ with the axis.

Here for the first refraction, since $\lambda/h_1 = \tan \phi$, $\Delta_1 \left(\frac{1}{\mu s} \right) = \frac{\mu - 1}{\mu^2 r_1}$, the deflexion of the primary focus is given by

$$\delta \left(\frac{1}{f_1} \right) = \frac{\tan^2 \phi}{2\mu} \left(3\Delta_1 - \frac{1}{r_1} \left(\frac{1}{\mu} - 1 \right) \right) = \frac{\tan^2 \phi}{2\mu^2} \frac{(\mu - 1)(\mu + 3)}{\mu r_1}.$$

At the second refraction similarly

$$\delta \left(\frac{1}{f_2} \right) = \frac{\mu^2 \tan^2 \phi'}{2} \left(3\Delta_2 - \frac{1}{r_2} \left(1 - \frac{1}{\mu} \right) \right)$$

$$\text{or} \quad \frac{\tan^2 \phi}{2} \left(\frac{3}{\mu} - \frac{3(\mu - 1)}{\mu^2 r_1} - \frac{\mu - 1}{\mu r_2} \right).$$

Hence the total deflexion of the primary focus is given by

$$\delta\left(\frac{1}{F}\right) = \frac{\tan^2 \phi}{2F} \left(\frac{3\mu+1}{\mu}\right) \text{ since } \frac{1}{F} = (\mu-1) \left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

Similarly for the secondary focus the deflexion $\delta\left(\frac{1}{F}\right)$ is

$$\frac{1}{2} \left(\frac{\lambda}{h_1}\right)^2 (\Delta_1 + \Delta_2) - \frac{1}{2} \left(\frac{\lambda}{h_1}\right)^2 \left\{ \frac{1}{r_1} \left(\frac{1}{\mu} - 1\right) + \frac{1}{r_2} \left(1 - \frac{1}{\mu}\right) \right\},$$

and since

$$\Delta_1 = \frac{\mu-1}{\mu r_1}, \quad \Delta_2 = \frac{1}{F} - \frac{\mu-1}{\mu r_1},$$

we have

$$\delta\left(\frac{1}{F}\right) = \frac{\tan^2 \phi}{2F} \left(\frac{\mu+1}{\mu}\right).$$

These formulæ were given by Coddington, but of late have been somewhat ignored by textbook writers, though used with success in the design of some modern photographic objectives.

XXV. Note on Reflexion from a Moving Mirror.

By W. GORDON BROWN*.

WITH regard to the question of alteration of amplitude by the Doppler effect, which was raised by Mr. Edser in October†, I should like to point out that the relation between the kinematics and electromagnetics of the problem is very clearly brought out by the Faraday tube theory of radiation.

In this theory a tube of force lying along the direction of the ray is supposed to transmit transverse vibrations like an elastic string. The displacement of the tube from its equilibrium position alters the direction of electric intensity. If it is assumed that the magnitude of the component parallel to the ray is unaltered, then the transverse component of electric force will be equal to this unaltered intensity multiplied by the tangent of the angle between the tube and the ray. Thus if r measures distance along the ray, z is the transverse displacement, and R , E are the r -, z -components of electric intensity,

$$E = R \frac{\partial z}{\partial r} \dots \dots \dots (1)$$

The magnetic force is the product of the velocity of the

* Communicated by the Author.

† Phil. Mag. vol. xxviii. p. 508.